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July 1927 Number 61

BULLETIN
OF THE
NATIONAL RESEARCH COUNCIL

TRANSACTIONS OF THE AMERICAN GEOPHYSICAL UNION
EIGHTH ANNUAL MEETING, APRIL 28 AND 29, 1927
WASHINGTON, D. C.

PUBLISHED BY THE NATIONAL RESEARCH COUNCIL
OF
THE NATIONAL ACADEMY OF SCIENCES
WASHINGTON, D. C.
1927

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of the
National Research Council

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INTRODUCTION

The American Geophysical Union is the American National Committee of the International Geodetic and Geophysical Union, and the Executive Committee of the American Geophysical Union is the Committee on Geophysics of the National Research Council. The object of the Union is to promote the study of problems concerned with the figure and physics of the Earth, to initiate and coordinate researches which depend upon international and national cooperation, and to provide for their scientific discussion and publication. In the accomplishment of this object, the Union is divided into six sections in accordance with the organization of the International Geodetic and Geophysical Union as follows: (*a*) Geodesy; (*b*) Seismology; (*c*) Meteorology; (*d*) Terrestrial Magnetism and Electricity; (*e*) Oceanography; (*f*) Volcanology.

Annual assemblies of the Union and annual meetings of its sections for the discussion of reports and symposia are held each year. Thus far there have been eight such assemblies and meetings, all held in Washington, D. C. The transactions of these annual assemblies and meetings have been published as bulletins of the National Research Council as follows: Second, 1921, Bulletin No. 17; fourth, 1923, Bulletin No. 41; sixth, 1925, Bulletin No. 53; seventh, 1926, Bulletin No. 56; and eighth, 1927, the present volume. The transactions of the first, third, and fifth annual meetings in 1920, 1922, and 1924, respectively, were devoted to discussion, there being no scientific papers presented, and, therefore, these transactions have not been printed and were mimeographed for limited distribution only.

The eighth annual assembly of the Union and the meetings of its sections were held April 28 and 29, 1927, in the building of the National Academy of Sciences and National Research Council at Washington, D. C. The assembly took place on the afternoon of April 29 and the meetings of sections on the morning and afternoon of April 28 and on the morning of April 29. The meetings were devoted chiefly to symposia and discussions on selected topics relating to current problems and developments in geophysics. An exhibit of geophysical instruments, researches, and applications was prepared and displayed during April 22 to April 29, 1927 (except Sunday, April 24), in the exhibit rooms of the Academy-Council building. Thanks to the efforts of the several institutions and individuals taking part, the exhibit proved of great interest.

The Union was favored by the attendance of an unusual number of visitors representing the National Committee of Canada and various Canadian governmental departments and universities, thus evidencing an active spirit of cooperation on the part of the two countries. Those in attendance from Canada were: A. Amos, Director of the Hydraulic Section of the Department of Lands and Forests of the Province of Quebec; A. S. Eve, Director of Physics, McGill University; Ernest A. Hodgson, Seismologist of the Dominion Observatory; A. G. Huntsman, Director of Atlantic Stations of the Biological Board of Canada; Noel J. Ogilvie, Director of the Geodetic Survey of Canada; John Patterson, Assistant Director, Canadian Meteorological Office, and Charles Savary, Chief of the Section of Geodesy and Cartography of the Department of Lands and Forests of the Province of Quebec. It will be noted that a number of these representatives also presented papers before the meetings.

In addition to the attendance of a majority of the members of the Union from all parts of the United States, the following guests and delegates representing various institutions and universities in the United States were present: W. S. Adams, Director, Mt. Wilson Observatory; Joseph G. Brown, Professor of Physics, Leland Stanford University; Perry Byerly, Seismologist, University of California; C. A. Heiland, Head of the Department of Geophysics and delegate from the Colorado School of Mines; J. S. Joliet, S. J., of the St. Louis University; L. A. Jones, Physicist of the Research Laboratory of the Eastman Kodak Company; E. H. Kincaid, Lieutenant, U. S. N., Hydrographic Office; G. W. Pickard, Consulting Engineer of the Wireless Specialty Apparatus Company; W. C. Repetti, S. J., of the St. Louis University; Parker D. Trask, Research Associate of the American Petroleum Institute, and George Winchester, Professor of Physics, Rutgers College and the State University of New Jersey. Authors other than members of the Union who presented or contributed papers at the meetings included: O. S. Adams, Coast and Geodetic Survey; L. B. Aldrich, Astrophysical Observatory; E. T. Allen, Geophysical Laboratory; H. C. Avers, Coast and Geodetic Survey; Gregory Breit, Department of Terrestrial Magnetism; J. P. Buwalda, California Institute of Technology; I. F. Hand, U. S. Weather Bureau; C. V. Hodgson, Coast and Geodetic Survey; Paul Kirkpatrick, University of Hawaii; F. B. Littell, Naval Observatory; W. C. Mendenhall, Geological Survey; J. T. Pardee, Geological Survey; T. Parkinson, Bureau of Standards; Edison Pettit, Mt. Wilson Observatory; E. S. Shepherd, Geophysical Laboratory; Philip S. Smith, Geological Survey; C. A. Swick, Coast and Geodetic Survey, and E. G. Zies, Geophysical Laboratory.

The success of the meetings was largely due to the active interest of the delegates, guests, and authors of papers presented, and this opportunity is taken on behalf of the Executive Committee of the Union to express most grateful appreciation to them.

RESOLUTIONS

RESOLUTION ON TRANSLATIONS OF REPORTS ON SEISMOLOGICAL INVESTIGATIONS PUBLISHED IN THE JAPANESE LANGUAGE

(Submitted by Section of Seismology and adopted at general assembly of the Union, April 29, 1927)

Whereas, it has become known that the reports of much of the seismological investigations carried on in Japan will hereafter be published in the Japanese language only, and

Whereas, This procedure is calculated to deprive most of the American students in this field of research of the advantages of this literature, be it

Resolved, That this matter be brought to the attention of the National Research Council in the hope that the Council may provide that this literature be rendered into English, also that provision be made whereby mimeographed copies of these translations be supplied investigators at research institutions gratis and to business concerns, insurance companies, and others interested at cost, and

Resolved, further, That, should such an arrangement be feasible, a committee of the American Geophysical Union be empowered to make a choice of the material to be so translated and distributed.

REPORTS AND PAPERS

GENERAL MEETING

(Washington, D. C., April 29, 1927)

REPORT OF THE DELEGATES OF THE AMERICAN GEOPHYSICAL UNION TO THE THIRD PAN-PACIFIC SCIENCE CONGRESS AT TOKYO, JAPAN, IN OCTOBER, 1926

(Delegates—Arthur L. Day, N. H. Heck, T. A. Jaggar, G. W. Littlehales, George F. McEwen, H. F. Reid, T. Wayland Vaughan, Bailey Willis)

The interest of the American Geophysical Union in the Third Pan-Pacific Science Congress held at Tokyo, Japan, in October, 1926, lies in the fact that the Division of Physical Sciences of the National Research Council included in its scope all the subjects of geophysics into which the work of the American Geophysical Union is divided. Furthermore, in establishing the Pacific Science Association as a permanent organization, the plan was adopted of having each country represented by a National Committee, in the case of the United States, the National Research Council. Inasmuch as the function of the American Geophysical Union is to keep the National Research Council informed of the activities coming within its field, there is an opportunity for service in this connection.

Abstracts of all papers were printed in advance of the meetings, but the complete papers are not yet available. All that can be done at present is to give a broad picture of the activities in regard to the six subjects represented by the sections of the American Geophysical Union.

OCEANOGRAPHY

This subject will be reported in considerable detail by Dr. T. Wayland Vaughan to the National Research Council and those interested may consult his report. The activities in oceanography were well organized as there was in existence a Committee on the Physical and Chemical Oceanography of the Pacific. This, in addition to its interim activities, provided a basis for informal action before and during the meetings and proved helpful by promoting informal discussion and preparing resolutions and recommendations.

About forty papers were presented on subjects directly relating to oceanography and there were others bearing on oceanographic problems. At least ten countries were represented. Both the physical and biological divisions of oceanography were discussed. The subjects relating to geophysics included: Oceanic circulation; mean sea-level; bottom configuration as obtained both by direct measurement and by acoustic methods; correct velocities for acoustic soundings: relations of seismology to oceanography; marine meteorology; proposed expeditions including that of the *Carnegie* and other related subjects.

In the case of oceanography, resolutions proposed by the Division of Physical Sciences and approved by the Congress covered a wide range. One of the most important was that calling for the formation of a Committee on Oceanography for the Pacific, replacing the committee which has been mentioned and broadening its scope. It is contemplated that the work of this Committee will eventually result in the formation of an international council for the exploration of the sea for the Pacific similar to that now in existence for the Atlantic. The president of the Congress appointed Dr. T. Wayland Vaughan as chairman of this committee. Other resolutions dealt with the organization of the committee and provided for subcommittees on special subjects.

Another resolution called for closer cooperation between the National Committees and between members of subcommittees and suggested the submission of programs and seeking of suggestion and advice in order to secure standardization and coordination, and provided for reports to the International Chairman to be submitted by him at future meetings.

Other resolutions related to obtaining more numerous and more accurate surface-temperature readings and called attention to the need for their publication; to the daily charting of the meteorology of the world; to the study of coral reefs, and to the need for complete determination of the bottom configuration, especially in the vicinity of islands.

The work of the Dutch Geodetic Commission through Dr. J. Vening Meinesz in making gravity-determinations at sea by means of a submarine was strongly endorsed, and the hope expressed that other nations would take up and extend this work.

A number of the delegates visited the Marine Biological station at Misaki, Sagami Bay, and the Hydrographic Office at Tokyo, thereby becoming familiar with some of the oceanographic work of Japan.

SEISMOLOGY

Naturally this subject received considerable attention, three meetings being given to various phases of the subject and ample opportunity being given to visit the seismological installations in the vicinity of Tokyo and elsewhere.

The meeting of the Section of Seismology included two symposia, one on the network of earthquake observations of the Pacific, the other on the transmission of earthquake waves in the Pacific region. Representatives from three countries discussed these subjects. Other papers described a long-period pendulum now in use in Japan and another the relation to the focus of the point of beginning of the long waves. One paper dealt with the correlation between earthquake-frequency and earth-tides and another on variation of latitude, tide, and earthquakes. That biology can not be eliminated from earthquake study was shown in the paper on the effect of the Kwanto earthquake on marine organisms.

The subject of tectonics, especially crustal movements connected with earthquakes, was discussed in connection with matters relating to geodesy and one meeting was given to the principles of earthquake-proof construction of buildings, involving, of course, the motions which have to be resisted. No resolutions were presented from this section, not so much on account of lack of need for international cooperation, but for the reason that there has been no machinery to provide for such coordination. A resolution originating in this section provided for interim committees for those sections which desired them, to deal with matters relating to the subject between meetings and serve as an informal means of preparing resolutions and recommendations at the meetings. It is hardly to be expected that progress will be made without such organization and it is hoped that the new plan will prove useful at the meeting in Java.

A number of the delegates were able to visit some of the institutions where seismological observations are made though the time available was totally inadequate. The Seismological Institute of the Imperial University of Tokyo has a very elaborate collection of instruments from the earliest to the more modern. The Central Meteorological Observatory at Tokyo also operates a number of seismographs including Japanese and European designs. Special instruments have also been developed at Kyoto and Kyushu universities.

It is of special interest to note that as a result of the great earthquake all the older instruments are being replaced with newer types having electro-magnetic damping. The extent of this undertaking is evident when it is realized that every institution in Japan which is capable of carrying on the work is operating seismographs.

It was learned that although there is considerable publication in English, much valuable work has never been translated; and it appears that it is a proper task for the Geophysical Union to find a means of translating the more essential parts of the available material.

The amount of work being done in Japan is so great and the problem so complicated that it is difficult to determine what progress has been made.

GEODESY

This subject was treated in connection with tectonics. There was a general presentation of the geodetic and geophysical problems of the Pacific. The need for gravity-observations at sea was emphasized and there was some discussion of isostasy. However, the chief subject on which geodesy has a bearing was the discussion of crustal movement especially in Japan. It will be noted that under the discussion of oceanography the extension of gravity-observations at sea was urged.

TERRESTRIAL MAGNETISM AND ATMOSPHERIC ELECTRICITY

Representatives of three countries discussed recent contribution to the knowledge of terrestrial magnetism in the Pacific. One paper gave the results of a study of the magnetism of volcanic rocks. The ocean work of the *Carnegie* was described with special reference to the work in atmospheric electricity. Atmospheric-electric observations in the Dutch East Indies were described.

A number of the delegates visited the Kakioka Magnetic Observatory, about seventy miles northwest of Tokyo. This is now being modernized, all of the buildings are being replaced and entirely new instrumental equipment is being installed. Atmospheric potential-gradient observations are being made. Special apparatus for the study of abrupt beginning of magnetic storms is in use.

METEOROLOGY

There was a symposium on typhoons with discussion of the subject as based on very complete studies in the Philippine Islands, China, and Japan. It was brought out that the great number of observing stations is making it possible to follow and study a typhoon from its inception till it disappears.

Upper-air observations from airplanes with some of the difficulties involved were discussed. Weather types of China and the Netherlands East Indies were treated. The distribution of accidents due to lightning in Japan was discussed, interesting statistics being given.

In addition to visits by most of the delegates to the Central Meteorological Observatory at Tokyo, a group visited an aerological station and one of the mountain weather-stations.

This is scarcely the place to discuss the hospitality and the elaborate social entertainment. It is important, however, that record should be made of the many courtesies extended. Great care had to be taken to express only reasonable wishes as they were certain to be gratified. Our questions were answered in detail, and every effort was made to explain to us, where necessary, the various things that we saw. Many of the delegates left Japan feeling regret that the time was so short that they could not do more than obtain a broad picture of what is going on in science. This is particularly the case in seismology, a subject of great practical importance in that country.

The contacts made with scientists from all parts of the Pacific are of value not only to the delegates but they are an asset to the Union in carrying on its work. It is important that the American Geophysical Union as it more and more adequately fulfills its functions should not overlook the importance of keeping in close touch with the rapidly growing scientific effort in the Pacific region.

SYMPOSIUM ON SOME FACTORS OF CLIMATIC CONTROL

INTRODUCTORY REMARKS

W. J. Humphreys

Numerous factors enter into the problem of the control of climate, and climate is subject to change. The geologists tells us that we have had many variations of climate in the past, and we know that we have many varieties, sometimes even in the same locality, in New England, for instance. What are the things that determine the climate for a particular place? The radiation from the Sun, the composition of the atmosphere that permits certain radiations to get through and shuts others out, the way the wind blows, the way the water flows, the heights of mountains and their locations in respect to the oceans and directions of prevailing winds, and a host of other influences. If we pulled down mountains, the climate of that region would be modified. If we erected high mountains, the climate in that section would be greatly changed, made more rainy on the windward side and drier to the leeward. Great quantities of fine dust in the upper air would make our climate colder. A variety of things may be factors, and they have not always been the same—constant or of equal importance. There was a time when there was free circulation from the Gulf of Mexico to the Arctic Ocean, and from the Persian Gulf and Caspian Sea also to the Arctic Ocean. Another factor is the nature of the covering of the Earth; if it were covered with white sand, the climate would be different than if covered with black soil. The white sand would reflect much insolation which the black soil would absorb. Again, there appears to be a critical extent for polar ice-caps such that a comparatively slight change in this extent, either increase or decrease, would lead automatically to a much greater change in the same direction. The combination of such factors obviously would produce effects which separately they would not. Many of these factors could be discussed here to good advantage, especially all those considered by C. E. P. Brooks in his "Climate Through the Ages." Obviously though, it is quite impossible to cover the whole of this subject in the short time available, so that we have arranged for papers on only the more important factors.

U. S. Weather Bureau,
Washington, D. C.

THE INFLUENCE OF THE ATMOSPHERIC CONSTITUENTS UPON CLIMATE

L. B. Aldrich

Climate is largely determined by the prevailing range of temperature. The temperature is maintained first, by the solar radiation which continuously falls upon our atmosphere at the mean rate of 1.93 calories per square centimeter per minute; and secondly, by the exceedingly complicated series of absorptions, scatterings, reflections, endless re-radiations, and convection- and conduction-effects which result from the passage of the solar beam through the atmosphere. Langley,[1] and more recently Dietzius[2] and others, have shown that the Moon, receiving solar radiation per unit area identical with the Earth, experiences within the brief interval of a lunar eclipse a change of temperature ranging from about that of boiling water to far below freezing. On the Earth the average daily range is less than 10° C and seldom exceeds 20° C. The difference is due to this: the Earth has an atmosphere; the Moon has not. Oxygen and nitrogen, which comprise the bulk of the atmosphere, have relatively little effect on temperature. From other constituents, however, comprising less than three per cent of the weight of the atmosphere are highly important as affecting temperatures. These constituents are water (both as water vapor and in the form of clouds), dust, ozone, and carbon-dioxide.

What happens to the solar beam after it enters the atmosphere? On a clear day at sea-level, with the Sun in the zenith, about seventy-five per cent of the incident solar-radiation is transmitted directly to the Earth through the atmosphere. The remainder is partly scattered and reaches the Earth in the form of sky-light, partly selectively absorbed, in the main by water vapor, and partly reflected out to space. For other altitudes of the Sun, as the air-path increases the amount directly transmitted decreases and the sky-brightness increases. On hazy days, due to a larger content of water vapor or fine dust or both, less radiation is directly transmitted and more is scattered, absorbed and reflected. To illustrate the variation in intensity of the directly transmitted beam even in a cloudless sky and at an altitude of 3,000 meters above sea-level, I have chosen more or less at random the following observations on four different days at the Montezuma station of the Smithsonian Institution.

The last column is a measure of the amount of scatter due both to dust and water-vapor haze. The first two days show that an increase of

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[1] Mem. Nat. Acad. Sci., Vol. 4, 9th Mem.: 193 (1890).

[2] Akad. Wiss. Wien, Ber., (2A), 132: 5-6, 193 (1924).

Date	Calories directly transmitted (per cm^2 per minute, at air mass 1.5 and mean solar-distance)	Precipitable water (determined spectroscopically)	Brightness of sky in a concentric ring 30° diameter around the Sun (in calories per cm^2 per minute)
Aug. 6, 1920.......	1.632	.074 cm	.0100
Aug. 10, 1920......	1.527	.916 cm	.0100
Aug. 21, 1924......	1.468	.260 cm	.0208
June 22, 1924......	1.579	.310 cm	.0103

water-vapor content depleted the solar beam seven per cent. The last two show that though the water-vapor content remained unchanged, an increase of dust sufficient to double the diffuse scatter also produced a seven per cent depletion. And comparing the first and third days, an eleven per cent depletion is found in a hazy, moderately wet sky as compared with a dry, clear sky. On cloudy days practically no radiation is directly transmitted and more than seventy-five per cent is reflected back to space from the upper surface of the cloud.[3]

The temperature at the surface of the Earth results from the balance between the incoming solar-radiation and the outgoing terrestrial-radiation. The albedo of the Earth is estimated to be forty-three per cent.[3] This leaves fifty-seven per cent available to warm the Earth. To maintain its mean temperature unchanged, the Earth must radiate to space as much as it receives. With a mean solar-constant of 1.93 calories, we have $1.93 \times .57 \times \frac{1}{4} = .28$ calories per cm^2 per minute being radiated to space by the planet Earth. Abbot[4] has shown that less than one-fourth of this is radiated by the Earth directly to space. The atmosphere is the principal source of the outgoing radiation—largely because of the absorptive power of water vapor and clouds, assisted in smaller degree by carbon-dioxide, ozone, etc.

The incoming solar-radiation lies almost wholly within the wave-lengths 0.3 μ to 3.0 μ with a sharp maximum at 0.47 μ. The outgoing terrestrial-radiation, since it comes from a far cooler source, lies almost wholly within the wave-lengths 4 μ to 50 μ with a maximum at 9.5 μ. The selective absorption of water vapor for solar-radiation occurs largely in a number of bands of wave-length greater than 0.8 μ. This is considerably beyond the maximum of intensity. On a clear but humid day the absorption of solar radiation by water vapor would rarely be so great at fifteen per cent. In contrast, water vapor absorbs terrestrial radiation almost completely, except in the region around 10 μ, and even there it is partially absorbing. Fowle[5] computes that in an atmosphere containing one cm precipitable water, seventy-two per cent would be absorbed. This "greenhouse effect"—transmitting solar radiation but screening off the terrestrial—is still strong even in a very dry atmosphere. As Abbot shows, with so little as 0.003 cm of precipitable water, nearly fifty per cent of terrestrial radiation is absorbed. This fact—that even a very small amount of water vapor so strongly absorbs terrestrial radiation—makes it evident that reasonable variations in the water-vapor content of the atmosphere would have little effect on climate. One important factor, however, should be mentioned in this connection. At present on the average about fifty per cent of the Earth's surface is cloud covered. C. E. P. Brooks[6] has shown that, other conditions remaining the same, if the total amount of cloud were increased by one-tenth it would probably suffice to lower the mean temperature by 5° C, and vice versa. This does not seem surprising since the upper surface of a cloud reflects seventy-eight per cent of the incident solar-radiation, so that the quantity of heat available to warm the Earth is greatly decreased by an increase in total cloud-surface.

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[3] See Smithsonian Misc. Collect., 69, no. 10.

[4] Proc. Nat. Acad. Sci., 4: 104 (1918).

Dust particles suspended in the atmosphere have a quite different effect on temperature from that of water vapor. In 1913 Abbot and Fowle[7] published a paper on volcanoes and climate which showed that following the great eruption of Mt. Katmai in Alaska in June, 1912, the skies of Bassour, Algeria, and Mt. Wilson, California, were unusually hazy. Daily observations at both places disclosed that the atmospheric transmission-coefficients were reduced fully twenty per cent. This condition continued, though in less degree, during the summer of 1913. Abbot and Fowle also showed that similar hazy skies followed some of the great volcanic eruptions of former years. Humphreys[8] has made a careful study of the effects of fine dust in the upper air. He points out that dust particles absorb more of terrestrial than of solar radiation and this would tend to increase the Earth's temperature. He shows, however, that absorption is not the controlling factor and that scattering and reflection from dust particles produce the opposite effect. Dust particles, being materially larger than air molecules, affect the long waves of terrestrial radiation in much the same way as air molecules act upon the shorter waves of solar radiation. That is, they merely scatter terrestrial radiation and reflect little. The solar radiation, however, is both scattered and reflected in large measure by the dust particles. On reasonable assumptions, Humphreys computes that the total effect of dust in preventing the transmission of solar radiation is thirty times as great as it is with terrestrial radiation. This he calls an "inverse greenhouse effect" which produces an eventual lowering of the surface-temperature. There is considerable evidence that lowered temperatures have followed volcanic eruptions. Brooks[9] computes that the twenty per cent reduction in the atmospheric transmission-coefficients such as occurred in the Katmai eruption of 1912, if long continued would lower the mean temperature of the earth over 5° C. On the other hand, he points out that the complete absence of volcanic dust would not raise the mean much over ½° C, an amount insufficient to account for the temperatures which prevailed during warm geological epochs.

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[5] *Ibid.*, p. 105.

[6] "Climate through the Ages," London, 1926, ch. 7.

[7] Smithsonian Misc. Collect., 60, no. 29.

[8] Physics of the Air, J. Franklin Inst., part 4, ch. 8 (1920).

Arrhenius,[10] in 1896, basing his computations on Langley's early work on the temperature of the Moon, estimated the effect of variations in the quantity of carbon dioxide in the atmosphere on the temperature of the Earth. He concluded that two and five-tenths to three times the present amount of CO_2 would increase arctic temperatures 8° or 9° C. If the CO_2 were decreased to 0.6 of its present value, he computes a lowering of temperatures in latitudes 40° to 50° of 4° or 5° C. Chamberlin[11] elaborated this theory in an effort to explain climatic variations during geological ages. Abbot and Fowle,[12] however, showed that as little as one-fortieth of the present amount of CO_2 exerted the maximum possible absorption-effect, and that in the two more important absorption bands of CO_2, at 4 to 5 μ and 14 to 15 μ, neither of which are in the region of maximum intensity of the terrestrial radiation energy-curve, water-vapor is nearly completely absorbing. Hence, even though the amount of carbon dioxide were to vary through a wide range the effect on the surface-temperature would be negligible. Humphreys[13] points out that a variation in CO_2 at levels above the water-vapor level, particularly a decrease, might alter temperatures, but that the total effect would be small.

The exact role of ozone in the atmosphere is still undetermined. The existence of ozone in the upper air has been shown by many observers.

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[9] Climate through the Ages, p. 134.
[10] Phil. Mag., London, 41: 237 (1896).
[11] J. Geol. 7: (1899).
[12] Ann. Astrophys. Obs. Smithsonian Inst., 4: 25.
[13] Physics of the Air, p. 567.

And there is general agreement that the amount present is equivalent to about three mm of ozone at atmospheric pressure and that the total quantity is somewhat variable. A recent paper by Fabry[14] entitled "The Absorption of Radiation in the Upper Atmosphere" summarizes our present knowledge of the effect of ozone upon solar radiation. He shows that ozone must occur principally at very high altitudes. His theory as to its formation is as follows:

"Radiation of very short wave-length (λ less than 1850) acts upon oxygen transforming it into ozone; at the same time this radiation is strongly absorbed by the air. It is probable that such radiation exists in the solar emission; it must produce ozone in the Earth's atmosphere, but only in the highest part because it cannot reach the lower strata. Radiation of $\lambda = 1850$ is completely absorbed by ten meters of air at atmospheric pressure and could scarcely penetrate lower than the forty-km stratum. On the other hand, radiation lying between $\lambda = 2000$ and $\lambda = 3000$ decomposes ozone, and between these two inverse actions a state of equilibrium must be established, the quantity of ozone depending on the relative intensities of the two ranges of radiation." Assuming that the solar ultra-violet emission is identical with that of a black body at 6000° K, Fabry computes that ozone in the upper air obsorbs about four per cent of the incident radiation when the Sun is in the zenith. As a result he concludes that the temperature of the upper strata may well be perceptibly higher than that of the stratosphere. Ladenburg and Lehmann[15] found an ozone absorption band at 10μ. Also, Fowle[16] found a distinct absorption at 10μ in the solar spectrum which could not be ascribed to water vapor or carbon dioxide and which did not appear in his tube-work. This absorption band, occurring close to the maximum of terrestrial radiation and in the place where water-vapor absorption is weakest, must have a distinct tendency to increase surface-temperatures by the action of the "greenhouse effect"; but from existing data it would be difficult to determine the magnitude of this effect. Doubtless the near future will tell us much regarding the absorptive qualities of atmospheric ozone.

SUMMARY

In considering reasonable or possible changes in the composition of the atmosphere the following three would seem competent, if sufficiently long continued, to produce marked changes of climate:

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[14] Proc. Phys. Soc., 39: part 1, Dec. 15, 1926.

[15] Ann. Physik, 21: 305 (1906).

[16] Ann. Astrophys. Obs. Smithsonian Inst., 4: 285.

(1) A change in total cloudiness, provided all other factors remained unchanged.

(2) An increase in fine dust, such as was produced by the eruption of Mt. Katmai in 1912. This would effectively lower the temperature. A decrease in dust-content would not, however, be competent to produce the temperatures of warm geologic epochs.

(3) A large increase in the amount of ozone in the upper air would probably effectively lower surface-temperatures. As yet, however, exact data are unavailable regarding the formation of atmospheric ozone and regarding its ability to transmit radiation.

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THE GENERAL CIRCULATION OF THE ATMOSPHERE AS A CLIMATIC CONTROL

ALFRED J. HENRY

Meteorologists of the first half of the nineteenth century thought of the general atmospheric circulation as consisting essentially of a rising current of warm air in the tropics with an overflow poleward in the higher levels, a descent at the poles, and a return current on the surface of the Earth.

The modern notion, although it holds to the convective interchange between the equator and the poles, discards the idea of warm air rising in the tropics and flowing poleward and substitutes therefor a poleward flow of disconnected masses of warm air side by side with similar masses of cold polar air flowing equatorward. This interchange takes place for the most part through the action of traveling cyclones and anticyclones of the temperate latitudes.

The major climatic control. The interaction of these two differently directed masses of warm air and cold air, respectively, causes variations in weather and climate, best exemplified in temperate latitudes of the Earth; and it may be truly said that the temperature contrast between equatorial and polar regions is the major climatic control. This control is largely modified by the unequal distribution of land and water surfaces, so much so, that climates are divided into two great classes, viz, continental and marine.

The Northern Hemisphere, with about forty per cent of land surface, has a moderate continental climate; the Southern Hemisphere, with but seventeen per cent, has a fairly typical marine climate.

Ocean currents, largely initiated by the winds, as pointed out by

Ferrel many years ago, are more effective as a climatic control than the winds themselves. The general effect of the atmospheric circulation is to equalize, so far as may be possible, the temperatures of continents and oceans, to cause large differences between the climates of the east sides as compared with the west sides of the continents—a difference that increases with the latitude—also, to produce wet and dry zones such as those in the trade-wind regions.

Semi-permanent cyclones and anticyclones. The general circulation of the atmosphere as modified by the rotation of the Earth on its axis creates in various parts of the globe six more or less permanent formations known as "Semi-permanent cyclones and anticyclones," each with its characteristic wind-circulation. Five of the six anticyclones are situated over the oceans, and the sixth over eastern Asia. The anticyclones are best developed over the oceans in late summer and over Siberia in winter. The semi-permanent cyclones are most fully developed in winter and over the oceans. The circulation of the atmosphere in these formations exercises an important influence upon the climate of the regions in which they are situated and over large bordering areas.

Unfortunately, the study of world weather, as such, is not yet fully within our grasp, although a beginning has been made in the publication of world-wide weather statistics in a single volume, that is, the *Reseau Mondial,* issued as Part V of the British Meteorological Yearbook.

It is a reasonable assumption that the general circulation of the atmosphere has not changed substantially in the last million years and that catastrophic changes in climate have not occurred within that time with the possible exception of a progressive desiccation in parts of Asia, reported upon by Huntington, and the pronounced lowering of the temperature of Greenland, which according to C. E. P. Brooks took place from about the fifth to the tenth or eleventh centuries of the Christian Era. Climatic oscillations of the present are confined to restricted areas and are ephemeral in character; thus, southern Europe has recently experienced a series of about ten warm winters and the last winter in the United States was what may be the final one of a series of five mild winters. Changes of this character are short-lived and are doubtless due to small changes in the general trend of the larger atmospheric currents—changes that for the present, elude systematic observation and analysis.

It is conceivable that an increase or decrease in the intensity of the general circulation, however brought about, must affect the climate of the globe. In quite recent years not a little speculation has been indulged in as to the effect of an increase or decrease in the amount of solar energy received on the Earth's surface. It has been suggested that a rise in temperature in the tropics, accompanying increased solar-radiation would produce a horizontal flow of cold air from high latitudes and thereby tend to cool temperate regions; also, that in years with high solar-radiation, land areas in summer would be warmed to a greater degree than oceanic areas and thus tend to produce lower pressure over the continents and therefore an exchange of oceanic and continental air. This, however, would be but the normal condition of the continents in the summer.

It has been further suggested that variations in solar radiation in all probability would make themselves first felt in the higher levels of the atmosphere and thereby produce disturbances which in turn would induce changes in the lower layers. As heated air tends to rise rather than to sink, it is difficult to follow this suggestion.

The effect on the weather of changes in solar radiation, although apparently a simple problem, is far more complex than it would seem to be. Before definitive results can be reached, much more study is necessary.

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THE OCEAN AMONG THE FACTORS OF THE CONTROL OF CLIMATE

G. W. LITTLEHALES

Overspreading more than two-thirds of the surface of the globe, throughout all longitudes and nearly from pole to pole, in basins whose cubical content is fourteen-fold greater than the bulk of all the land of the world above sea-level, is a medium whose capacity for heat far outruns the like property in any other abundant substance in Nature. The place of water in the economy of Nature is most remarkable. Everywhere natural conditions are seen to be molded and influenced by its wonderful properties. Although it is nowhere told how much water there is in the world, there is no doubt that the presence in the oceanic basins of 324,000,000 cubic miles of this substance must constitute the ocean a dominant factor in geophysics, exercising through its circulation a governing influence upon heat transference and upon climate. Throughout the ocean's vast expanse of one hundred and forty million square miles, exceeding, by eighty million square miles, the total area of the land surface of the Earth, it is never in equilibrium, and, consequently, forever in reaction under stresses both external and internal, resulting in movement to bring about a redistribution of mass.

Internal forces are generated in the ocean whenever any change takes place in the physical properties of the water itself; that is, if either the temperature or the salinity is changed, then a dynamical reaction must arise and a redistribution of mass must ensue. The internal system of forces in an ocean are disturbed whenever it becomes the abode of organisms whose life processes alter the salinity, whenever the mass radiates or absorbs heat or undergoes evaporation, whenever precipitation from the atmosphere is received or the inflow of rivers takes place, or whenever internal physical transformation occurs as a result of turbulence and mixing.

It is in respect of temperature that the ocean exhibits some of its most remarkable characteristics. Over eighty per cent of its bulk remains at a temperature below 40° F. More than ninety per cent of the sea bottom is thus covered, including all those great tracts, in depths greater than 2000 fathoms, where the temperature is uniformly found to be within the range of a few degrees above the freezing point of fresh water. Salt water freezes at a lower temperature than fresh water: sea water with a salinity of thirty-five parts per thousand freezes at 28° F, and it is not uncommon to find temperatures but little above this in the waters of the polar seas. The office which the ocean performs in the economy of Nature by virtue of its saltness, it could not perform were its waters wholly fresh. About sixteen per cent of the entire ocean surface has a temperature below 40° F. In the surface waters of the lower latitudes there is a progressive rise in temperature, reaching the average of 81° near the Equator; but the position of the isothermal lines is much modified by prevailing winds, by the situation of land-masses, and by the distribution of areas of barometric maxima and minima. The warmer surface waters occupy only a comparatively small depth even in the tropics.

Although the mean daily variation of temperature in the surface waters of the open ocean is probably less than 1° F, the range from season to season may amount to 50° F in those regions, like the oceanic areas around Newfoundland, Japan, and the Cape of Good Hope, where the surface is occupied successively by waters of polar and equatorial origin; but on the other hand, there are very extensive regions both in the tropical and polar waters where the range does not exceed a few degrees during the course of the year. With descent into the depths, the seasonal changes tend rapidly to diminish.

It is in the aerial ocean overlaying the marine ocean that the prime mover in oceanic circulation is found. Operating around regions of barometric maxima in the eastern part of the temperate zone, both north and south of the equator, in all the oceans, where action-centers exist through the accumulation of more than the average amount of the Earth's atmosphere, the winds induce surface currents in the ocean that move in a clock-wise circuit in the northern hemisphere and in an anti-clockwise circuit in the southern hemisphere, with modifications brought about by the influence of atmospheric centers of action where there is less than average amount of the Earth's atmosphere, as obtains near Iceland, near the Aleutian Islands in winter, and in the polar borders of the south temperate zone, and yet others where there is alternately less or more of of the Earth's atmosphere, as obtains over the continent of Asia. In the trade-wind regions, evaporation is very great and hence the salinity of the surface waters is high; but as the temperature is also high in these regions, the highly saline water remains at the surface during its westward drift in the tropical region on both sides of the equator and thence poleward in the western part of the oceans until, by reduction in temperature, the density becomes greater than that of the surrounding waters. The systematically varying distribution of salinity and temperature giving rise to a systematic distribution of density or heaviness of the waters, is one of the basic factors in the sub-surface or inner circulation of the ocean, causing a systematic descent from the surface of the heavier waters which characterize the middle latitudes and a slow movement thence to the deeper waters, entirely distinct from the oscillations of the mass of the ocean set up by seiches and tides both toward the pole and toward the equator, where ascending currents rise from the depths. The characteristics of this system of circulation may be interpreted by profiles showing the distribution of temperature in the depths and confirmed by profiles showing the distribution of oxygen in the depths. In the former, the isothermal lines show a pronounced downward trend in the middle latitudes and a pronounced upward trend in the equatorial regions; and in the latter, it is evident that the waters in the depths of the ocean in the equatorial regions have been much longer out of contact with the atmosphere than the waters of the temperature latitudes which, descending from recent contact with the atmosphere, contain by absorption about three times as much oxygen as the deep equatorial waters which have ascended from the depths at the close of the long circuit of the bottom waters toward the equator. The pressure caused by the weight of the elevated masses about to be described, being transmitted to the bottom of the ocean, has the effect of intensifying these deep poleward and equatorward movements.

These massive movements are in striking contrast to the relatively rapid rate of motion of the surface currents impelled for the main part in general accord with the anti-cyclonic circuit of the winds, which, in their direct and indirect effects taken in conjunction with the configuration of the continents, constitute the prime agency in originating and maintaining the circulation of the ocean.

The movement of the surface drift is communicated by friction and vertical interchange of water to the underlying layers, but owing to the rotation of the globe, the surface current caused by a steady wind is inclined to the wind's direction at an angle of 45°, to the right in the northern hemisphere and to the left in southern hemisphere. With descent beneath the surface, the currents deviate more and more from the winds, and the main mass of the water moves at right angles to the direction of the wind. Consider now the case of the action-centers of high barometric pressure around which the winds circulate in a clockwise direction in the northern hemisphere and in an anti-clockwise direction in the southern hemisphere. In the northern hemisphere, all these winds will be driving the water to the right and in the southern hemisphere to the left, that is, toward the center of the system in each case. The result is that the water is piled up in the center into an elevation down the slopes of which the water will run. The rotation of the globe will deviate the water to the right in the northern hemisphere and to the left in the southern hemisphere, and when a steady state is reached it will be flowing around the central elevation in a clockwise direction in the northern hemisphere and in an anti-clockwise direction in the southern hemisphere.

As a consequence of this circulation, heat is conveyed by the route of the western and polar borders of the oceans across the planetary zones. For instance, in the North Atlantic Ocean, the waters of the tropics is born far beyond the Arctic Circle to the shores of Spitzbergen. The world-isotherms show how the temperature along the northwest coast of Europe is thus affected.

In concomitance with the penetration of warm water into the polar regions, there is a penetration of polar water into the lower latitudes in all the oceans. Where the warm waters that reach Spitzbergen and Novaya Zemblya are disappearing beneath the lighter waters, discharged upon the Arctic Ocean from the great rivers of the northern continents, these ice-bearing waters are moving westward to form the East-Greenland Current which becomes an important accessory to the Labrador Current, whose waters reach the vicinity of the 40th degree parallel of latitude and there in conflict with the equatorial waters create vigorous convection and a true action-center of oceanographical influence.

It has long been known that the mere nearness of warm water has not a decisive action on climate, but that it is made effective by the winds that blow over it. During eight months of the year from the end of

August to the beginning of May, in nearly all parts of the North Atlantic Ocean the temperature of the surface water is higher than the temperature of the air whose sweep in middle and northern latitudes is toward the European side.

An impression of the order of magnitude of these quantities may be gathered from estimates that have been made in relation to the Atlantic Ocean. The Antilles Current conveys nearly forty cubic miles of water an hour past Porto Rico; and, in the Florida Strait, the Gulf Stream conveys about fifteen cubic miles an hour, moving with a speed of three miles per hour. The combined Florida and Antilles currents move northward to Cape Hatteras with an average velocity of seventy miles per day in the center and half this amount on the edges. Off Cape Hatteras is the so-called "Delta of the Gulf Stream," where it begins to separate into several branches. South of Nova Scotia the velocity is about thirty-eight miles a day. At the southeastern and southern edge of the Grand Bank of Newfoundland, the Gulf Stream comes into conflict with the cold southward-flowing Labrador Current, which greatly lowers its temperature. The average temperature of the water flowing towards Europe after passing the Grand Bank is 10° to 15° F lower than the temperature of the Gulf Stream off Cape Hatteras.

Off the Grand Bank of Newfoundland, the Gulf Stream divides into several branches: the most northerly flows toward west Greenland, another flows toward Iceland, and a third containing the main body of the water flows toward Europe and the Mediterranean Sea, again dividing in about 40° north and 40° west into two branches. The southern branch turns southeastward skirting the coast of southwestern Europe and Africa as a cold current. The northern branch gaining renewed velocity from the prevailing southwesterly winds crosses the Atlantic with an average speed of twelve miles a day and bathes the shores of western and northwestern Europe. The bulk of the water passes north of Ireland and Scotland. Under the influence of the prevailing southerly winds, the larger part of this water drifts along the Norwegian coast to Novaya Zemblya where it is largely overlain by colder, but fresher water and loses its identity, but some of it remains on the surface to reach Spitzbergen giving rise to the relatively favorable climate of that land and keeping almost free of ice its western coast. The volume of this stream of Atlantic water passing along the coast of Norway is about one-quarter of a cubic mile an hour in the latitude of the Lofoten Islands. Even this amount, small compared with the volume of the Gulf Stream off the Atlantic coast of the United States, exercises climatic influences that have important effects upon the Norwegian harvests.

The influence of the marine ocean upon climate is exercised jointly with the aerial ocean above it. The former is the conservator of heat and the conveyor by which it is delivered to distant regions, to be distributed by the latter through the agency of the winds.

A tract of the ocean 450 miles square and one-tenth of a mile deep, in being reduced one degree in temperature, would give off enough heat to raise the temperature of the atmosphere 10° over the whole of the United States up to a height of two miles.

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THE CRITICAL EXTENT OF POLAR ICE-CAPS

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(Abstract)

This is a brief review of our present knowledge concerning one aspect of climatic control. The subject is of importance in connection with certain phases of geology such as glaciation and the Wegener hypothesis. According to this hypothesis the continents at various times have drifted apart in an east-west direction and have also experienced large movements from south to north and from north to south again. This latter feature of Wegener's theory is supported by climatological evidence, and loses its only support if other means of securing large variations in climate are found.

Considerable attention has been given to the possible physical factors which might explain variations in climate during past ages. Among the factors studied have been: Variations in the eccentricity of the Earth's orbit and in the inclination of its axis; changes in the amount of water vapor, carbon dioxide, and dust in the Earth's atmosphere; and variability in the amount of heat radiated from the Sun. Most of these factors are doubtless of importance, but it is not easy to understand how they can produce the radical changes in the Earth's climate, which have occurred so many times during geologic history.

The theory of polar ice-caps as developed by Kerner and by Brooks is as follows: The climate on the Earth is either glacial or non-glacial. The non-glacial is the usual or normal type of climate. During non-glacial periods the polar region had no great ice-sheets; the temperature in these regions was approximately the same as we have today in the temperate zone. The rarer glacial periods were distinguished by a much lower temperature. The extreme north and south parts of the Earth were covered with ice-caps which sometimes extended to middle

latitudes. Now suppose that at the North Pole the temperature is low enough for a small ice-cap to form. As soon as the sheet of ice formed it would cause the temperature to fall because of the high reflecting power of the ice (about four-fifths of the Sun's radiant energy being reflected back into space). This would cause more ice to form, thus increasing the area of the ice-cap, which would still further lower the temperature, and so on.

A small initial lowering of temperature would, therefore, be capable of producing a polar ice-cap of very considerable extent. The exact effect depends primarily upon the normal horizontal temperature-gradient; that is, upon the normal variation of temperature with latitude, and upon the additional cooling produced by each square kilometer of ice newly formed. The first effect is proportional to the radius of the cap and the second to the square of the radius, and it is for this reason that a cap, having once been well started, would grow at an increasing rate.

From such data as are available it appears that, starting with a non-glacial condition, that is, with an Earth having no large permanent ice-fields, it is possible for an initial lowering of temperature of only 0.3° C to produce eventually an ice-cap extending to 65° latitude. In general, whenever the mean temperature at the poles is close to the freezing point of sea-water, a critical condition exists such that a very small change of temperature can cause a most profound change in the extent of ice-fields. A change in temperature initiated by any one of the supposed factors, such as atmospheric dust or change in the solar constant, causes the temperature to jump from one state to the other, and it is only the very slow change in these factors which prevents the transition from glacial to non-glacial states from being more rapid. It is of interest that according to this theory an increase in the Earth's mean annual temperature of one degree or two degrees centigrade would cause the present polar ice-caps to disappear.

Such calculations as these are subject to many uncertainties. Many of the factors involved are not amenable to precise evaluation. The theory of the critical extent of polar ice-caps is to be considered as a qualitative generalization rather than as a quantitative law. And yet, it would seem that a recognition of the general applicability of this theory would allow the problem of ice-ages and of radical changes in climate to approach a satisfactory solution.

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CHANGES IN ELEVATION OF THE ROCKY MOUNTAINS IN PLEISTOCENE TIME AS POSSIBLE CLIMATIC FACTORS

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It is well known that the Sierra Nevada and Cascade ranges intercept moisture from the Pacific, causing it to be precipitated on their western slopes with resulting arid or semi-arid conditions east of them. In a less degree the Rocky Mountains effect a similar result, particularly north of the latitude of Yellowstone Park. Precipitation on their western sides or westernmost ranges, such as the Clearwater Mountains, Cœr d'Alene, Cabinet, and Purcell ranges is much greater than on the ranges farther east or the plains beyond. It follows that if the intercepting ranges were removed or even lowered considerably, the interior plateaus and the great plains in northern United States and Canada would receive an increased precipitation which, of course, would be one of the factors necessary to renewed glaciation. In this connection, it is noteworthy, perhaps, that in the Olympic Mountains, which receive a heavy precipitation, permanent snow and ice-fields are rather general above an altitude of five thousand feet. Farther east in the same latitude, the present lower ice-limit rises as the precipitation decreases. In Glacier Park it is at least seven thousand feet.

The present heights of the Cascade Range and the Rocky Mountains are due almost wholly to crustal movements that have occurred since the Miocene. There were older mountains, but they had been generally lowered by erosion before the movements producing the present mountains began. According to Smith and Willis,[1] the present Cascade Mountains were elevated in Pleistocene or at least post-Pliocene time, and this view is supported by the uplifted position of the Pleistocene Satsop formation described by Bretz.[2] Since then there have been oscillations of at least two hundred and ninety feet in the level of the strand-line in the Puget Sound region.[3]

In the Columbia Valley, evidences are found of a water body that existed briefly during the later or Wisconsin part of the Pleistocene; and this body is interpreted as the result of depression which reached a maximum of twelve hundred feet or more.[4] The depressed area doubtless included the Cascade Mountains and possibly parts of the Rocky Mountains also.

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[1] George Otis Smith and Bailey Willis. Contributions to the geology of Washington. U. S. Geol. Survey Prof. Paper 19: 28, 38, 92 (1903).

[2] J. Harlen Bretz. The Satsop formation of Oregon and Washington. J. Geol., 25: 446-458 (1917).

[3] J. Harlen Bretz. Glaciation of the Puget Sound region. Wash. Geol. Survey Bull., 8: 231-239 (1913).

The northern Rocky Mountains are made up of many more or less detached or individual ranges and plateau-like masses. All of them have been rather deeply and maturely dissected by streams, but most of them still preserve even crest lines or flat-topped summits that suggest ancient peneplains or other surfaces of small relief that were developed by erosion at low altitudes. To form the present mountains, the old surface was elevated differentially. In southwestern Montana, it was warped and broken into fault-blocks, some of which were raised and others depressed. These movements were almost wholly post-Miocene and locally at least they have continued into Pleistocene and Recent time.[5]

The studies of Alden[6] and others in the northern Great Plains show a succession of regional uplifts during late Tertiary and Pleistocene time that in places aggregate one thousand feet or more. One of these, probably of less magnitude, was an early Wisconsin uplift that immediately preceded or attended the advance of the Keewatin ice. There has also been a post-Wisconsin uplift, the influence of which is still felt by the streams. The movements described by Alden were regional and occurred in the same period as the differential uplifts in the mountains. The regional elevation was periodic and progressive and since the beginning of the Pleistocene has caused a net gain in altitude of the mountain crests, for, in the more western ranges at least, they still carry remnants of the old low erosion surface.

Uplift may be regarded as a cause of glaciation provided the uplifted and glaciated areas are the same. On the other hand uplift might, by cutting off or diverting the flow of moisture, stop the accumulation of ice in a neighboring area or shift the locus of glaciation to another place.

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[4] J. HARLEN BRETZ. The Late Pleistocene submergence in the Columbia Valley of Oregon and Washington. J. Geol., 27: 489-506 (1919).

[5] A paper in preparation by the author, entitled "Late Tertiary and Quaternary faults in southwestern Montana," describes the Continental Fault near Butte, a fault at the east foot of Bitterroot Range, and a fault at the west foot of Madison Range, Montana, on all of which upward movements of the adjoining mountain-masses have occurred in very late geologic time. A scarp produced by movements on the fault at the base of Madison Range cuts across moraines of the Wisconsin stage, later alluvial deposits, and has an appearance of newness that suggests an age to be measured in scores or at most a few hundreds of years.

[6] W. C. ALDEN. Physiographic development of the northern Great Plains. Geol. Soc. Amer. Bull., 35: 385-424 (1924).

The evidence so far available shows that during the Pleistocene, the Cordilleran area here considered was far removed from a condition of crustal stability and is even yet, as it were, quite alive. It seems most probable, therefore, that the changes in the mountains' heights from time to time must have exerted a strong, possibly a controlling, influence on the growth and changes in position of the great glaciers in the regions of Keewatin and Hudson Bay.

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SOME POST-TERTIARY CHANGES IN ALASKA OF CLIMATIC SIGNIFICANCE[1]

PHILIP S. SMITH

Discussion of relatively late changes in Alaska that may have had climatic significance is much hampered by the lack of specific information relating to many of the pertinent factors, but even then it is too large a topic to touch more than the fringes of in the short time allotted.

One of the speculations that crops up periodically both in its relation to climate and to migrations of animals and plants, is the inquiry into what would be the result if the region near Bering Strait had been at one time more depressed or more elevated than at present. The details of the situation that would have been produced can not be stated with finality, but it seems doubtful whether possible changes of this sort would have effected as radical physical differences as might be conjectured from the casual inspection of an ordinary map. The reason for this conclusion is, that unless depression exceeded 2,300 feet the Strait would maintain practically its present width as the portals at Cape Prince of Wales and at East Cape rise abruptly to elevations of 2,300 and 2,521 feet, respectively, and in the midst of the Strait are three islands, the largest of which is from three to four miles in diameter and has a height of 1,759 feet above the sea. In Seward Peninsula there are no deposits or other evidence that even remotely suggest that the region has been submerged two thousand feet below its present stand during Tertiary time or later.

Furthermore, much of Bering Sea and the Arctic Ocean between Alaska and Asia is really part of a land-mass rather than part of an oceanic area. This can be clearly seen from the map (Figure 1) in which areas lying between sea-level and a depth of thirty fathoms, between thirty fathoms and one hundred fathoms, and from one hundred fathoms down, are indicated by distinct tints. From this map it will be seen that, on the whole, the two continents are separated by only a shallow sea; or,

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[1] **Read by permission of Director, United States Geological Survey.**

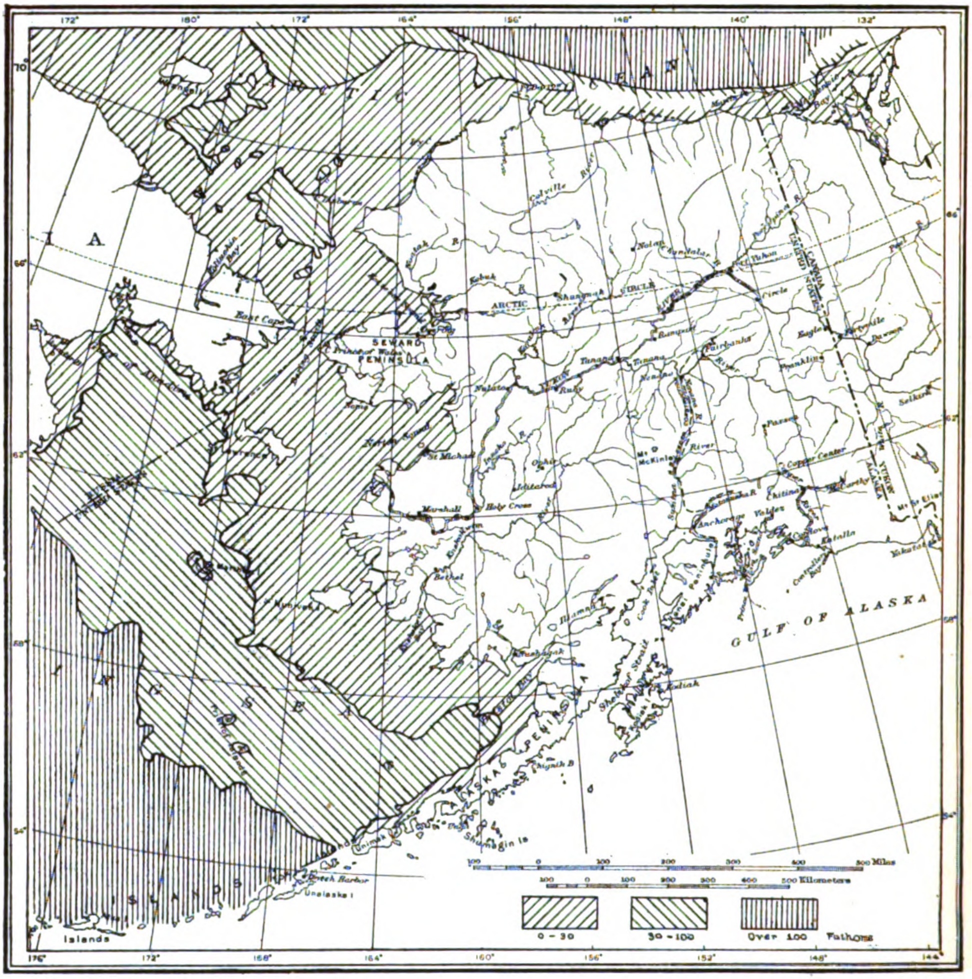

Fig. 1

Depths of water in Bering Sea and parts of Arctic Ocean.
(Depths in Pacific Ocean not shown)

to put it another way, are practically connected except for a slight submergence. Therefore, even if the Aleutian Islands were completely removed, the "continental area," if I may use that term, would extend as far south as the Pribilof Islands and as far north as latitude 72°. This land-mass is believed to be of relatively great age. Though the surface of some of the submerged area is doubtless formed of relatively late detritus brought down to the sea by the Yukon and Kuskokwin rivers, that source is entirely inadequate to explain the larger feature which is more than one thousand miles wide.

The evidence of past glaciation contrasted with present glaciation in extent and relationship to the mountain-masses in which it originated leaves little room to doubt that both were controlled by essentially the same general conditions as now prevail in the matter of wind-direction, and of atmospheric and oceanic circulation. In other words, the glaciers of the recent past were relatively proportional to those of the present, and on opposite sides of ranges had in general the same differences as are observed in existing glaciers. There is, therefore, small likelihood that in the late past, Pacific-Ocean circulation into the Arctic Ocean was notably different from what it is today.

It is, however, a significant fact, but one which still awaits explanation, that the maximum glaciation in the past in Alaska covered in area only fifteen to seventeen times as great as the existing glaciation. Few of the past glaciers in Alaska appear to have extended for an airline distance of more than 150 miles from the crest of the ranges in which they originated, whereas in eastern North America they extended many times that distance. This striking and apparently anomalous difference in amount of glaciation warrants further search for data regarding the synchroneity of glaciation in the two regions. Capps, who is the only one to publish on this subject, has given facts which suggest that the maximum extent of ice in one of the valleys in eastern central Alaska was about 125,000 to 150,000 years ago, which he considers indicates general contemporaneity with the Wisconsin stage in the States.

In suggesting the general stability of Bering Straits it was not intended to imply that Alaska has long stood inert in its present physical form. Evidence is almost universally present showing that in different parts it has been subjected to movements of uplift and depression. These movements mark both broad regional warpings and localized mountain building. Former higher stands of the land are relatively obscure because many of the deposits and other kinds of geologic evidence that mark these movements are now buried under the sea or masked by later processes and products. Certain it is, however, that near Nome the country stood, in late Tertiary times, at least 150 feet higher than it now does because at that place buried beaches have been recognized to that distance below sea-level.

In the present inquiry, however, former higher stands of the land are not of as much immediate significance as former lower stands and evidence of these is much more legible and widespread. From many parts of Alaska, data have been obtained that prove relatively recent uplifts amounting to hundreds if not thousands of feet. This does not mean that uniform uplifts of these amounts are recognized, for even within short distances the amount of the same uplift varied greatly. For instance, Brooks described an old marine plain in southwestern Seward Peninsula that in places stands as much as 600 feet above sea-level, but that in other places, not more than 50 miles distant, now stands below sea-level. Obviously the evidence regarding former lower stands of the land becomes progressively less distinct the higher the old land-surface has been uplifted; because the higher ones are usually more vigorously attacked by erosive processes and generally are older than the lower ones and consequently have been longer exposed to attack. At many places all the way from southeastern Alaska to the extreme northern part, evidences of relatively recent uplift are fairly distinct, up to six hundred feet, and in selected places up to one thousand feet. The highest uplift that has been definitely determined is in the St. Elias region, where Russell and Maddren describe a marine deposit containing fossils of Pleistocene age that now stands three thousand or possibly five thousand feet above the sea.

Although by no means susceptible of precise measurement relative to sea-level, the geologic evidence within the Alaska Range indicates an uplift, in that tract, of many thousand feet in Tertiary and later time. Briefly, the great arc now marking the Alaska Range had relatively low relief in the early Tertiary, as coal-beds and fine sediments extend far into the present mountain-mass. Subsequently deformation took place whereby these early beds were bent until they now stand vertical or even overturned. These beds are now preserved mainly in the bowls of synclines, the higher parts of the folds which at one time formed the mountain peaks having been removed by erosion so that it is impossible to determine the maximum height to which these beds at that time rose. Concomitantly with the growth of the mountains, erosion became vigorous and great bodies of coarse material were eroded from the growing mountain-mass and deposited near their flanks. In places these gravels have a measured thickness of two thousand feet, so that uplift within the ranges amounting to at least that amount must have taken place, though, of course, the gradients of streams that laid these deposits down were probably steeper than those of the present streams. Although these deposits were not laid down at sea-level, the fact that they now occur in peaks rising to elevations of more than four thousand feet above the sea, while it does not give the exact measure of the uplift, does serve to indicate that they have been raised several thousand feet. The growth of the mountains is still going on, as is shown by the earthquakes that occur in them and by the recently raised beaches, such as those described by Tarr and Martin, on their flanks. It would strain the credulity of the geologists who have worked in the Alaska Range to believe that that mountain chain had only slight topographic strength in the late Tertiary; but that parts of it stood several thousand feet lower than now, is in entire accord with all the known facts. Analysis of the effects on climate which such a decrease in elevation would produce must be left to our brethren skilled in climatological research.

U. S. Geological Survey,
Washington, D. C.

PLEISTOCENE AND RECENT TOPOGRAPHIC CHANGES IN THE PACIFIC COAST STATES

John P. Buwalda

Advance Summary.—In comparison with the eastern half of North America, great topographic changes have occurred in the Pacific Coast States during the last three periods of geological time—the Pliocene. Pleistocene, and Recent. During these later periods of geological history the eastern mountains and plains have apparently suffered only relatively gentle upbowing or downwarping, but since the beginning of the Pliocene the western mountains and plateaus have not merely been uplifted thousands of feet but have been strongly folded and broken by faults. In the West, neighboring areas have moved in opposite directions. Some have gone up great distances and now form the high ranges; others have sunk and now either form the valleys into which the erosional detritus from the ranges is being in part deposited or they form estuaries or sounds if, lying near the Coast, they sank below sea-level.

While mountains existed in the west before the Pliocene, the great barriers of today have come into existence mainly during and since the beginning of that period. The duration of the Pleistocene and the Recent together was very much less than the Pliocene. The larger part of the growth of our present mountains was accomplished in the Pliocene, but very notable topographic changes occurred also in the Pleistocene and Recent.

It seems obvious that if the West Coast mountains have any effect on the climate of the interior of North America, marked changes in their altitude would result to at least some degree in changes in the climate they influence.

GENERAL GEOGRAPHY OF THE PACIFIC COAST REGION

As is well known, that part of the United States lying west of the Rocky Mountains is divisible topographically into four north-south belts. Skirting the shore of the Pacific lie the Coast Ranges, extending from southern California to the Canadian boundary. Their width is usually not over a few tens of miles and their height, excepting in three areas, is usually less than three thousand feet. To the east of the Coast Ranges lies a north-south line of valleys; these from Canada southward are the Puget Sound, Cowlitz, Willamette, Rogue, Sacramento-San Joaquin, Mohave Desert, and Salton Sink depressions. In Washington and Oregon the Cascades border this depression on the east; in California the Sierra Nevada occupy a corresponding position along most of the inland margin of the State. Extending eastward from the Cascades and Sierras is the desert region, usually known as the Great Basin, areally larger than the whole country to the west of it, and reaching to the base of the Rockies.

MOVEMENTS ALONG CALIFORNIA COAST

Pleistocene changes in relative level of land and sea along the California Coast are indicated by physiographic evidence and by position of Pliocene strata bearing marine fossils. At numerous places, notably south of San Francisco, at Santa Cruz, at San Pedro Hill near Los Angeles, and in the San Diego region, old strand-lines and marine terraces representing the ancient sea-floor, form steps from the present strand well up toward, or even attaining, the hill- or mountain-tops. They indicate intermittent uplift totalling not less than 1500 feet. The highest marine terraces are faint in their features, and it is possible that still higher ones originally existed but have been destroyed by erosion. They are not continuous at the same level along the Coast, however. Some stretches of the shore show evidence of submergence of at least some hundreds of feet. San Francisco Bay resulted from drowning through subsidence of a considerable stretch of the Coast. These uplifts and depressions were mainly Pleistocene in date. On the terraces are found the shells of organisms which lived on the ocean-floor before its emergence; they are largely living species. Where submergence has occurred it has been so late in geological time that the waves have modified the newly-established shores but little.

Pliocene marine sediments along the Coast bear out the evidence of the terraces, although the movements which tilted these strata were apparently often earlier than the terrace-forming uplifts. The uplifts and depressions were, however, largely post-Pliocene, and were of magnitudes comparable to those indicated by the terraces.

Less is known of the changes of level of the Washington and Oregon coasts. Terraces have been recognized, however, at many places and marine strata in a tilted position and of late geological age. It is clear that the northwest coast has undergone vertical movements comparable to those of the California Coast. The old strand-lines occur many hundreds of feet above sea-level. Their Pleistocene age is attested by their preservation, even in the heavy rainfall of that region, and by the fossils found on them.

MOVEMENTS WITHIN THE COAST RANGES

Three very precise surveys in the California Coast Ranges by the Coast and Geodetic Survey, spaced approximately twenty-five years apart, have shown that peaks serving as triangulation points have moved in the intervals as much as twenty feet horizontally. These movements have probably been proceeding for a long time in the past and the aggregate displacements may be measured in miles or tens of miles. Vertical displacements of neighboring blocks with reference to each other are apparently not occurring at the present time so actively as horizontal shifting; but high scarps, particularly in southern California, attest to such displacements in very recent times. North of Pasadena and north of San Bernardino, and at other points, mountain-faces rise very steeply and boldly to heights of five to eight thousand feet above the adjoining plain. The slight degree of dissection by the erosional agencies indicates clearly that the uplifts occurred largely in Pleistocene and Recent time. Middle Pliocene strata cut by the fault of plane of breaking corroborate the late Pliocene and post-Pliocene dates of the movements. However, the peaks of these mountains had already reached approximately their present height in late Pleistocene time, for glaciation affected them. The evidence indicates, therefore, that most of the vertical mountain-making movements occurred during later Pliocene time, and only a smaller part during the Pleistocene or Glacial period and the Recent period.

There is some reason to think that parts at least of the Coast Ranges may have been higher in late Pliocene time than they are now, and that hence during the Pleistocene they have been reduced somewhat in elevation. The folds raised in middle Pliocene time, of which only the roots remain now, would form high mountains, and it is possible that since about middle Pliocene time the general change in relief has been one of reduction of heights by erosion at a rate more rapid than the rate of uplift.

PACIFIC VALLEY DEPRESSION

While the Coast Ranges of California, Oregon, and Washington experienced uplift during the Pleistocene, the valleys lying to the east of them clearly were depressed. Puget Sound, in northwestern Washington, is a great river-valley system, with its branching arms, which was depressed and drowned by the sea. It probably sank not less than a thousand feet. The Willamette Valley has likewise been bowed down to some degree, in northwestern Oregon, and Pleistocene sediments spread thickly over its floor as a consequence.

In California, the Sacramento-San Joaquin Valley has been warped downward while the adjoining mountains rose. Part of this depression doubtless occurred during the Pleistocene, although the larger part of it seems to antedate this period somewhat. And the same may be said for the Salton Sink depression; it is in part Pleistocene in date of origin.

CASCADE MOUNTAINS AND SIERRA NEVADA

With the possible exception of the Rocky Mountains, these two ranges form the most effective barrier in the western United States for influencing climate. Lying end to end, they extend from the Canadian boundary to southern California, the south end of the Sierras terminating approximately at the southern margin of the belt of prevailing westerly winds.

The Cascades maintain a height of five to seven thousand feet practically throughout their length from Canada to northern California and the Sierras continue southward with about that elevation from northern California to central California, in the region about Lake Tahoe. From this point southward they rise from seven to about fourteen thousand feet in the Mt. Whitney region, south of which they descend again to three or four thousand feet.

The west slopes of these two ranges are heavily timbered and receive heavy precipitation. The east slopes are relatively dry and look out over the vast dry expanses known as the Great American Desert, the arid condition of which results mainly from the abstraction by these ranges of the moisture which the west winds from the Pacific would otherwise bear farther inland. The ranges create rain-shadow areas. The temperature conditions on the two sides of these ranges are likewise very different.

The geologic date of uplift of these two ranges is probably about the same, if we may judge by the canyons cut in them since the beginning of their growth, and by stratigraphic evidence. They are probably both still rising, moreover, as attested by earthquakes. In 1872 a slip of several feet on a fault at the east base of the Sierras in Owens Valley was doubtless a detail of this upward movement. Lower Pliocene, and perhaps younger strata, have been disturbed along the bases of the ranges, so that it appears that the initiation of their latest elevation occurred in post-lower Pliocene time. Both ranges had, however, reached approximately their present height by the Glacial period, at least by the latter part of it, for they show strongly the evidence of having borne large ice-streams in their higher parts. Moraines found along their eastern base indicate that the Sierras probably stood at approximately their present height during the last two, and perhaps the last three, of the advances of the Pleistocene ice. The Cascades are likewise strongly glaciated. It is clear that the major part of the uplift of these ranges had been accomplished by middle Pleistocene time.

In connection with the effect of these ranges upon the climate of the country to the east, it is of interest to note that the level of maximum precipitation is about seven thousand feet on the west slope of the Sierras. This would apparently indicate that for each additional thousand feet increase in height of the mountains above the seven thousand feet level, the corresponding additional effect upon the climate to the east would be less.

GREAT BASIN PROVINCE

The region between the Cascades and the Sierras on the west, and the Rockies on the east, lies in general lower than the western ranges bordering it, and it hence has no such marked effect upon the climate of regions to the east. But it is interesting to note that this immense desert province, larger in area than the whole region lying west of it, is apparently still being actively deformed with resulting changes of level. One range in Nevada has been uplifted within historic times, with slipping on the fault bounding it to the extent of several feet. Bold fault-scarps in southeastern and northeastern Oregon, and occasional earthquakes in eastern Washington, indicate clearly that the growth of the separate mountain-masses lying within the Great American Desert is still in progress. But here again, as in the case of the Cascades and Sierras, the larger part of the movements which created the ranges appear to have occurred before the last glacial advances. The summit regions of a number of the ranges were vigorously glaciated. Clear evidence of post-glacial faulting and uplift is presented at a number of places, but the movements were not large in magnitude.

CONCLUSION

Evidence from physiography, structural geology, and paleontology combines to demonstrate that uplifts of hundreds of feet, and in some areas, probably thousands of feet, have occurred in our Pacific Coast mountain belts since the beginning of Pleistocene time. Depressions of hundreds of feet have occurred in some other localities. The movements are continuing today, possibly as actively and rapidly in many places as at any time in the past. This latter fact merely tends to emphasize the recency of many of the larger changes in level and topography. A large part of the changes in level had been accomplished before at least the last two dates of advance of the ice during the Pleistocene.

California Institute of Technology,
Pasadena, California.

SECTION OF GEODESY

(Washington, D. C., April 28, 1927)

RECENT PROGRESS OF CANADIAN GEODESY

Noel J. Ogilvie

Exhaustive field tests and study have been made during the past two years of a 3¾-inch transit (see Figure 1). This small instrument, 12¾ pounds, is capable of measuring angles with primary accuracy when used within the limits of its telescope-power, maintains its adjustments under exceptionally strenuous conditions and can be manipulated in about one-half the time occupied with twelve-inch, three-micrometer transits.

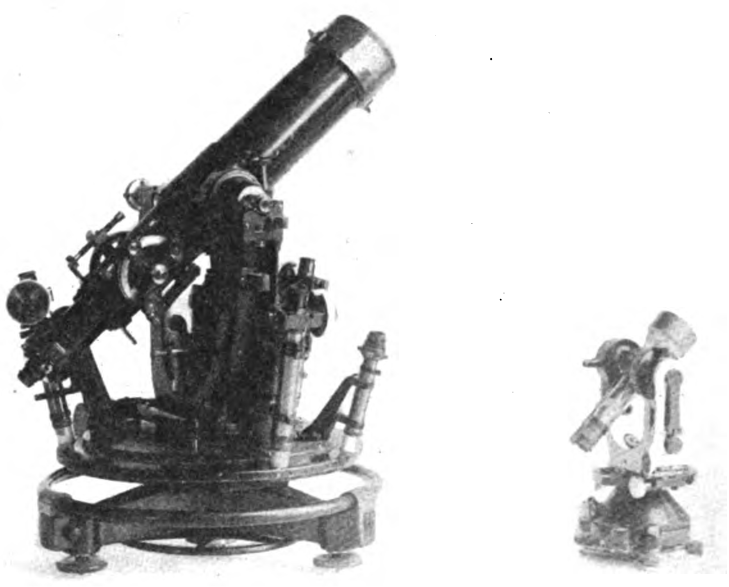

Fig. 1

Twelve-inch theodolite and three-and-three-quarter-inch theodolite, Geodetic Survey of Canada.

The telescope objective is only 40 mm in diameter, which is inadequate under hazy atmospheric conditions, but four transits of a similar design with 60-mm objectives, 5½-inch glass horizontal circles and weighing in their cases about 30 pounds each will shortly be delivered (see Figure

2). It is anticipated that this design will fulfil all requirements of work for which a twelve-inch instrument was formerly thought necessary.

In precise levelling a development, which it is believed will ensure greater permanency and convenience to the public is the construction, at certain specified points, of monuments of a new and enlarged type to be known as the "fundamental bench-mark" (see Figure 3).

The intention is to build these in the cities and more important towns and junction points, and each monument will be considered as the governing bench-mark for the town and surrounding district. The monument, whenever possible, will be constructed in a park or public square, the selection of the site being carried out in collaboration with the municipal authorities in order that, as far as can be foreseen, the location chosen shall be free from disturbance in the future by reason of building, grading operations, or other changes.

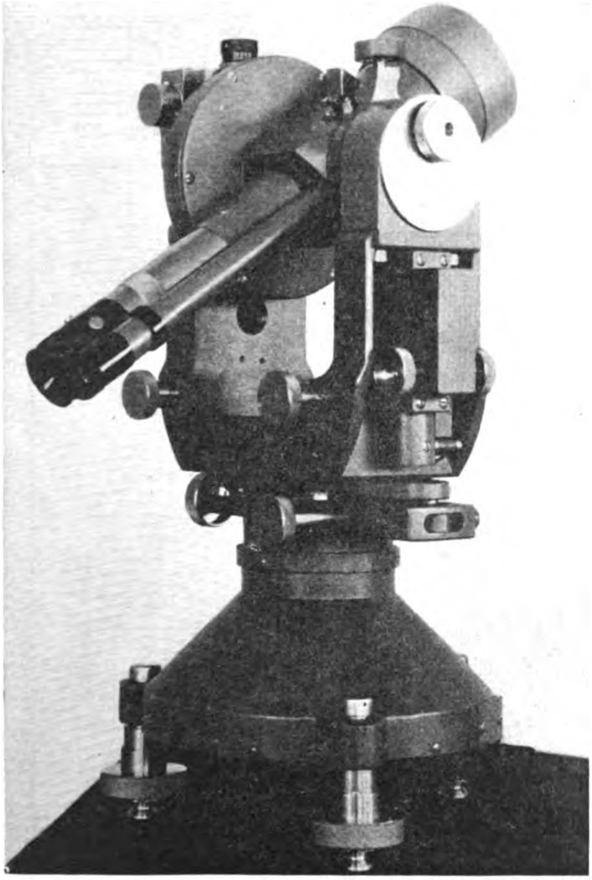

FIG. 2

New large model five-and-one-half-inch theodolite (weight 23 pounds), Geodetic Survey of Canada.

The actual monument or "fundamental bench-mark" is constructed of concrete and takes the form of a frustum of a pyramid, two feet square at the base, one and one-half feet square at the top, and seven feet in

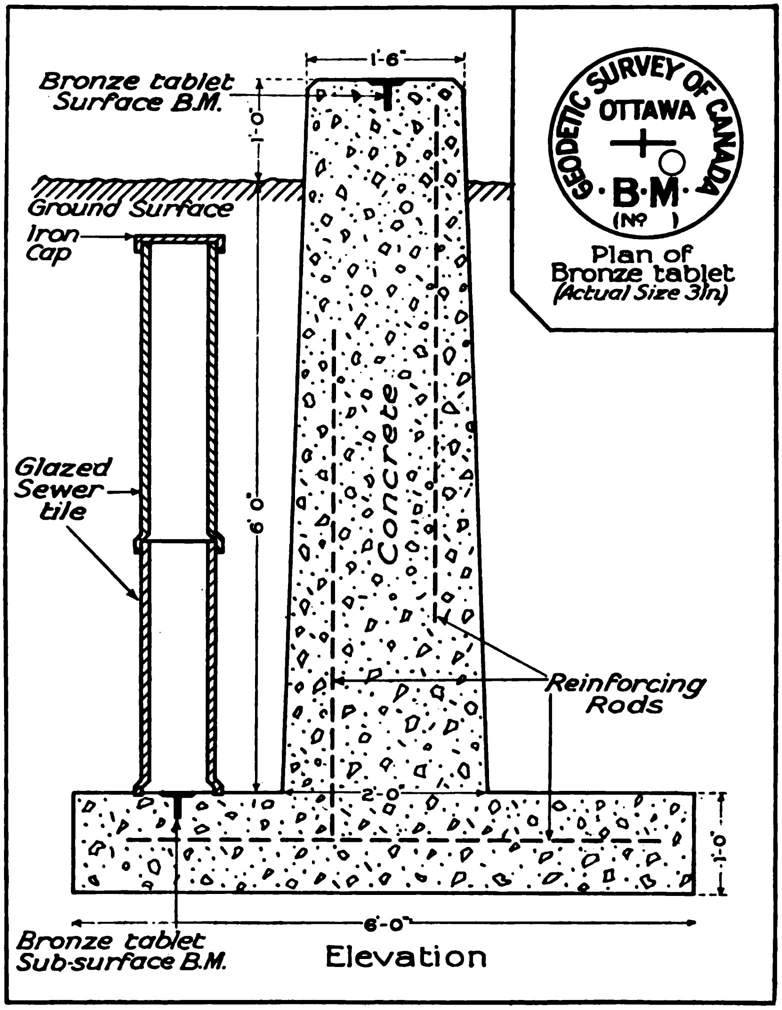
1'-6"
Bronze tablet Surface B.M.
1'-0"
Ground Surface
Iron Cap
Glazed Sewer tile
6'-0"
Concrete
Reinforcing Rods
2'-0"
1'-0"
6'-0"
Bronze tablet Sub-surface B.M.
Elevation
GEODETIC SURVEY OF CANADA
OTTAWA
B.M.
(No)
Plan of Bronze tablet (Actual Size 3In)

FIG. 3

Fundamental bench-mark, Geodetic Survey of Canada.

height, and is reinforced with four twisted steel rods. The excavation for the structure is carried to such a depth that only from twelve to fifteen inches of the column projects above the natural ground-surface; it rests on a circular concrete base six feet in diameter and one foot thick, reinforced by steel rods placed radially, the base and column being constructed as a monolith.

A bronze tablet of standard form is set vertically in the top of the monument and the elevation for this is listed as the elevation of the "fundamental bench-mark." In order, however, to provide for the recovery of the elevation, should the tablet or the top portion of the monument be damaged, a second tablet, known as the subsurface mark, is set in the concrete base at one side of the column. This is protected by means of two sections of six-inch sewer tile, reaching to within a foot of the ground surface and closed with a metal or concrete cover. The filling in and grading of the earth around the column conceals this from sight.

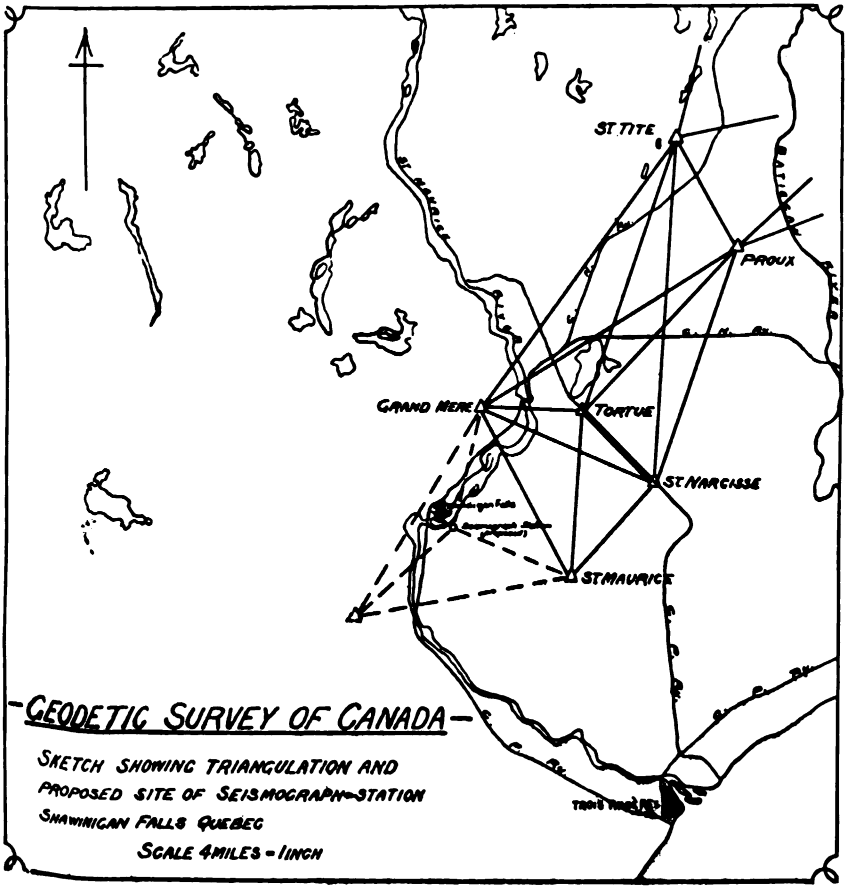

FIG. 4
Sketch showing triangulation and proposed site of seismograph-station at Shawinigan Falls, Quebec, Geodetic Survey of Canada.

Two schemes of triangulation were observed in 1926 at Seven Falls and Shawinigan Falls (see Figure 4) in the Province of Quebec with the express purpose of cooperating with local industrial concerns in a study of seismic disturbances and resulting earth-movements. The triangulation is continuous with the first-order work of the St. Lawrence river, and covers in each case the industrial locality and surrounding area of about seventy-five square miles, where slight earth-movements and damage to buildings resulted from the earthquake of 1925.[1]

The geodesist has proved the theory of isostasy to be a scientific fact from the result of measurements made in various areas of the Earth's surface, and the general conclusion is justified that isostasy prevails throughout the Earth's crust.

During the coming summer a comprehensive programme of astronomic determinations, extending across the continent, will be initiated at various triangulation stations covering a non-continuous belt of about four thousand miles (see Figures 5 and 6).

The results of these astronomic determinations are expected to sustain the value of the depth of compensation; but in addition an important field of study will be opened with regard to the influence of local topography and in consequence the extent of the regional compensation.

Considerable progress has been made along the lines of research and adjustments. Publications are being prepared dealing with the following new subjects: Reduction of Traverses, Nature of the Differential Adjustment, and the Subject of Geodesy Treated From a New Point of View.

The traverse-theory is particularly interesting, as by it the ordinary surveyor's latitudes and departures on the plane are added together as latitudes and departures and converted by the simple application of Legendre's Theorem to true geodetic latitudes and longitudes. It is hoped that this will prove the means of linking up many of the old surveys with geodetic work.

The differential adjustment of observations is a very easy mathematical presentation, and it affords a rigorous method of keeping all the level-nets of this Survey up to date as from time to time they include new observations, and of showing the effects of the inclusion of these new observations. Similarly, in its effects on triangulation the old data are maintained. By this system it may be said that one net of adjustment can be added to another net or scheme by considering the connecting lines as differentials.

The treatise on geodesy, which has been delayed, but is now in page-proof form, possesses many unique features showing how the symbol

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[1] See Bull. Nat. Res. Counc., 56: 41-46, Washington, D. C., 1926.

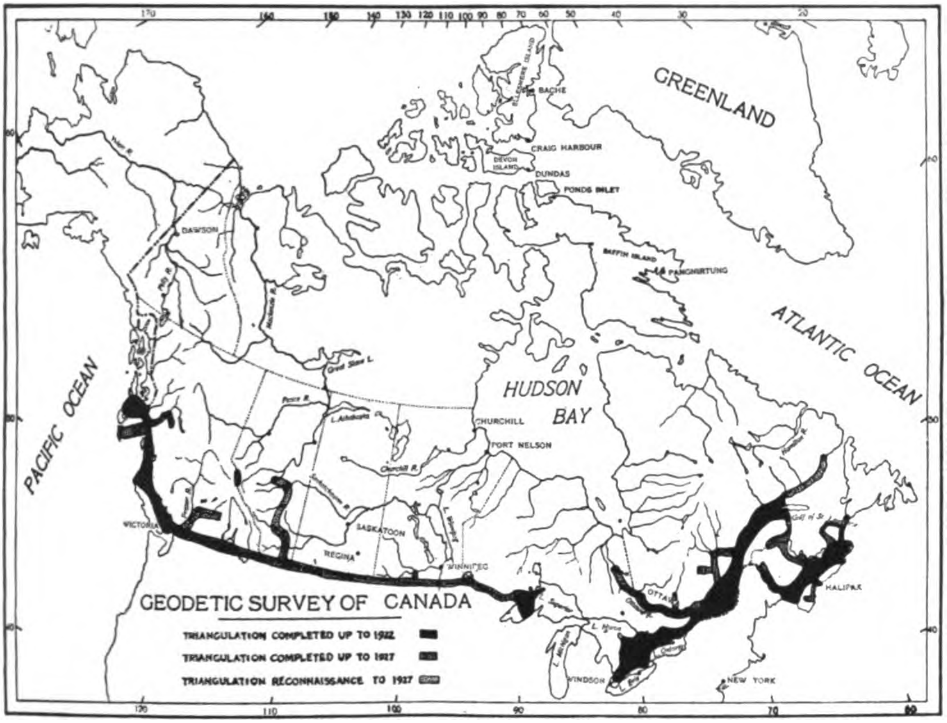

Fig. 5—Triangulation to 1927, Geodetic Survey of Canada.

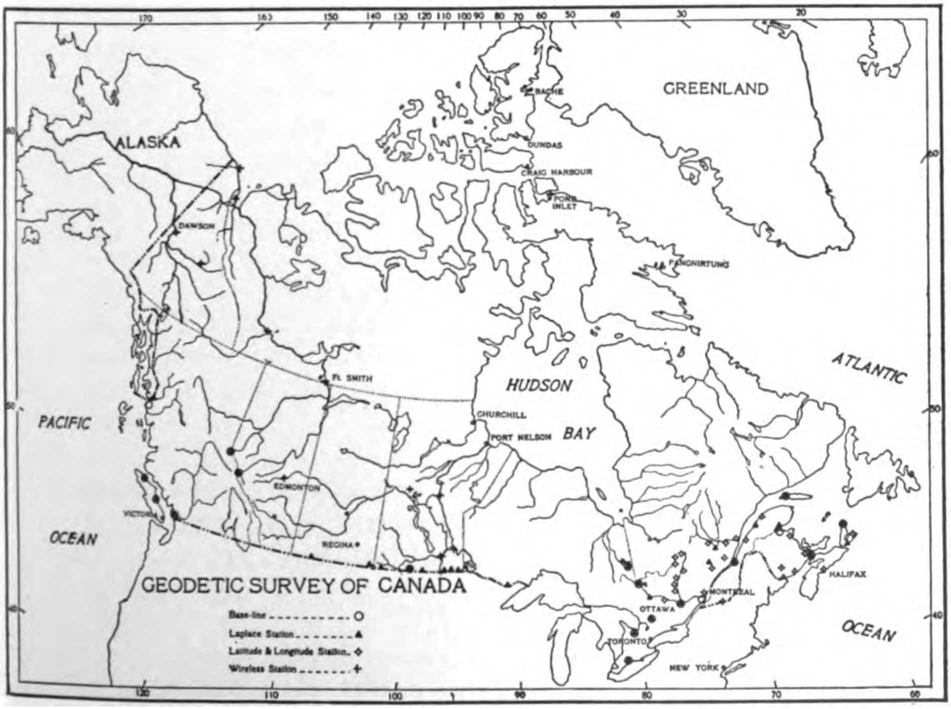

Fig. 6—Base-lines and stations to 1927, Geodetic Survey of Canada.

"δ" has a magic relation between the ellipsoid and the sphere. Also, an attempt has been made to obtain rapidly converging power-series to express values which formerly were obtained by the lengthy evaluation of very unwieldy formulæ.

The progress of the field operations of the Survey through 1926 is summarized in Table 1.

Table 1—*Summary of Field Operations to 1926, Geodetic Survey of Canada*

Field operation	Prior to 1924		1924-1926		Total	
	Miles	*Km*	*Miles*	*Km*	*Miles*	*Km*
Completed first-order triangulation, length	3,575	5,753	1,441	2,319	5,016	8,072
Completed second-order triangulation, length	500	805	302	486	802	1,291
Reconnaissance; observing not yet started, length	482	776	731	1,209	1,233	1,985
First-order traverse, length	75	121	305	491	380	612
Precise leveling	21,134	34,011	1,526	2,456	22,660	36,468
Secondary leveling	9,261	14,904	47	76	9,308	14,980
	Number		*Number*		*Number*	
Fundamental bench-marks	....		23		23	
Standard precise bench-marks	6,722		905		7,627	
Standard secondary bench-marks	2,891		26		2,917	
Base-lines	14		7		21[a]	
Laplace stations	18		16		34	
Astronomical stations, apart from Laplace stations	....		7		7	

[a] Four of these are of second order.

Geodetic Survey of Canada,
Ottawa, Canada.

WORLD LONGITUDE-NET: U. S. NAVAL OBSERVATORY STATIONS AT WASHINGTON, D. C., AND SAN DIEGO, CALIFORNIA

F. B. Littell

The world longitude-campaign by the use of radio signals, which was originally proposed by General Ferrié in 1919, was carried out in 1926.

For several years the matter had received the attention of a mixed commission representing the International Astronomical Union and the International Geodetic and Geophysical Union. In its final form the plan called for the simultaneous determination of the longitudes of twenty-three stations of the first order, representing thirteen countries, and of as many, or more, secondary stations (see Figure 1). Those of the first order comprise the original round-the-world-polygon composed of San Diego, Algiers, and Shanghai, and the stations at Washington, Ottawa, Vancouver, Greenwich, Edinburgh, Paris, Madrid, San Fernando, Naples, Genoa, Mogadiscio, Dehra Dun, Tokyo, Melbourne, Sydney, Adelaide, Wellington, Lembang, Manila, and Honolulu. It was desired to determine the longitudes of these stations with the highest precision possible, both as a contribution to our knowledge of longitudes, and as a basis for future studies relating to possible crustal movements of the Earth's surface. In order that there might be sufficient material even if weather or radio conditions should be adverse, the plan called for work extending over a period of two months.

The radio stations as originally planned were Annapolis, Bordeaux, and Honolulu, to which were added Bellevue, Eiffel Tower, Nauen, Saigon, and Malabar. Both long-wave and short-wave signals were used. The American stations sent out signals three times and the others usually twice a day. Nearly all the signals were of the rhythmic type, sixty-one ticks to the mean time minute, but the regular American time-signals were also used.

As the arrangements at Washington and San Diego were quite similar, only the latter need be described. There was a small observer's room, with a cement-lined clock-vault beneath. Immediately outside were two cement piers, one for the transit instrument and one for the meridian-mark lens, and 160 feet (48.8 meters) to the north was another pier for the meridian-mark. As these were constructed in 1923, they were well seasoned and in fine condition for use. The transit was mounted practically out in the open air. It was protected from the Sun's heat by a wooden covering, which was removed just before sunset each day. The transit having an aperture of three inches and a focal length of thirty-three inches was made by Prin, of Paris, in 1913. It was equipped with an impersonal self-registering transit-micrometer, driven by a small electric motor. The speed of the traveling wire is adjustable, according to the declination of the star, and the fine-motion is by hand, operating through a planetary gear. The device for recording on the chronograph was changed from a make-circuit to a break-circuit, thus eliminating a relay from the circuit, and making one less lag to investigate. The in-

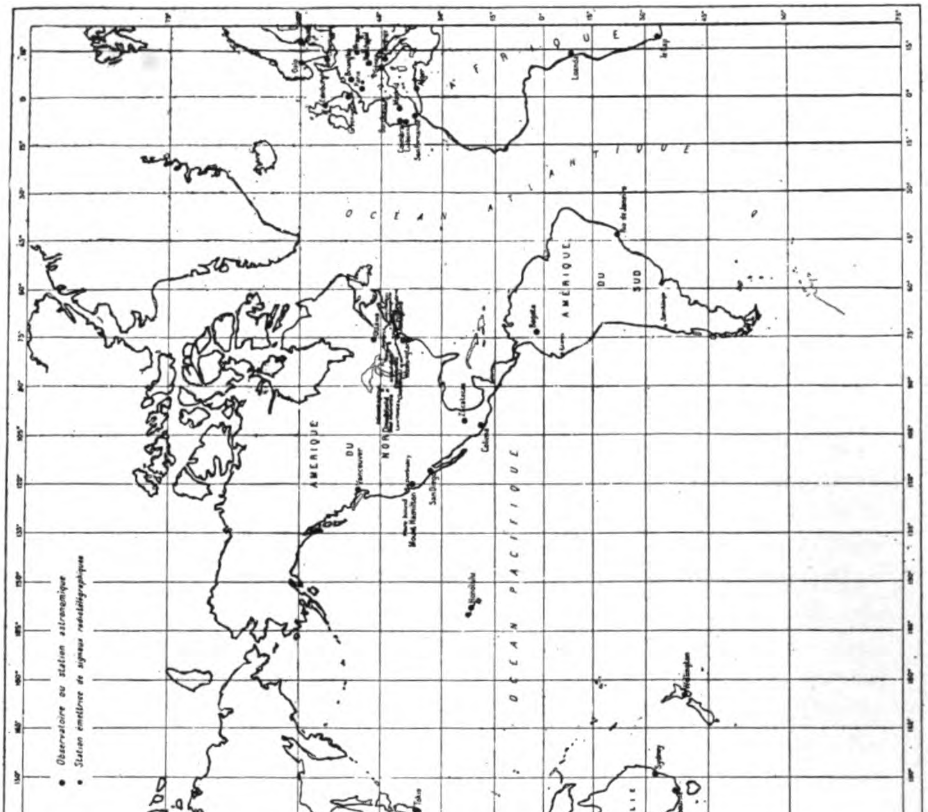

FIG. 1

Observatories and astronomical stations taking part in the world longitude-campaign.

strument is quickly reversible, and each star was observed direct and reversed at the same transit, thus eliminating immediately the effect of collimation-error and pivot-inequality.

The clock was a Riefler in an air-tight case in the clock-vault where the temperature was kept nearly constant by means of four small electric heating-units controlled by a kerosene-mercury-glass thermostat similar to the ones used at the Naval Observatory. During October the temperature-range was from 86° to 88° F, and during November, except for a few days when an electric connection was broken, the range was from 85° to 86° F. The clock performed excellently and the clock-correcting could be interpolated over intervals of a few days, with great security. This made it possible to utilize the radio signals on nights when no stars were observed.

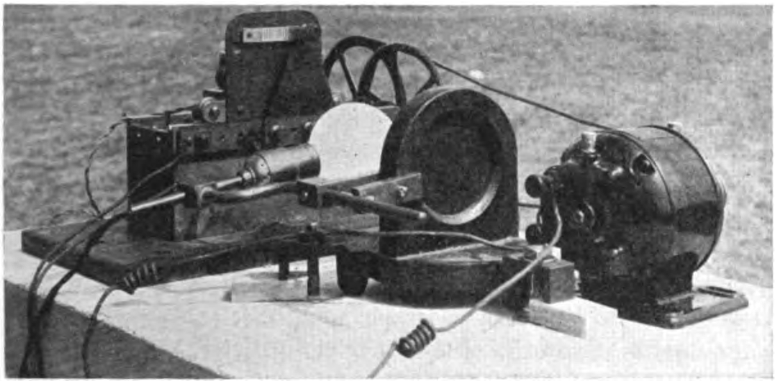

FIG. 2

Personal-equation machine on meridian-mark pier at San Diego, California.

The meridian-mark consisted of an artificial star collimated by a lens of 160-feet (48.8-meter) focus. As the Prin instrument moves very considerably in azimuth, such a mark was very necessary, even though the stars used were as near the zenith as practicable. A reading on the mark was taken near the time of each star-observation.

The level of the instrument was determined by a striding spirit-level. The vial was an excellent one by Pessler, and the value of a division was measured several times during the progress of the work by means of a level-trier, without removing the vial from the frame in which it was used. As a check on the level of the instrument, a nadir-observation over mercury was taken each night when possible. Even when there was no wind the nadir-images were almost always very poor, possibly due to vibrations caused by the surf, the location being very near the bay, although the ocean was several miles away. While the individual nadir-levels were very erratic there was no indication of any systematic difference between such levels and those obtained by the spirit-level.

A small personal-equation machine (see Figure 2) was mounted on the meridian-mark pier, and each observer measured his personal equation several times during the work. In the case of Mr. Hammond it was quite constant and small, the adopted correction being $-0^{s}.004$, while in the case of the writer it was variable, the adopted correction ranging from $-0^{s}.036$ at the beginning to $-0^{s}.008$ at the end. This change, which was corroborated by the comparison of the observed clock-corrections, may have been due to lack of recent experience in the use of the Prin transit. In the case of Mr. Watts, the Prin observer at Washington, the adopted equation was $-0^{s}.007$.

The probable error of a clock-correction from a single star was $\pm 0^{s}.016 \sec \delta$. This is quite materially less than in the 1913 to 1914 work, when the average for four observers was $\pm 0^{s}.028 \sec \delta$. The difference seems to be very largely accounted for by the improvement in the star-positions as given in the catalog of W. S. Eichelberger, recently published by the U. S. Naval Observatory, though some of it is due to improvements in the instrument and accessories.

The radio receiving-sets designed by Mr. Sollenberger were adapted to record on the same chronograph with the star-observations, thus eliminating the effect of its relay lag. The long-wave signals were most successfully received, Malabar and Saigon 9,000 miles and 8,000 miles away, respectively, being regularly recorded. In order to receive Annapolis, with which there was strong interference from a commercial station in Honolulu operating on exactly the same wave-length, a radio barrage was successfully employed. The combined lag of the receiving set and of one intervening relay was measured and corrected for. The corrections at San Diego ranged from $-0^{s}.019$ to $-0^{s}.028$. At Washington they were larger, but well determined. The correction was a function of the current-strength from the radio set, and this was recorded for each observation.

Fog and condensation interfered with observing stars at San Diego in the latter part of the night during October, but sufficient work for a time-sight was obtained on fifty-four out of the sixty-one nights, and on twenty-six of these nights two clock-corrections were obtained. At Washington time-sights were secured on forty-three nights.

The following preliminary results have been obtained. The Greenwich results are based on data kindly furnished by the Astronomer Royal for England, those for Paris are based on data in the Bulletin of the B. I. H., Vol. II, No. 32. In the Washington-San Diego longitude, nine different radio signals were used, and there seems to be very little difference either in accuracy or in the end result, as to whether the signal be long-wave or short-wave, American or rhythmic style, day or night.

Preliminary Results

	h	m	s	s
Washington-San Diego, 1926, radio............	2	40	32,641±	0.002
Washington-Greenwich, 1926, radio............	5	08	15.753±	.002
" " 1913 to 1914, radio.....	5	08	15.721±	.014
" " 1866 to 1892, cable.....	5	08	15.78 ±	.050
San Diego-Greenwich, 1926, radio.............	7	48	48.390±	.002
Washington-Paris, 1926, radio................	5	17	36.659±	.003
" " 1913 to 1914, radio........	5	17	36.653±	.003

Velocity of Radio Transmission

Washington-San Diego.............	179,000±14,000 miles per second
Washington-Greenwich	139,000± 5,000 miles per second
Washington-Paris	112,000± 8,000 miles per second
Washington-Paris, 1913 to 1914.....	179,000±12,000 miles per second

In the Washington-Greenwich determination of velocity there were used two sets of signals from Annapolis and one set from Bordeaux, all recorded automatically, and all reduced by direct readings of the times of the signals except the Bordeaux signal received at Greenwich, which was reduced by an estimation of coincidence times. This difference in the method of reduction may be the cause of the low value obtained for the velocity. The transmission-times are quite accordant, and a systematic error of about one one-hundreth second would be sufficient to account for the low velocity.

In the Washington-Paris determination of velocity, there were three sets of signals from Annapolis and two sets from Bordeaux available, all recorded automatically. The resulting velocity seems to be much too small. When the transmission-times from an individual night's work are plotted, they show marked variation with the time of a somewhat periodical nature.

U. S. Naval Observatory,
Washington, D. C.

A STUDY OF THE VARIATION OF MEAN SEA-LEVEL FROM A LEVEL SURFACE

HENRY G. AVERS

It was stated at a meeting of this Section two years ago that the first-order leveling seemed to be developing the fact that the planes of mean sea-level at adjacent tidal stations along the same coast are not, necessarily, in the same level surface.

This condition was first brought to our attention when the tidal stations at Fort Hamilton, New York, and Portland, Maine, were connected by a line of first-order leveling. The results of the leveling showed the plane of mean sea-level at Portland to be 169 millimeters higher than the same plane at Fort Hamilton. Similarly, on the Pacific Coast the leveling showed that the plane of mean sea-level at Vancouver, B. C., was 103 millimeters higher than the same plane at Seattle, Washington. In each case, mean sea-level at the northern station was the higher.

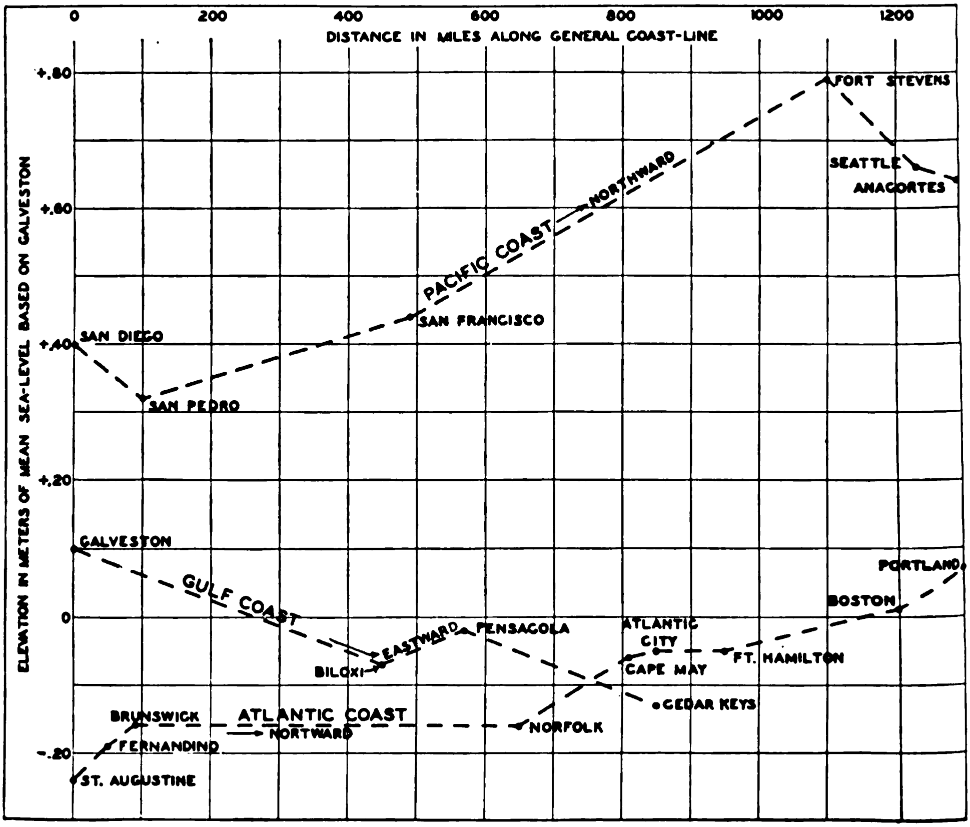

Fig. 1

Graphs showing adjusted elevation of the plane of mean sea-level at each tidal station above the same plane at Galveston, Texas.

These differences are about three times as large as those which could reasonably be attributed to the systematic or accidental errors in the leveling and a study has been made to determine if, in general, there is a slope in the mean sea-level surface.

For the purpose of this study a special adjustment of the first-order level-net has been made. Before adjusting the net, certain level-lines were omitted which had received very low weights in the previous general adjustments. These lines included all the leveling run by the U. S. Coast and Geodetic Survey previous to 1899 and some of the leveling run by the U. S. Geological Survey previous to 1905. The closing errors of the circuits of which these lines formed a part also showed that they were weaker than the more modern lines of leveling. The total length of the leveling used in the adjustment was about 40,000 miles.

In the adjustment the entire net was allowed to swing free on one mean sea-level connection, that at Galveston, Texas. There are 104 circuits in the net and the adjustment was carried through by the method of conditions, each circuit-closure requiring an equation. There were no conditions other than the closing errors of the circuits.

The adjusted elevation of the plane of mean sea-level at each tidal station above the same plane at Galveston, Texas, is shown in Figure 1.

Starting at Galveston, Texas, and proceeding eastward along the coast of the Gulf of Mexico the planes of mean sea-level at Biloxi, Mississippi, at Pensacola and Cedar Keys, Florida, are respectively 0.07 meter, 0.02 meter, and 0.13 meter lower than the plane of mean sea-level at Galveston.

Starting at St. Augustine, Florida, where mean sea-level is 0.24 meter lower than at Galveston, and proceeding northward along the coast of the Atlantic Ocean the elevation of the plane of mean sea-level at the following places is, Fernandina, Florida —0.19 meter; Brunswick, Georgia, —0.16 meter; Norfolk, Virginia, —0.16 meter; Cape May, New Jersey, —0.06 meter; Atlantic City, New Jersey, —0.05 meter; Ft. Hamilton, New York, —0.05 meter; Boston, Massachusetts, +0.01 meter; and Portland, Maine, +0.07 meter.

On the coast of the Pacific Ocean, starting at San Diego, California, where mean sea-level is 0.40 meter higher than at Galveston and proceeding northward the elevation of mean sea-level at the following places is, San Pedro, California, +0.32 meter; San Francisco, California, +0.44 meter; Ft. Stevens, Oregon, +0.79 meter; Seattle, Washington, +0.66 meter; and Anacortes, Washington, +0.64 meter.

The foregoing values would indicate that along the coast of the Gulf of Mexico there is a general slope downward in the mean sea-level surface from west to east. Along the coasts of the Atlantic Ocean and the Pacific Ocean a general slope upward of the mean sea-level surface from south to north is indicated.

Comparison of mean sea-level on the Pacific Ocean with mean sea-level on the Atlantic Ocean shows that at San Diego it is 0.59 meter above Fernandina, Florida; at San Francisco it is 0.49 meter above Atlantic City; and at Seattle it is 0.59 meter above Portland, Maine. In each case the leveling indicates that mean sea-level on the Pacific Ocean is higher than on the Atlantic Ocean. This result agrees in sign with that shown by the leveling across the Isthmus of Panama where it was found that mean sea-level on the Pacific Ocean is 0.583 foot (0.178 meter) higher than on the Atlantic Ocean for that locality.

The values given for the differences in level between the various planes of mean sea-level are from the results of the leveling on the basis that the tidal observations determine the actual mean sea-level surface. All of the leveling involved in the adjustment is of the first-order and of such accuracy that, on an average, a correction of about 0.15 millimeter per kilometer is necessary to close the circuits. It is believed, therefore, that the differences in level shown by the adjustment are real differences and that they are not due to any peculiarities in the leveling.

U. S. Coast and Geodetic Survey,
Washington, D. C.

WORLD LONGITUDE COMPUTATIONS AND ISOSTATIC REDUCTIONS OF GRAVITY AT SEA

C. H. Swick

Two stations of the world longitude-net, Honolulu and Manila, were occupied by observers of the U. S. Coast and Geodetic Survey. Satisfactory observations were obtained at both stations, although at Manila there was some delay in getting started owing to difficulties with the radio apparatus. An interesting feature of the work at both stations was the use of the gravity-pendulum in place of an accurate astronomical clock.

A precision time-piece was needed in this work because the various radio signals came in at different times of day and night and it was obviously impossible to make an independent time-determination for each signal received. Instead, a set of star-observations was made on each clear night, and the precise rate of the chronometer, during each interval between the mean epoch of these observations and the time of the radio signals, was determined by means of the gravity-pendulum.

At Honolulu, a very good check on the accuracy of this method for carrying time was obtained in the following manner. It happened that one set of signals emitted each day by the Honolulu radio station came about midway between the time-observations on the preceding and the following nights. It was possible, therefore, in many cases to obtain the local sidereal time of this particular set of signals both by computing ahead from the preceding time-determination and backward from the following one. In several instances the discrepancy thus obtained was less than one one-hundredth of a second of time, and the maximum only slightly greater than two one-hundredths. This indicates not only that the gravity-pendulum is a very accurate time-piece but also that the time-observations themselves are remarkably consistent.

The chief objection to this use of the gravity-pendulum is the rather excessive amount of labor involved both in the field observations and in the office computations. For example, at Honolulu the pendulums were kept going almost continuously for more than fifty days and several extra observations were made each day to compare the pendulum with the chronometer at the times of the different radio signals. Altogether the observations and computations of the pendulum part alone amounted to as much as would be required at twenty-five or thirty ordinary stations.

The final results of the world longitude-work will not become available until all the differences which were determined last October and November have been adjusted into one great net around the world. It has been partly arranged that the U. S. Coast and Geodetic Survey shall have at least the Pacific Ocean part of this net to adjust. If one looks at a map on which the stations are plotted[1] this looks like a simple task; but if one considers that at San Diego alone the Naval Observatory determined the local sidereal time of 650 different sets of signals and that each of these signals gives a longitude-difference with every station recording the same one, the work involved in getting all the data coordinated begins to look formidable.

In December Dr. Meinesz arrived at Java and thus completed his circumnavigation of the globe in a submarine of the Dutch Navy. His belt of gravity-determinations at sea now extends entirely around the world. In February, after a brief rest, he made an additional series of determinations across the edge of the continental shelf south and west of Java. This trip, by the way, came near to being disastrous as something went wrong with the machinery and the ship submerged to a depth of eighty meters before an adjustment could be made. As this was twenty meters beyond the maximum depth for the submarine the

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[1] See map given in connection with the preceding paper in this Bulletin by F. B. Littell.

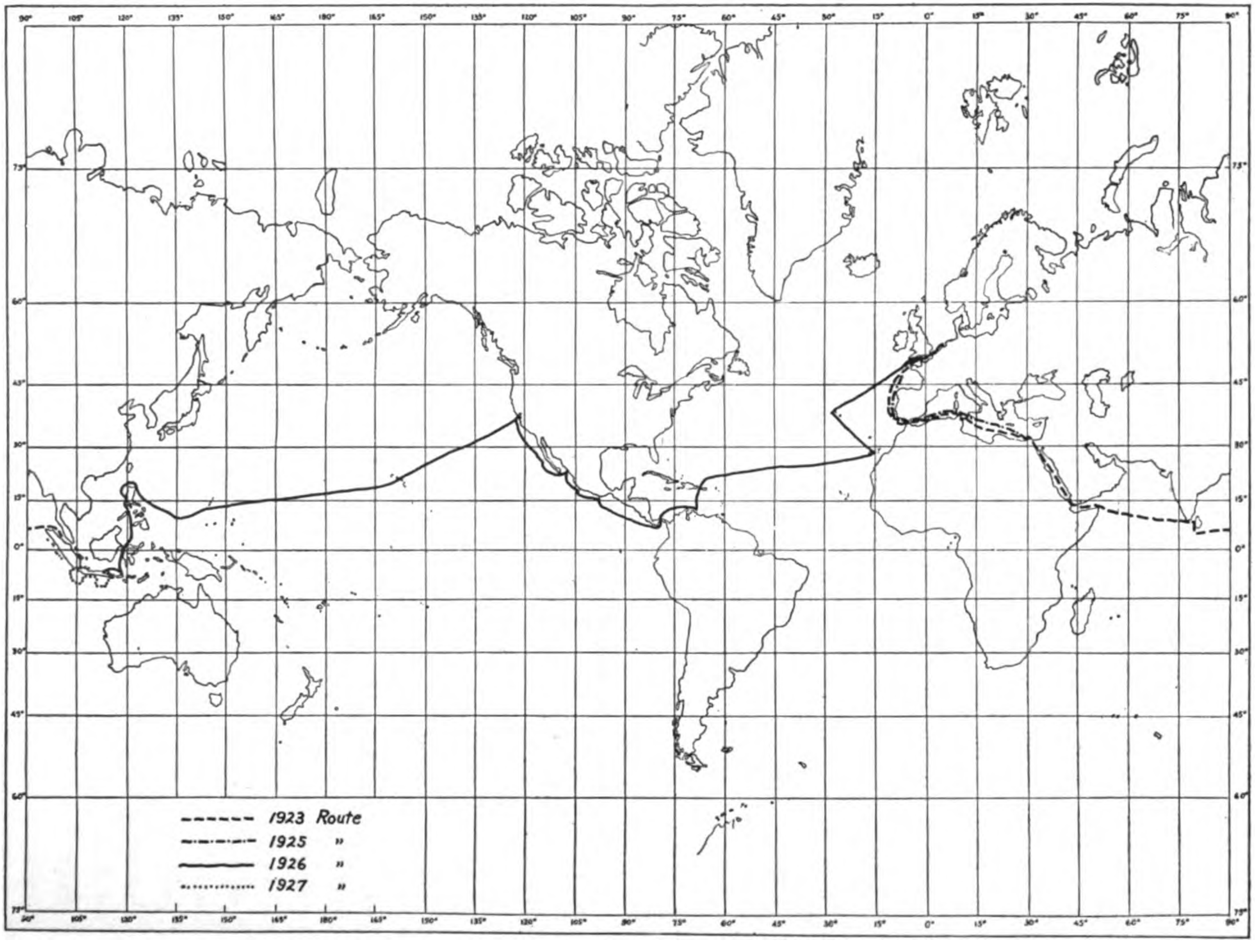

FIG. 1

Routes followed by the Dutch submarine on which Dr. F. A. Vening Meinesz made his gravity-determinations at sea.

danger of collapse was very great, the pressure at this depth amounting to more than eight tons per square foot.

On his trip across the Atlantic and as far as Honolulu in the Pacific, Dr. Meinesz was able to make preliminary computations of his observations for many of the stations; these he forwarded to the U. S. Coast and Geodetic Survey whenever he had an opportunity. Beyond Honolulu the extreme heat and other hardships of the trip affected Dr. Meinesz' health to such an extent that he was unable to make the computations and it was only with much difficulty that he kept up the observations.

Dr. Meinesz returned to Holland in March and within a few weeks he will probably send us the results for the rest of his stations. There are about one hundred stations for which he has not yet sent the data. We hope to complete the reductions within a short time after the data become available, perhaps in time to permit Major Bowie to give a preliminary report at the Prague meeting. It is expected that the complete report of the reductions will be published by Holland as was done for the thirty-one stations[2] of the first trip in 1923.

The isostatic reductions have already been made for fifty-six stations between Holland and Honolulu. Except for two large negative anomalies near Porto Rico, nothing about the results warrants special comment except the tendency of the anomalies to be positive, especially over the deep water. Of the twenty-five stations in the Atlantic which have been reduced, all have positive anomalies except the two just mentioned. In the Caribbean Sea there are an equal number of positive and negative anomalies, but in the Pacific the tendency is strongly positive again, only two out of twenty-five stations having negative anomalies. The average isostatic anomaly for the fifty-six stations is +0.026 dyne.

In studying over these anomalies, Major Bowie and I have come to the conclusion that positive anomalies over the oceans are due to a large extent to the depression of the geoid over these areas and to the fact that the gravity-formula is based on determinations at land stations alone, at many of which the geoid surface is considerably above the spheroid. If it were possible to apply an elevation-correction to reduce each observed value of gravity to the spheriod, the formula derived from these values would undoubtedly have quite different constants from those in our present formula, and the anomalies over the oceans would then have very nearly normal values.

A computation of the depression of the geoid in the Atlantic, off the coast of South Carolina, showed that at a point 525 miles east of Charleston the geoid is more than twenty-four meters lower in relation to the spheroid than at Charleston. A correction for even this amount of depression would change the anomalies by 0.008 dyne. In the middle of the Atlantic the depression of the geoid is undoubtedly greater than at the point computed.

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[2] **Isostatic reductions by the U. S. Coast and Geodetic Survey of the results of the pendulum-observations at sea made in 1923 between Holland and Java. Publication of the Dutch Geodetic Committee, Delft, 1926.**

U. S. Coast and Geodetic Survey,
Washington, D. C.

LOOP-CLOSURES RESULTING FROM THE READJUSTMENT OF THE FIRST-ORDER TRIANGULATION IN THE WESTERN PART OF THE UNITED STATES

O. S. Adams

At the meeting of this section of the Union last year a partial report on the readjustment of the first-order triangulation in the western part of the United States was given. The work of the adjustment has now been advanced to the point at which some interesting and important results can be given.

At the time of the report last year two sections remained to be observed. These were the western end of the 49th parallel and the arc in southern California and Nevada joining the coast arc to the one extending from the 39th parallel in western Utah southward to the Mexican boundary. The field work on these sections was completed in due time and the office work of the figure-adjustment was made as quickly as possible. It was the desire of the Coast and Geodetic Survey to finish all the work of the preliminary adjustments at as early a date as possible so that the loop-closures could be computed and the readjustment of these closures could be made so as to fix the final positions in the junction-figures.

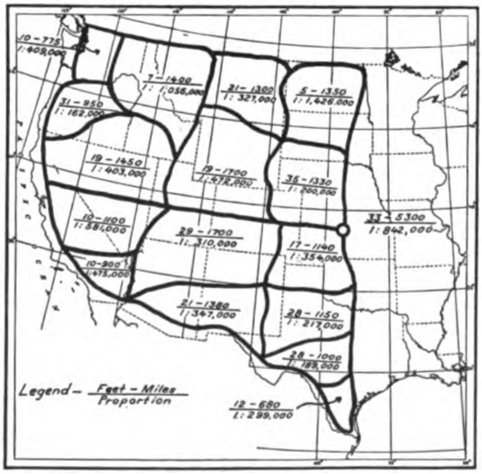

Fig. 1

Loop-closures in feet with number of miles in loop and approximate proportional part of length of loop that closure represents.

Figure 1 with the loop-closures shows some surprising results regarding the accuracy of triangulation when length and azimuth-control are introduced into the adjustment. There are only two loop-closures greater than one part in 200,000 and the largest closure in the whole scheme is only one part in 162,000.

The mean of all the closures is approximately one part in 450,000. The closure around the entire outer boundary is thirty-three feet in a distance of 5,300 miles or approximately one part in 842,000. In the whole network there are between twelve and thirteen thousand miles of triangulation, or enough to reach half way around the Earth at the equator.

The adjustment of these closures has been finished and the final position of a station in each junction-figure has thus been determined. Since an azimuth in each of these junctions is fixed by the Laplace azimuth-observations, it is now possible to compute the positions of all of the stations in each junction-figure. The exact latitude and longitude of these control-stations in the junction-figures are of no especial interest in themselves, but the change in these positions from previously adjusted values is of interest and is shown in Figure 2. It will be seen that all of the west-coast stations have been moved south and east approximately one second of arc in each direction. At the northwest corner the change is —1."056 in latitude and —1."374 in longitude. This junction will furnish the data that is to be carried northward through

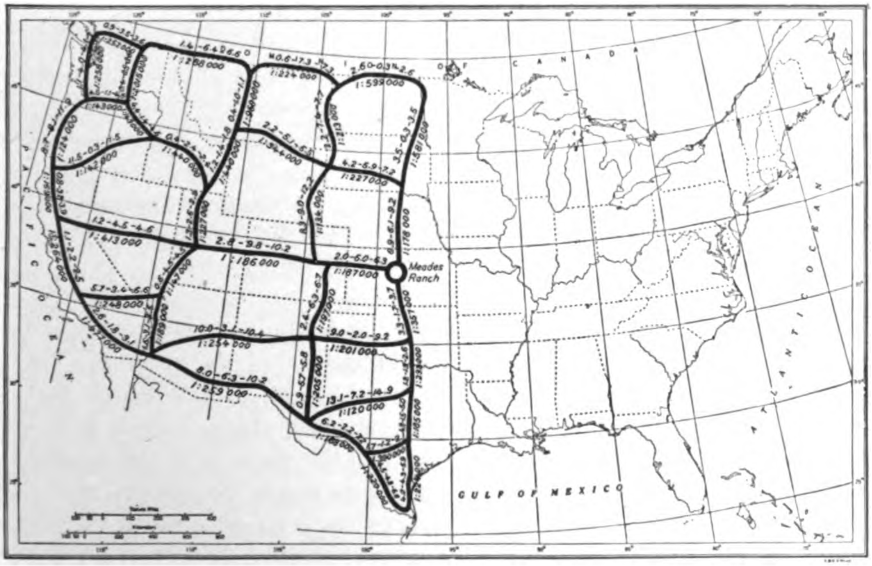

FIG. 2

Readjustment changes in latitude and longitude of points previously adjusted.

western Canada to our triangulation in southeast Alaska. These data have already been tabulated and a copy has been sent to the Geodetic Survey of Canada so that that organization may be able to use this control in the computation of its work which connects directly with the first-order work in Alaska. It is expected that the computations in Alaska will soon be put on the North American Datum of 1927.

The adjustment of the loop-closures distributes these closures among the various sections in such manner as to do the least possible violence to the various parts of the general scheme. Figure 3 shows in feet, first the amount that must be absorbed by the latitude, and secondly the amount by the longitude, and the total closure of the section. The approximate proportional part of the total length of the section that this closure represents is also given in each case. In no section is this closure greater than one part in 120,000 and the majority of the closures are near one part in 200,000 or better. This means that no undue violence will have to be done to any parts of the entire scheme in the adjustment of the sections to absorb these still outstanding discrepancies.

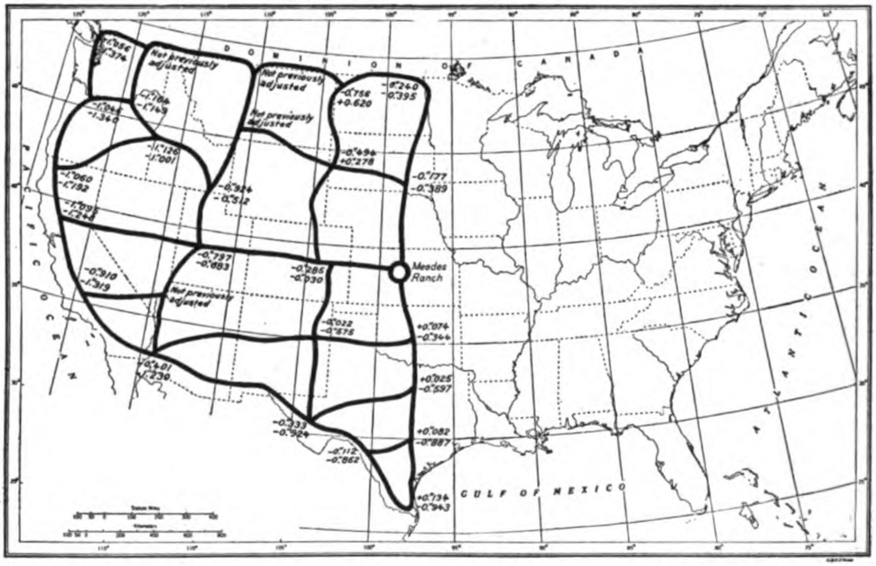

Fig. 3

Closures in feet to be absorbed by the sections: (1) Closure in latitude, (2) closure in longitude, (3) total closure in distance, and (4) approximate proportional part.

It will now be the task to add the conditions for closure in latitude and longitude to the various sections of the network and, in this way, to adjust the whole into one continuous framework. In this way, part of the closure of each loop is put in each section and no section will have to absorb an undue amount. By the former method of piecemeal adjustment, any new section closing a loop was required to absorb all of the closure of the loop and hence oftentimes severe violence was done to arcs of excellent observational data. The fact that the orientation has been maintained throughout by Laplace azimuth-observations has obviated one of the weak elements in triangulation, and this fact has contributed greatly to the excellent closures that we have found in our work of adjustment.

U. S. Coast and Geodetic Survey,
Washington, D. C.

A PROGRESS REPORT ON THE GEODETIC WORK OF THE UNITED STATES COAST AND GEODETIC SURVEY

C. V. Hodgson

Since the last meeting of the Section one major arc of first-order triangulation has been completed, a minor arc has been begun and completed, and a beginning has been made upon another major arc.

The major arc completed was the west end of the 49th parallel, an unusually difficult piece of work; the stations were so inaccessible that five observing parties were used in order to finish the work before the glaciers below the stations should become dangerous. Even with that number of parties, smoke from forest-fires again so delayed the observing that the stations were finished only with the greatest difficulty and with frequent risk of serious injury to the observers and their assistants.

The minor arc completed extended from Las Vegas, Nevada, across the south end of Death Valley westwardly to a junction with the Pacific Coast triangulation at about the 36th parallel. It serves the two-fold purpose of dividing up a large area devoid of first-order control and will also, we hope, help to localize some of the large apparent changes in geographic position of points in the Pacific Coast triangulation. Both this project and the one completing the 49th parallel triangulation were finished in time to be included in the adjustment described by Dr. Adams.

The third arc was east of the 98th meridian where we have an area of about 200,000 square miles, including all of Iowa and parts of the adjacent States, with no first-order horizontal control points in it. A start was made last year toward the development of that area by observing for about 100 miles along a north-and-south arc in northern Iowa, which will eventually extend from northern Minnesota to the Gulf. At the same time about 200 miles of first-order traverse were run in Wisconsin as a part of the same scheme. During the coming summer we hope to complete the part of the scheme lying in Minnesota.

Passing now for a moment to the future, several belts of triangulation are needed to develop the area east of the 98th meridian to a point where a study can be made of the errors of position similar to that described by Dr. Adams. About 5,000 miles of triangulation and traverse will be required for this initial development of the eastern half of the country which will require about $375,000 for the field and extra office work. At the present rate of appropriation for geodetic work it will require seven or eight years to complete the horizontal control outlined. From a purely scientific standpoint we might be content to let the mills of the gods grind on at their present rate, but from an economic standpoint that rate is entirely too slow.

Several short lines of first-order leveling were run during the year but none had any unusual importance in the development of the system of precise levels. The principal lines were in South Dakota, Colorado (two projects), and California. The appropriation of $10,000 for geodetic survey in regions subject to seismic disturbances was spent entirely on leveling in California and, while the results thus far obtained are not startling they will serve as an excellent foundation for future investigations.

It is always interesting to examine a progress map and to do "paper reconnaissance" on it. In Colorado and Utah there is an immense area without cross lines of levels. A line across Utah will, we hope, be completed during the coming summer, and our plans for next year include a line of levels from Albuquerque to Pueblo, which will subdivide another large blank area on the map. Other lines of importance will be run as rapidly as funds permit. Thus the level-net grows in intricacy of design; turning back the years and looking at the level-net of 1916 we can see that some progress has been made even though it is not as rapid as we should like to see.

U. S. Coast and Geodetic Survey,
Washington, D. C.

SUGGESTIONS FOR THE IMPROVEMENT OF PENDULUM OBSERVATIONS

C. A. Heiland

In the location of mineral deposits or geological subsurface structures by means of gravity-measurements, the Eötvös torsion balance has hitherto been used almost exclusively. Pendulum measurements have been but little used in the solution of such problems and then chiefly when the problem was one of surveying large buried tectonic units, extending over a considerable area. What has hampered the use of the pendulum measurements in most cases has apparently been the time required to make them, apart from the fact that the gravity anomalies produced by most types of subsurfaced structures do not greatly exceed the errors of the results.

The comparatively large amount of time necessary for pendulum observations is, as is generally known, due to the fact that the oscillations of the pendulums must be compared with those of an astronomic clock carried around in the field; the rate of the clock in turn must be checked at each station, either by astronomic time-determinations or by the reception of radio time-signals. In other words, it is necessary to spend at each field station a time at least as long as the interval between two consecutive time-determinations, that is, about twenty-four hours.

In order to reduce this time, it has been suggested (A. Berroth, Zeitschrift für Geophysik, 1927, Heft I) that the astronomic clock be left permanently at a central station and connected by wires to the flash-box of the various field stations. With this arrangement, it is not necessary to wait at the field stations for the time-signals; one field station may then be finished in about four hours. However, Berroth found that this advantage in time was compensated to a considerable extent by the time wasted in the repair of the wiring between the central station and the field stations. He, therefore, suggested the broadcasting by radio of the beats of the astronomic clock, the reception of these signals at the field stations and their conversion into beats of the flash-box.

Unquestionably a central arrangement of the clock would save considerable time. Moreover, the accuracy of the results may be expected to increase somewhat, since the rate of the astronomic clock, when set up permanently at some one place, will probably be more nearly constant than it would be when carried around in the field.

There remains, however, one disadvantage, because the results are still affected by any faults in the behavior of the clock between two time-signals; in other words, it would not be possible to find out whether within twenty-four hours the astronomic clock showed, for instance, a sudden change of its rate instead of a continuous variation. Moreover, another source of error is the mechanical contact made in the astronomic clock to set the flash-box to work; the operation of this calls for the use of electromagnets and springs and, therefore, involves also a time-lag, which can not be expected to be always constant.

I therefore offer for discussion the following suggestion, one aim of which is first to dispense entirely with astronomic time-determinations when radio time-signals are received. Second, the proposed method does away with the flash-boxes and, third, it does away with the visual observations of the moment of coincidence. It, therefore, may be expected to improve the accuracy of the results.

It is possible to make the measurements at two stations exactly simultaneous, so that any variations of the rate of the central chronometer during the time of observation are eliminated.[1]

If we denote by T the time of oscillation of the central chronometer, by T_o its value when observations are made at the central station having the gravity g_o, by $T_{,}$ and $T_{,,}$ its values at the time when measurements are made at the field stations having the respective values of gravity $g_{,}$ and $g_{,,}$, and if we let m (with corresponding subscripts) denote the factor $\frac{n}{2n \pm 1}$ in the coincidence formula, and l the length of the pendulum we have

$$\frac{g_o}{\pi^2} = \frac{l}{m^2_o\, T^2_o}$$

$$\frac{g_{,}}{\pi^2} = \frac{l}{m^2_{,}\, T^2_{,}}$$

$$\frac{g_{,,}}{\pi^2} = \frac{l}{m^2_{,,}\, T^2_{,}}$$

whence

$$\frac{g_{,}}{g_{,,}} = \frac{m^2_{,,}\, T^2_{,,}}{m^2_{,}\, T^2_{,}}$$

There are two possible ways of eliminating $T^2_{,,}$ and $T^2_{,,}$ in this equation: (1) by observing with a single pendulum-apparatus at two stations and keeping T constant during the time required for both observations; (2) by making the measurements at the two stations exactly *simultaneous* with *two sets of apparatus* (any variations of T would thus be eliminated). Since the first method is practically unattainable, we choose the second one.

The constant difference of the two sets of pendulum apparatus 1 and 2 employed may be determined by simultaneous standardization at the central station. For (omitting now the subscripts of T for the case of simultaneous observations) we have

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[1] The advantage of making simultaneous observations at two field stations may have occurred to others, but no methods have been published which could be used for this purpose.

$$\frac{l_1}{m^2_o\,T^2} = g_o = \frac{l_2}{M^2_o\,T_2}$$

where M is the factor m for the instrument No. 2

$$m^2_o = M^2_o$$

or putting

$$\frac{l_1}{l_2} = q$$

$$m^2_o = M^2_o\,q$$

Then

$$\frac{g_{,}}{\pi^2} = \frac{l_1}{m^2\,T^2}$$

and

$$\frac{g_{,,}}{\pi^2} = \frac{l_2}{M^2\,T^2}$$

T cancels out, giving

$$\frac{g_{,}}{g_{,,}} = q\,\frac{M^2_{,,}}{m^2_{,}}$$

Putting

$$M^2 = M^2_o\,a = \frac{m^2_o}{q}a \text{ and } m^2_{,} = m^2_o b,$$

we get

$$\frac{g_{,}}{g_{,,}} = \frac{a}{b}.$$

This simple relation expresses the ratio of the values of gravity at two neighboring stations which have been determined simultaneously.

The question now is, how to synchronize the observations at the two neighboring stations, or, strictly speaking, how to arrange matters so that exactly the same beats of the central clock may be used at both stations for comparison with the pendulums.

The proposed mechanism is as follows: At a central station we place, not the usual astronomic clock, but a pendulum apparatus, the invariable pendulum of which is made to swing continually in a vacuum. In front of the pendulum mirror in the receiver there is a lens instead of a plain glass, at the focus of which a diaphragm is placed illuminated by a strong electric light. A beam of light, therefore, goes through the diaphragm, then through the lens in the receiver, is reflected by the pendulum mirror and passes through the lens again. At the focus of the lens there is also another diaphragm, behind which is placed a photoelectric cell, the position of which is so adjusted that the reflected beam of light just hits this second diaphragm if the pendulum is in perfect rest or if it passes through this zero-position while swinging. Every exposure of the photoelectric cell produces an impulse of an electric current, which in turn acts upon the circuit of a radio transmitting-set. In other words, the radio set at the central station transmits a dot whenever the pendulum passes its zero-position. The two field stations at which the observations are to be made simultaneously are each equipped with a receiving outfit. The variations of the arriving wireless energy are there converted into fluctuations of an electric source of light, and the signals of the central pendulum are recorded photographically on the same sheet as the oscillations of the field pendulum on each station.

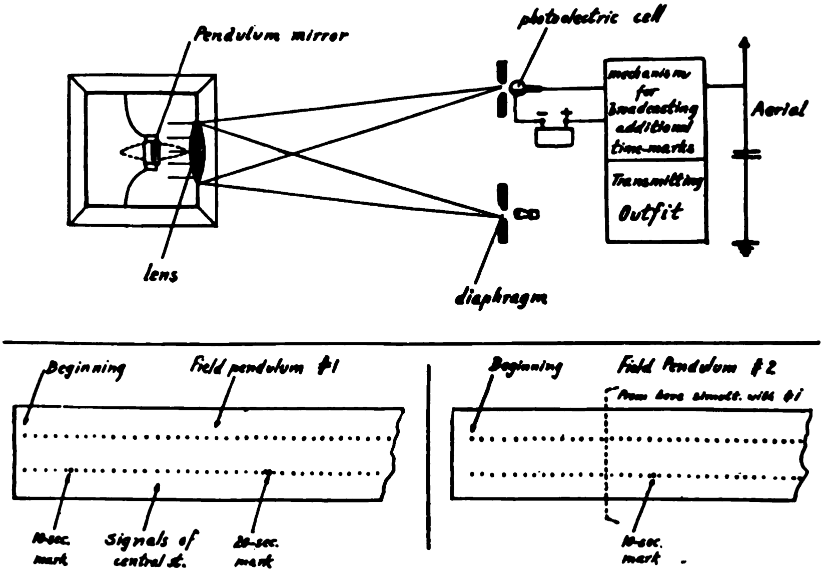

FIG. 1

Schematic diagram of apparatus for proposed central transmitting-station and of records at receiving field-station (started simultaneously within one minute).

In order to use exactly the same beats of the central pendulum for the comparison with the pendula at both stations, the signals must be marked in some way. This is done by the arrangement described in the next paragraph.

The circuit, which is closed whenever the photoelectric cell is exposed, also operates an electromagnet, which moves a dial in a counting mechanism. This mechanism is in turn connected with another mechanical arrangement which effects the broadcasting of six different special signals, consisting of dots and dashes. The counting mechanism brings it about that when twenty dots are transmitted, i. e., after ten seconds, the first special signal is broadcasted. After twenty seconds, there follows the second special signal, and so on. Since, in other words, sets of sixty seconds each can be readily distinguished on the photographic sheet, it is only necessary to start the records on each field station within one minute simultaneously. Any ordinary watch furnishes this accuracy. In reading the results from the photographic sheets, exactly the same beats of the central pendulum can be used for comparison with the field pendulums.

In applying the preceding method, it is no longer necessary to compare the central clock with absolute time-determinations or with radio time-signals. I believe, therefore, that it should be possible in the future to occupy with the proposed method about four field stations a day, including the establishment and dismantling of the stations.

Colorado School of Mines,
Golden, Colorado.

SECTION OF SEISMOLOGY

(Washington, D. C., April 29, 1927)

REPORTS OF PROGRESS IN SEISMOLOGICAL WORK IN THE UNITED STATES

A—CARNEGIE INSTITUTION OF WASHINGTON

ARTHUR L. DAY

The activity of the Carnegie Institution of Washington in the field of seismology in 1926 may be reviewed briefly somewhat as follows:

One major task, jointly elaborated at the outset between the Institution and the U. S. Coast and Geodetic Survey, contemplated fixing the position both in horizontal and vertical dimensions of significant landmarks throughout the entire zone of earth-movement in California. The obvious purpose, which has often been alluded to before, was two-fold, (1) to fix with precision the actual displacements occurring during earthquakes, (2) the much more novel, and, eventually, perhaps more useful, purpose of attempting to evaluate indications of accumulating stress or "creep."

"The pursuit of this object has made necessary a new primary triangulation extending southward from Ukiah to the Mexican boundary, and east and west across the State to suitable points beyond the Sierra Nevada Range which is believed to be nearly free from movement. This triangulation is now nearly completed, north and south, and three lines have been finished in the east-west direction, namely, at San Francisco Bay, San Luis Obispo, and the Mexican boundary.

"For the detection of vertical movements lines of precise leveling have been undertaken in the southern portion of the State, chiefly with reference to known fault-zones. The leveling is still in progress. Eventually it is hoped to extend it over substantially all of the region covered by the recent triangulation."

The second major task which the Institution confronted was an adequate, detailed study of the geology of those portions of the west coast region in which the movements more commonly occur. This is a task of very considerable dimensions covering a detailed geological examination of the southern two-thirds of the State of California. It can therefore only proceed slowly, in collaboration with the U. S. Geological Survey and the geological departments of the universities and technical schools of central and southern California. The first published report of this work was prepared by Dr. Levi F. Noble, of the U. S. Geological Survey, and contains detailed studies of portions of the master-fault of

California, the San Andreas Rift. This report is printed in the Annual Report of the Advisory Committee in Seismology of the Carnegie Institution of Washington for 1926. Several years are expected to be occupied with these geological studies.

The third item on our program has been the primary responsibility of the Carnegie Institution of Washington alone, namely, the development of suitable instruments for the study and evaluation of local earth-movements in the region under investigation and the erection of suitably located stations for their installation. The Wood-Anderson seismograph has been devised for this purpose and the two horizontal components have now been in service for three years or more. Instruments of this type are now being installed in the new central station in Pasadena and in subordinate stations at Riverside, at La Jolla, at Santa Barbara, and at the Mt. Wilson Observatory. Other stations are also under consideration. Several designs of vertical-component instruments have been considered and three different types have been set up at different times, but this phase of the problem is not yet completely solved. With this group of stations in southern California there will probably be affiliated a second group about San Francisco Bay in charge of Professor Byerly, of the University of California. The cost of these stations has been contributed by citizens of San Francisco, and instruments of the same type as those used in the south are in process of manufacture. Probably another year will see this northern group of stations in successful operation as well.

It is a particular pleasure and privilege to announce that the central laboratory building in Pasadena, built for this purpose by the California Institute of Technology, has just been completed and the apparatus is at this moment being installed in it. It is a beautiful building for the purpose, well located, simple in architecture, but with abundant space for the investigations to be undertaken in it. Several piers erected upon the underlying granite afford the best of facilities for the transmission of earth-vibrations to the instruments and two tunnels leading into the granite hillside provide opportunity for tilt-measurements or other experimental studies. This laboratory is to be in charge of Mr. Harry O. Wood, who has conducted the work in Pasadena since its inception.

Of the problems to be taken up in this laboratory it is perhaps too early to give any extended account. Its first concern is obviously to develop a vertical-component instrument which shall be equally as competent for the purpose as the Wood-Anderson horizontal-component instrument. The second is almost equally obvious, namely, to provide for identical minute-to-minute time-signals for all of the stations which are to participate in this investigation. This year is likely to be fully occupied with these two problems alone.

Meanwhile the development of a driving mechanism for the chronographs, which receive the seismometer-records, is now practically completed. A beautiful electric mechanism driven from a tuning-fork or vibrating reed with power derived from a storage battery has been devised which provides a uniform speed of rotation with the minimum expenditure of power. The synchronous motors operate gear-work which is designed to run in oil and to give a speed of rotation of the recording drums of 60, 30, 15, or 7.5 mm per minute. Under favorable conditions an accuracy of one part in a hundred thousand has been attained.

These devices are likely to prove of the greatest convenience in all chronograph-work. They appear to be thoroughly trustworthy in use and inexpensive to operate. They have been offered for patent and will presently be described in print.

The Committee has continued to administer during the past year the grant of $5,000 provided by the Carnegie Corporation of New York in aid of seismological publication. Twenty-four reports of recent seismologic research, including the full description and theory of the torsion seismometer by Anderson and Wood, have been printed in the Bulletin of the Seismological Society of America, either wholly or partly at the expense of this fund, which has been of the greatest service to seismology in its hour of need. Amount expended $3,358.96. The available balance of the fund on July 1, 1926, was only $1,641.04. It will presently be exhausted and it is understood that a renewal of the grant is not to be expected.

Advisory Committee in Seismology,
Carnegie Institution of Washington,
Washington, D. C.

B—JESUIT SEISMOLOGICAL ASSOCIATION

James B. Macelwane, S. J.

The Jesuit Seismological Association, organized in the summer of 1925, is composed of twelve seismographic stations built and equipped by Jesuit universities and colleges in the United States. These are Canisius College, Buffalo, New York; Loyola University, Chicago, Illinois; St. Xavier College, Cincinnati, Ohio; Regis College, Denver, Colorado; Fordham University, New York, New York; Georgetown University, Washington, District of Columbia; Spring Hill College, Mobile, Alabama; Loyola University, New Orleans, Louisiana; St. Louis University, St. Louis, Missouri; University of Santa Clara, Santa Clara, California; Gonzaga University, Spokane, Washington; Holy Cross College, Worcester, Massachusetts. All but one of these stations originally owed their existence to the energetic initiative of Reverend Frederick L. Odenbach, *S. J.*, of the John Carroll University, Cleveland, Ohio, who during the years 1909 to 1911 persuaded the faculties of the various institutions of their opportunity to make substantial contributions to scientific knowledge in this field at comparatively small cost. They were at first united into a Jesuit Seismological Service with a central station in Cleveland. But this arrangement lasted only until 1912.

The details of the reorganization into the present Association partly as the result of the kindly interest of Dr. Arthur L. Day and Mr. Harry O. Wood, were published in a recent issue of the *Bulletin* of the Seismological Society of America (16: 3, 1926).

During the time from 1912 to 1925 Father Francis A. Tondorf, *S. J.*, of Georgetown University, did more than anyone else in the Jesuit group to keep alive public interest in seismology. By his lectures, writings, and especially by his regular and prompt reports of earthquakes, he stimulated the press, and through it scientific men and the people at large, to seek further information on seismological subjects. To keep pace with the growing interest, he steadily improved the equipment of the Georgetown station until he had in the Galitzin vertical component the most sensitive instrument in the country. In return for the data given them, the Associated Press furnished Father Tondorf copies of all reports on earthquakes that came over the wires. These reports were printed and distributed by Father Tondorf under the name of Press Dispatches and were much appreciated, especially by foreign seismologists. In addition to the monthly reports of earthquakes recorded by the Georgetown station, Father Tondorf also printed an Annual Report. He also found time to undertake such fundamental research as that carried through with Mr. Stuart B. Moore on the nature and causes of traffic vibrations in roadways. Father Tondorf is now planning to add to the equipment of Georgetown two horizontal-component Galitzin seismographs.

The second station in importance in the group which organized the Jesuit Seismological Association is that at Fordham University, New York. Until three years ago this station was equipped with only an eighty-kilogram Weichert seismograph. But that year a gift of ten thousand dollars for the purpose by Mr. William Spain of New York made possible the erection of a new station-building and the importation of two Milne-Shaw seismographs. One of these was installed in the autumn of 1924, the other late in 1925. Very recently, Mr. Spain has generously donated to the Fordham station three Galitzin seismographs. These are now under construction in Esthonia. Fordham will shortly take its place beside Georgetown among the best-equipped stations in the world. Besides publishing monthly reports of the earthquakes recorded at the station, Fordham communicates by cable with the international center at Oxford.

The station at Regis College, Denver, under the direction, since its beginning, of the Reverend A. W. Forstall, *S. J.*, has maintained an uninterrupted series of records. A number of minor improvements have been made at the station. The monthly reports are distributed to a much wider circle of observatories throughout the world than formerly.

The stations at Gonzaga University, Spokane, and at the University of Santa Clara have remained in continuous operation and now issue reports regularly. The small Weichert seismograph at Spokane normally registers unusually large amplitudes. Its records are of great importance for the earthquakes in Montana.

The stations at Spring Hill College, near Mobile, and at Loyola University, New Orleans, have furnished valuable data particularly for Caribbean and Central-American earthquakes. The station at Canisius College, Buffalo, has been in continuous operation though it has not been possible to resume publication of its reports. With the appointment last fall of Reverend George J. Brunner, *S. J.*, as director, the station at Loyola University, Chicago, which had been idle since 1912, was put back into continuous operation and has obtained some excellent records.

A new first-class station is being established by St. Xavier College, Cincinnati. It will be equipped with two long-period and two short-period Wood-Anderson horizontal-component seismometers, with a short-period Wood-Anderson vertical, and with a long-period Galitzin vertical-component seismograph. The first of the instruments has just arrived.

The St. Louis University station has kept the eighty-kilogram Weichert in continuous operation and has issued monthly reports for the years 1924, 1925, and 1926. During the years following the retirement of the first director, Reverend John B. Goesse, *S. J.*, the credit for maintaining the station in condition for operation is due entirely to the devoted Meteorologist of the University, Brother George E. Rueppel, *S. J.* At present the St. Louis University has under construction two new stations which will be described in some detail at the Cambridge meeting of the Eastern Section of the Seismological Society of America.

The Jesuit Seismological Association has established at Saint Louis a Central Station which serves as a clearing house for information for the benefit of the member-stations. It has arranged to cooperate very closely with Science Service and with the United States Coast and Geodetic Survey in the preliminary determination of the epicenters of all important earthquakes recorded in this country. Science Service defrays the cost of telegraphic reports to Washington and to Saint Louis, on which a double determination is made and then averaged. The determination thus becomes available to the press through Science Service very soon after each earthquake. The arrangement was made at the beginning of the year 1926 and has worked well. Great credit is due the officials of Science Service for this public-spirited undertaking. Altogether, the stations belonging to the Jesuit Seismological Association and also its Central Station have met with the heartiest cooperation, not only on the part of Science Service and the United States Coast and Geodetic Survey, but also on the part of other organizations, particularly the Canadian Government stations and individual stations in other parts of the world.

St. Louis University,
St. Louis, Mo.

C—SEISMOLOGICAL WORK OF THE U. S. COAST AND GEODETIC SURVEY[1]

N. H. Heck

The U. S. Coast and Geodetic Survey has been operating six seismological stations: San Juan, Porto Rico; Cheltenham, Maryland; Chicago, Illinois (at the University of Chicago, formerly operated by the U. S. Weather Bureau); Tucson, Arizona; Sitka, Alaska; and Honolulu, Territory of Hawaii (station formerly at magnetic observatory has been removed to the University of Hawaii where it is operated as a cooperative station). A Wood-Anderson torsion seismometer of the teleseismic type has been in operation at Tucson for a year and we are now able to appraise the results. There are Milne-Shaw seismographs at Chicago and Honolulu and recently the recording apparatuses have been rebuilt so that the time-recording is greatly improved. At the other observatories there are obsolete instruments, the Bosch-Omori, giving good records of severe earthquakes but missing practically all others. Replacements at Sitka and Porto Rico are part of the program as these are seismic regions of great importance without first-class instruments anywhere. Replacement at Cheltenham is not contemplated at present, since it is but a short distance from the first-class station at Georgetown. Probably the operation of seismographs at Cheltenham, except continuation of the Bosch-Omori, will be for experimental purposes.

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[1] Abstract of Report to Prague Meeting, International Geodetic and Geophysical Union.

The collection of data on felt earthquakes is becoming more complete. The Weather Bureau has continued to make reports and this is the back-bone of the reporting system. In addition, state geological surveys, various universities, and other institutions are making reports, and at times of earthquakes great numbers of miscellaneous reports come in. Even with all this, there is scarcely enough, in many cases, for proper delineation of isoseismal lines.

The *Seismological Report* of the Coast and Geodetic Survey summarizes all earthquake activity felt within the United States and the regions under its jurisdiction, and earthquakes recorded in the same region. Since with sufficiently sensitive instruments the latter class would include earthquakes from all parts of the Earth, in general they are confined to those in the American continents, adjacent waters, and the Pacific region. Only in exceptional cases are others included since it is felt that existing stations cover well all the remaining parts of the Earth. Annual summaries give tables and maps showing the activity during the year. Preliminary epicenters are determined when practicable. The report includes analysis of records of stations of the Coast and Geodetic Survey and of cooperating institutions which do not publish reports.

Immediate determination of epicenter, chiefly for the region that has been outlined, is made in cooperation with Science Service and the Jesuit Seismological Association, and numerous seismological stations throughout the region. A first determination is made for press purposes within twenty-four hours and another, when all reports are in, is sent to all cooperating stations. The latter is useful in interpretation of phases at the various stations since gross errors are avoided in determining phases.

The report does not ignore the important geodetic operations in California in connection with the earthquake investigation there but this subject is adequately covered by Dr. Day, as chairman of the Advisory Committee in Seismology of the Carnegie Institution of Washington.

Tidal records of stations of the Coast and Geodetic Survey and cooperating organizations throughout the region outlined above are examined for evidences of seismic sea-waves.

Other government activities in seismology include the geological investigation of the more important earthquakes by the U. S. Geological

Survey; the U. S. Bureau of Standards develops instruments; and the work of the Weather Bureau has already been noted. The U. S. Navy contributes reports of seismic waves experienced by its vessels and those of the merchant marine which send reports to it.

It has been found that much valuable information regarding various phases of earthquakes is in the Japanese language and is not likely to be translated by the Japanese. The Coast and Geodetic Survey has found that Government facilities for translation of Japanese do not extend beyond letters and short pamphlets. Accordingly, it has asked the American Geophysical Union to consider suggesting to the National Research Council the desirability of arranging for translation of highly important seismological information not now available. It is believed that a number of organizations are sufficiently interested to be willing to pay for such material.

U. S. Coast and Geodetic Survey,
Washington, D. C.

REPORT OF PROGRESS IN SEISMOLOGICAL WORK IN CANADA

THE SEISMOLOGICAL WORK OF THE DOMINION OBSERVATORY, OTTAWA

ERNEST A. HODGSON

Seismology is essentially a science requiring international research; furthermore, it has come to be rather minutely subdivided and specialized; finally, it may be observed that the data of the science are scattered and that a stock-taking of amounts and values of these is in order. We have in these circumstances the several reasons for presenting this paper, at this time, and before this, the Section of Seismology, of the American Geophysical Union.

The seismological work of the Dominion Observatory may be viewed in the light of (1) the records compiled, (2) the publications issued, (3) the present routine followed, and (4) the research-program in contemplation.

Considering this work in its relation to international effort it may be said that copies of any of the records at Ottawa are available to research workers of other countries on application. The publications are also available on request of those to whom they are likely to be of value. As regards the routine and research, these will be modified, whenever possible, to fit in to greater advantage with the programmes of other countries.

Speaking of seismological work as a phase of the modern specialization in science, it should be observed that the peculiar limitations and advantages of any station should be recognized and the work undertaken there be chosen so as to take full advantage of the situation obtaining. It does not pay to attempt work which requires too great a change in present equipment. This is the policy which has been followed at Ottawa, although increasing facilities are now making it possible to undertake more extensive research.

As a step toward eliminating the scattering and waste in seismological resources it was thought well to take advantage of the opportunity here presented of outlining those of the Observatory, in the way of records, publications, routine, and research, in order that other seismological organizations may know in what respects this institution may be able to cooperate with them in their work.

The records compiled date back to 1905, when the late Dr. Otto Klotz, afterwards Director of the Observatory, was given charge of the division of seismology, terrestrial magnetism, and gravity, of that institution. At that time a pair of Bosch, horizontal, photographic seismographs were purchased and installed; a Wiechert, eighty-kilogram, mechanical-registration, vertical seismograph was added in 1912. In 1915 a two-component Mainka, mechanical-registration, horizontal seismograph was set up at Dalhousie University, Halifax, and a similar instrument was placed, the same year, at the University of Saskatchewan, Saskatoon. In 1922, two Milne-Shaw seismographs began recording at Ottawa, in a specially-constructed vault, the floor of which is fifteen feet below ground and the roof six feet below ground-level. In 1925 a single component, Milne-Shaw seismograph was secured for a new station at the Experimental Farm at Ste. Anne de la Pocatiere, on the south shore of the St. Lawrence, near the centre of the seismic area.

The records from the Bosch seismograph have been carefully preserved whether earthquakes were registered or not. From April 1, 1908, to the present, this series is practically complete. The records are carefully preserved. Ten sheets are stored, without roll or fold, in each of the heavy manila filing envelopes provided. These are well indexed and are filed, on edge, in a special steel cabinet. The time-notations are especially good. Time, as recorded, is in general correct to within less than a second throughout. The constants of these seismographs were not determined very often during the earlier years and they can be known only approximately for most of the interval covered up to December 1, 1923. One component has been air-damped, the other magnetically damped throughout. The readings of the earthquakes and all memoranda for the series up to November 30, 1923, are preserved in a set of ledgers. Since that time another system has been in use, which will be described later in connection with the Milne-Shaw records.

Until October 15, 1920, the records from the Wiechert vertical seismograph were preserved only in the case of large earthquakes. Up to that time, a great deal of difficulty had been experienced with the temperature-control of the seismograph. The control having been attained, the records have been preserved ever since, whether earthquakes were registered or not. The same remarks as to filing, time-markings, constants, and recording of memoranda apply to these seismographs as were outlined in the preceding paragraph.

The records from Halifax have been preserved at Ottawa, only in the case of large earthquakes. When stored at the Observatory they are filed flat in the manila envelopes designed for that purpose. The time-markings at Halifax have been checked for some years by a telegraph company's signal, recorded once an hour on the seismograph itself along with the minute signals from a fairly good contact-clock. The time cannot be depended on, however, nearer than within two or three seconds, even if the company's signals were correct. We have reason to believe these are also out at times. The constants of the seismographs are known within rough limits only. They were determined at the time the instrument was placed, and again in August, 1920. The data were recorded in the Ottawa record ledgers up to November 30, 1923. They are since taken care of by the system now in force.

Until July 1, 1925, the seismograms from Saskatoon were preserved at Ottawa, only in the case of large earthquakes. Since that date all have been forwarded to Ottawa and filed there along with the records of the Ottawa seismographs. The time as recorded has not been very good and no efficient check was provided except for the first year or so. Within the last few months, however, a radio set has been installed and steps are now being taken to insure an efficient time-service for the records at Saskatoon. The constants are not well known. They were determined when the instrument was set and again in September, 1922, when it was moved to the new Physics Building of the University. The notes with regard to this instrument were entered in the Ottawa record ledger until November 30, 1923, and subsequently in the system at present in use.

The records from the Ste. Anne instrument have all been preserved. They, too, are filed vertically along with the other records in the special envelopes. The constants are well determined and with sufficient frequency to insure the reliability of the determined figures. The time is checked at least once a day by radio and a signal recorded on the seismogram by the observer listening in. The records are developed at Ste. Anne, but forwarded to Ottawa to be read. The data are cared for in the card-system in use at the Observatory.

The Milne-Shaw seismographs are, at present, the chief instruments of the Ottawa station. No. 17 has been recording continuously since May, 1922. No. 23 was used as an experimental seismograph until November, 1923. (See Pub. of the Dom. Obs., Vol. 7, No. 1.) Since then both instruments have recorded continuously, No. 17 as an east-west component and No. 23 as a north-south component. The records are also filed flat in the heavy envelopes. The constants are determined frequently and are always known. The time-system is the same as that used for the other seismographs of the Observatory and may be depended on, as recorded, to within a fifth of a second. The data to November 30, 1923, were entered in the record ledgers. Since then the entries have been on cards.

In each of the paragraphs referring to records, mention has been made of the fact that in November, 1923, a new system of recording was adopted. It is not necessary to describe this in detail here. It was published in the *Bulletin* of the Seismological Society of America, Vol. 14, No. 4, December, 1924, pp. 265-273, and in *Travaux Scientifiques,* Serie "A," Fascicule 2, pp. 89-120, of the Section of Seismology of the International Geodetic and Geophysical Union. The method adopted December 1, 1923, has been maintained rigorously ever since with but minor changes. It may be stated here that the system has been defined in minutest detail in a set of typewritten instructions covering over 200 pages, the whole fully covered by index and cross references. This insures uniformity of procedure, and a record of that procedure for future reference. All changes when adopted are at once reduced to writing and form part of the supplementary pages of the organization instructions. It was intended that this outline should be published, but this has been abandoned because of the rather obvious reason that it can serve well only the station for which it was designed. It should be pointed out, however, that this reduction of routine to writing cost a great deal of time and effort but that it has already paid and will long continue to pay large dividends in efficiency of operation and the elimination of repetition. (It is proving very useful just now in the emergency created by a change in personnel in the case of the stenographer of this division.)

The change in the system most worthy of note is that in the method of filing the record cards. The change was effected, beginning January 1, 1926. A whole month is now represented by a series of cards (one to each quake as before) filed vertically on end in a single drawer of a specially constructed cabinet. Tabs insure that a 3/10-inch lap reveals the index-edge of each card, front and back. Any card or group of cards may be removed at will for the entering of data; the whole month's records may be carried to a desk by simply pulling out the drawer, which, standing upright, forms a holder for the cards, preserving the index-edges in plain view.

Besides the seismograms, the daily records of the anemograph, the microbarograph, and the thermograph are preserved for reference. The weather maps, supplied by the Meteorological Service, are also preserved on file. An interesting correlation of the records from the Milne-Shaw seismographs and from the thermograph with the daily weather maps, all laid out in order and photographed, one week at a time, for the year 1926, is awaiting careful study for the possibilities which may lie therein as regards the phenomena of microseisms.

Supplementing the records of the instruments and the data entries in the ledgers or on the cards now used, monthly seismological bulletins have been issued since April 1, 1908. A complete set of these covers the time from then until now. Every earthquake, of which a trace appeared on an Ottawa seismogram, received a serial number and was reported by bulletin entry. Up to April 20 of this year a total of 2783 quakes had been recorded and their seismograms read for the bulletin. These are, of course, all on file.

Bulletins are also received at Ottawa from other seismological stations. A special file preserves these for use in the location of epicenters. Some of the reports from other stations date back to 1903; the file is fairly complete after 1908; practically all that were received at Ottawa since 1914 are there available. These records are of value for reference. They are carefully indexed by stations and by years and are stored in heavy manila envelopes in a convenient file. They serve, moreover, to remind us of the very excellent work which has been done in past years and which compares favorably with the best of our programs of today.

A most important part of the records assembled at the Observatory is represented by its library. Over 15,000 volumes are to be found there, about a quarter of which may be considered as dealing with geophysics. These are being regularly augmented by fresh purchases of new books required in the various divisions. Practically all geophysical (and astronomical) publications of value (in English, French, and German), and certainly all that have been brought under review and found of value in the work there, are subscribed for regularly. Besides these paid subscriptions there is a very considerable exchange-list. Nearly all the observatories in the world exchange publications with the Dominion Observatory. As the volumes of paid or exchange periodicals are completed they are bound, catalogued, and added to the library. The series of many of the periodicals extend back for one or two decades. The librarian, Mr. Labbe, has at his hand a great store of valuable records. The current material is kept fully up-to-date. All are fully catalogued and are instantly available.

The publications of the Dominion Observatory, dealing with seismology, have been written by Dr. Otto Klotz, by Ernest A. Hodgson, or by W. W. Doxsee. They have appeared, either in the series known as the *Publications of the Dominion Observatory* or in outside journals.

We may first consider the Publications list. At first the seismological numbers were scattered through the various volumes, along with the publications on astronomy and on other branches of geophysics. From now on seismological numbers will be confined to certain volumes. At present Volume 7 is being reserved, to be completed with such numbers only. The list of those numbers of the Publications dealing with seismology, to date, is as follows:

(1) Stereographic Projection Tables, Otto Klotz, LL. D. Pub. Dom. Obs., Vol. 1, No. 1 (1913).
(2) Earthquake of April 28, 1913, Otto Klotz, LL. D. Pub. Dom Obs., Vol 1, No. 5 (1913).
(3) Earthquake of February 10, 1914, Otto Klotz, LL. D. Pub. Dom. Obs., Vol. 3, No. 1 (1915).
(4) Seismological Tables, Otto Klotz, LL. D. Pub. Dom. Obs., Vol. 3, No. 2 (1916).
(5) The Effect of Cooling on a Cement Pier, Ernest A. Hodgson, M. A. Pub. Dom. Obs., Vol. 5, No. 2 (1921).
(6) The Location of Epicenters, 1917-18, Ernest A. Hodgson, M. A. Pub. Dom. Obs., Vol. 5, No. 4 (1921).
(7) The Location of Epicenters, 1919, W. W. Doxsee, M. A. Pub. Dom. Obs., Vol. 5, No. 9 (1922).
(8) The Location of Epicenters, 1920, W. W. Doxsee, M. A. Pub. Dom. Obs., Vol. 8, No. 2 (1922).
(9) Report of the Seismologic Division for 1923, Ernest A. Hodgson, M. A. Pub. Dom. Obs., Vol. 7, No. 1 (1925).
(10) The Location of Epicenters, 1921, W. W. Doxsee, M. A. Pub. Dom. Obs., Vol. 7, No. 2 (1925).
(11) The Location of Epicenters, 1922, W. W. Doxsee, M. A. Pub. Dom. Obs., Vol. 7, No. 3 (1926).

In No. 9 of this list (the Report of the Seismologic Division for 1923), pp. 55-57, appears a complete list of all the papers of the three authors published in outside journals up to January 1, 1924. Dr. Klotz did not publish any papers later than the period covered by that list. Subsequent publications in outside journals by Messrs. Hodgson and Doxsee will be found listed in the *Bibliographical Bulletin* of the Eastern Section of the Seismological Society of America, in which bulletin all subsequent papers by these authors will be listed as published.

The *Bibliographical Bulletin* of the Eastern Section of the Seismological Society of America has just been mentioned. This is also, in part, a contribution to seismology by the Dominion Observatory. The publication is edited at that institution, which has also, so far, furnished the supplies and postage required. It is a branch of international seismological work which, it is felt, has an important part to play. Coöperation from authors in all parts of the world is being earnestly sought. The possibilities are great. It is hoped that the bulletin will receive the support of all interested in seismology or its applications, not only in the way of subscriptions, but also in the reporting of papers which should receive mention therein. Brief reviews of the papers reported add greatly to the value of such entries.

The seismological work of the Dominion Observatory is, at present, carried on under the general supervision of R. Meldrum Stewart, the Director, by the following staff: Seismologist, Ernest A. Hodgson; Assistant Seismologist, W. W. Doxsee; Stenographer, Miss A. I. Dearth. It may be dealt with as (1) routine, (2) research.

The present routine is defined very rigidly by the typed outline previously mentioned. For present purposes it may be classified as (*a*) seismic records, (*b*) locations of epicenters, or (*c*) publications to be completed. These will be considered in the order named.

As regards seismic records it may be said that the various seismographs (the Bosch, two components, photographic; the Wiechert vertical, mechanical registration; the Milne-Shaw, two components, photographic, and an obsolete earth-tide instrument known as the "Deformation Instrument" and run for experimental and check-purposes) are maintained in constant operation. Since December 1, 1923, all of these instruments have been carefully checked for constants at frequent intervals and the results published in the monthly bulletins. The time-service of the Observatory furnishes minute-to-minute signals, correct to within a fifth of a second. Seismological bulletins giving a report of the readings, made from the seismograms registered, are mailed to over two hundred outside seismological stations within the first few days of every month, covering the records for the month immediately prior. This part of the work has been directly under the care of Mr. Doxsee since September 1, 1925.

The second division of the routine is that of the preparation of a series of publications on the location of epicenters. The series was begun in 1911 by Dr. Klotz, carried from 1914 to 1919 by Mr. Hodgson, and has since been in care of Mr. Doxsee. The method of stereographic projection has been used throughout. Every earthquake of which a trace was recorded at Ottawa has been included in the investigations. From 1911 to 1922 an extended list of the data collected was published, even in cases where no epicenter was determined. Beginning with the Location of Epicenters for 1923—now in preparation—the practice will be to preserve all the data received but to include in the analysis only those which yield an unquestioned identification of at least one principal phase. This relieves the pressure of work due to the entering of a great number of data which are never to be used; but it does not appreciably reduce the value of the resulting publication.

Of the publications to be completed at the present time, that which is most in arrears is the series of reports of the seismological division for 1924, 1925, and 1926. Most of the material, to date, has been prepared, but opportunity for completing the work has been lacking. The issue for 1923 was the first of a new series of yearly publications on the work of the seismological division. It is hoped that this may be brought up to date in the near future. Another publication most pressing for completion is that of the St. Lawrence earthquake investigation. There are already some four hundred pages of notes carefully typed out on forms to facilitate analysis. The whole will run to at least five hundred pages of such notes. This material must be analyzed and reduced as far as possible to essentials for the publication proper. The notes must be in good form, however, for preservation and future reference. The work of investigation is practically complete with the exception of one item (the evidence furnished by the tide-records taken in 1926 as to any changes in level just below the mouth of Riviere Ouelle, subsequent to 1900). This work is now being completed by the Tidal Survey of Canada. Besides these publications which are to be completed, the *Bibliographical Bulletin* requires regular attention five times a year. It involves a considerable amount of reading and also much clerical work, which, however, have their compensations in the regular work of the division. It is essential that this publication appear regularly and promptly. A system has been developed whereby much of the work done incidentally from day to day may be automatically made a part of the set task of preparing this publication. Thus when it comes time to get out another issue much of the material is already in shape.

The research programme in contemplation at the present time involves the installation of two single-component Wood-Anderson seismographs in or near the seismic areas of Quebec. The instruments have been purchased and have been delivered to the Observatory where they are now being given a series of tests. The investigation in Quebec is a cooperative one, the general direction of which is being undertaken by the Dominion Observatory. During the summer of 1926 the Geodetic Survey of Canada undertook the work of placing a triangulation-net and a series of bench-marks in each of the two areas—one on the St. Maurice River, near Shawinigan Falls, the other on the Ste. Anne River near Seven Falls (the Ste. Anne River flows south into the St. Lawrence near Beaupre, P. Q., about twenty-five miles below the city of Quebec). That work was completed during the season of 1926.

The Topographical Survey of Canada had already begun a map of the St. Maurice area; they hope to issue this in the near future. They have undertaken also to make a special map of the area in the region of the Ste. Anne River. The aeroplane work was completed during the winter. The topography shown in these air-pictures of the snow-clad valley is truly remarkable. Anyone familiar with the cadastral plans of that part of Quebec knows of the long, narrow farms, each abutting on a river or a roadway. To look on one of the aeroplane-maps is to see a copy of the cadastral plan traced out faithfully by the fences and hedges with the added feature of visible relief.

The Geological Survey of Canada arranged a reconnaissance of each of the areas by Dr. J. W. Goldthwait, of Dartmouth College, Hanover, N. H. This was carried out during the summer of 1926. It is Dr. Goldthwait's opinion that neither of the vault sites chosen is near fault-lines which give evidence of having been active within recent geologic time. They are, however, close enough to faults which are active, that records may be obtained to advantage on the sensitive Wood-Anderson seismographs.

Commercial interests in the province have agreed to cooperate in this investigation. They have attended to the construction of the vaults. One of these is now ready for the seismograph and the other will be ready soon. They leave nothing to be desired as sites for seismographs. Time will be recorded by means of box chronometers manufactured by Nardin. These are now undergoing rating tests at the Observatory. They will be checked, when on location, by means of daily signals recorded on the seismogram by the observer listening in on the wireless signals.

The work of installing the instruments will begin as soon as they have been checked at Ottawa. This should be within the next few weeks. The records are being awaited with considerable interest.

As soon as the Wood-Anderson instruments are in place the Milne-Shaw will be removed from Ste. Anne de la Pocatiere. An attempt will be made to use it during the summer to test out the effect of wind and of traffic on some of the larger buildings which are used for triangulation-stations in the city surveys. This work is being undertaken at the request of the Geodetic Survey.

Incidentally it is hoped that further data will be secured as to the cooling effects in the case of a relatively high building which result when a hot afternoon is followed by a cool evening. A paper given by the author before this Section of Seismology at the meeting last year indicated the curious tilting oscillations which seem to be the result of such thermometric changes. The paper of last year referred to these oscillations in the case of a rock mass. Presumably the same phenomenon will be found in the case of buildings.

Having, then, outlined the records on hand, the papers published, the routine and the prospective research, nothing remains except to repeat that those responsible for the seismologic work of the Dominion Observatory are most anxious to place their resources, as far as possible, at the disposal of those in a position to make use of them, and are more than ready to receive from their co-workers in the science suggestions of such changes as may be considered advantageous to the work as a whole.

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SCIENTIFIC PAPERS

SEISMIC WAVES AND THE SURFACE-LAYERS OF THE EARTH

PERRY BYERLY

In California at present the lack of seismographic stations sufficiently closely spaced forbids the drawing of travel time-curves for local earthquakes and thus allows no direct conclusions to be drawn as to the structure of the surface-layers of the Earth. The most that can be done is to examine our local earthquake records from the one or two stations at which they are recorded, and to compare the actual wave-groups registered with those computed on the bases of structures of the surface-layers as concluded for Europe by seismologists there.

For certain California earthquakes, the travel time-tables of A. Mohorovičić, based on a discontinuity at depth fifty-seven kilometers, have offered a satisfactory solution, i. e., one which agreed with field evidence.[1] But certain nearby earthquake records offer a complexity of early arrivals which cannot be explained by these tables.

The recent work of Harold Jeffreys[2] offers another solution of European surface-structure, and it is worth while to see if such a structure will explain our complex records in California. It is noteworthy that Jeffrey's interpretation calls for simpler records for earthquakes at depths greater than about thirty-five or forty kilometers than for earthquakes of higher focus.

On April 27, 1924, a quarry at Corona, California, fired a large charge of dynamite. This caused an earthquake which registered at the Pasadena seismographic station. The exact time of occurrence of this earthquake was observed by Mr. Harry O. Wood and his associates, and a good record of it was obtained by Mr. Wood at his Pasadena station, a distance of 67.5 kilometers from the focus. Thus we have a California earthquake of which the focus and time of occurrence are known, but of which we have only one record. Mr. Wood and the writer have studied the record in some detail, and with Mr. Wood's permission it is discussed briefly here.

An interesting comparison to this seismogram is offered by the record obtained at Tacubaya of the earthquake which accompanied the eruption of Popocatapetl on October 16, 1920.[3] The distance to the station from the crater was 72.4 kilometers. The similarity of the two records is marked. On both records there are a number of marked new arrivals between the beginning of the record and the arrival of the maximum group. The tables of Mohorovičić account for no such groups.

The first arrival is very weak at Tacubaya and this group did not register on the Pasadena seismogram. The *iM* on the Pasadena record satisfies the S_g travel-time of Jeffreys, velocity 3.2 kilometers per second (where S_g represents the transverse wave which has travelled its whole path in the granite layer). The interval between the beginning of the Tacubaya record and its *iM* satisfies Jeffreys' (S_g—P_g) time.

The first arrival on the Pasadena record and the second on the Tacubaya record have the travel-times of Jeffreys' *P* which has been twice refracted at the base of the granitic layer (or perhaps reflected at this surface, since at this distance the travel-times for these two cases are nearly the same), if the granitic layer has a thickness of fifteen kilometers and Jeffreys' velocities in the two upper layers hold. No *S* which corresponds to this strictly is found, i. e., none which shows a travel-time nearly 1.7 times that of this group.

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[1] Bull. Seis. Soc. Amer., 16: (1) 1, March, 1926.

[2] London, Mon. Nat. R. Astro. Soc., Geophysical Supplement, 1: (8) 385, Dec., 1926.

[3] H. CAMACHO, Anales del Instituto Geologico de Mexico, 2: Nos. 1, 2, and 3, 1925.

The next arrivals are larger than the previous groups, and therefore it is difficult to attribute them to a *P* which has been twice refracted at the base of the second layer, and their travel-times might be explained as such only if the thickness of this basaltic layer were fifteen kilometers, Jeffreys' velocities being assumed, whereas Jeffreys finds a thickness of twenty-five kilometers.

One of the large late arrivals might be explained as having travelled through the upper or sedimentary layer.

However, no definite conclusions can be drawn save that between the beginning of the records and the arrival of the maximum there are a number of marked new arrivals which are not called for by the tables based on a single discontinuity at a depth of fifty-seven kilometers. And the large amount of energy in some of these is difficult to explain on the assumption that they are *P*-waves travelling through Jeffreys' layers, even though they be his diffracted waves.[4] It is interesting to note that the apparent surface-velocities of some of these groups are the same as those of certain groups of surface-waves for distant earthquakes.

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SOME NEW WAVE-GROUPS OBSERVED ON THE RECORDS OF THE SOUTH PACIFIC EARTHQUAKE OF JUNE 26, 1924

James B. Macelwane, S. J.

(Abstract)

The study of the records of the earthquake of June 26, 1924, which occurred in the south Pacific Ocean, has revealed a large number of groups of waves which cannot be accounted for by the tables of travel-times now in use. Hence it was found necessary to calculate the time required to traverse other possible paths. This was done by the Wiechert-Herglotz method using much of the data published by Geiger and Gutenberg and applying Snell's law. The new groups are P_cS, P_cSS_cP,

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[4] Cambridge, Proc. Phil. Soc., 23: (4) 472-481, Oct. 1926.

PPPS, PPSS, PSSS, PPPS', PPSS', and *PSSS'*. These waves are served on many records. A few examples are here given.

Station	Δ°	New phases observed
Sydney	23.3	P_cP
Apia	49.8	P_cP, P_cS, S_cS
Batavia	63.4	P_cP, S_cS
Honolulu	87.3	P_cP, S_cS, P_cSS_cP (late), *PPPS, SPS*
Hyderabad	98.2	P_cSS_cP, *PPPS, PPSS, PPSS'*
Victoria	124.8	*PPPS, SPS, PPSS*
Cartuja	152.9	*PPPS, SPS, PPSS, PSSS*
Vienna	154.2	*PPSS, PPSS'*
Munich	156.8	*SPS, PSSS*
Zurich	157.9	*PSSS, PPSS, PPSS'*
Strasbourg	159.1	*PPPS, SPS, PPSS, PSSS, PPSS'*
Feldberg	159.6	*SPS, PPSS*
Paris	162.0	*SPS, PPSS, PPSS'*
Uccle	162.1	*SPS, PPSS, PPSS'*

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SIMPLE APPARATUS FOR SEISMIC MEASUREMENTS

Paul Kirkpatrick

The failure of the falling columns of Mallet, West, Milne, Omori, and Galitzin to attain to the dignity of precise seismometrical instruments appears to have been due to the fact that the true relation connecting the constants of the columns with the constants of the earth-motion necessary and sufficient for their overthrow was not known to these workers. This ignorance is not surprising since the columns used were subject to resonant disturbance by earth-vibrations of certain frequencies, and hence the mechanical problem involved was very complicated. The present paper describes a type of column, or as we shall say, a bar, which is braced at one side and suitably fitted with a bearing so that it is free to fall in one direction only. This arrangement has been found to eliminate all resonant vibration of the column and greatly to simplify the theoretical statement of the conditions governing its downfall.

By making the assumptions that the motion of the quake when the bar falls is simple harmonic, and that the bar in its erect position is not far from vertical it has been possible to obtain a complete solution of the problem of the motion of the bar. This solution is presented in a supplement attached to this paper so that the development may be consulted in its entirety if desired, but the only portion of which we have need at present is the conclusion that a bar will just be thrown down by a horizontal oscillation lying in its plane of rotation if the maximum acceleration of the motion has the value

$$a = g\,\theta \sqrt{1 + \frac{4\pi^2 R^2}{g L T^2}}$$

In this equation R is the radius of gyration of the bar, L is the distance from its center of mass to its axis of rotation, and T is the period of the oscillation. The angle through which the erect bar must be rotated before coming to the position of unstable equilibrium is ϕ.

This equation is substantiated not only by a rigorous derivation but also by an exhaustive experimental test. An oscillating table capable of horizontal simple harmonic motion of adjustable period and amplitude was employed. Every variable quantity represented in the above equation was varied widely, and of over five hundred observations every one conformed to the equation within the leeway of two or three per cent which the uncertainties of observation permitted.

If we consider a family of bars of similar construction and with parallel axes of rotation but differing in their angles of inclination it is evident that a quake of moderate intensity will overturn those bars which are inclined at the smaller angles, leaving the remainder standing. The critical value of ϕ which divides the fallen from the standing bars can therefore be determined with a precision depending only upon the number of bars used. But the equation shows that a knowledge of this critical angle will not, in general, suffice to determine the maximum acceleration, since this acceleration is also a function of the period of oscillation. The influence of the period can be suppressed only by making the whole second term under the radical sign a small quantity. Experiment shows that this may be done by a suitable choice of the form and dimensions of the bar, so successfully, at least, that for all oscillations with periods greater than one-fourth of a second the maximum acceleration is fully determined by the critical angle, with an error which can not be greater than six per cent and which is much less in nearly all cases. It is probably not worth while at present to strive for greater accuracy at this point, since the errors resulting from the assumption that the earth-motion is simple harmonic have not been investigated.

Further information may be derived from the bars. Referring again to the equation, one sees that a knowledge of the critical angle, as obtained by observing a given set of bars, gives an equation in which maximum acceleration and period alone are unknown. From a second set of bars, different from the first in their dimensions, another critical angle is observed, giving another relation between acceleration and period. The two equations are independent and together suffice to determine both the acceleration and the period. From these quantities the approximate amplitude is readily computed. These determinations are subject to three errors only. The first is the error in determining the critical angles; it may be made as small as desired by the use of a larger number of bars. The second is the error involved in the assumption that θ is small; this is entirely negligible for quakes of small intensity and rises only to about one per cent with very disastrous quakes. The third error is that of the assumption that near the time of maximum acceleration the earth-motion is simple harmonic. Assumptions of this kind have often been made, but never, so far as the writer knows, critically investigated. It is impossible at the present time to make a numerical statement of the exactitude of this assumption, but its validity becomes of less and less importance as the radius of gyration of the bars is diminished. A personal view is that errors of this sort can be kept below five per cent in the acceleration-determinations.

The bars which are now being used are of an inverted *T*-shape (see Figure 1). Nearly all of the mass is in the crosspiece, which is of lead. Near the ends of the crosspiece steel cups, taken from watch bearings, are located, and these cups rest over vertical steel needles fixed in a horizontal base-board. The upright member of the inverted *T* leans against an adjustable stop, which controls the magnitude of θ. The upright is two or three inches in length and the horizontal member about four. The center of mass of the system is perhaps two millimeters above the points of the supporting needles, giving the small radius of gyration required by theory. The bars are readily reset by tilting the entire base-board. No attempt has been made to determine the utmost sensitivity which is possible with such a bar but it has been shown repeatedly that shocks well below the threshold of perceptibility are dependably registered.

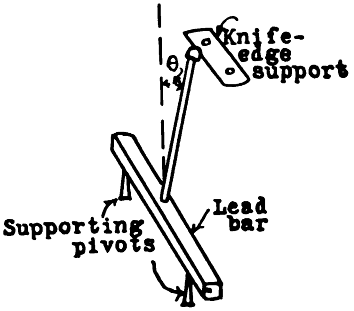

FIG. 1

Schematic diagram of simple apparatus for seismic measurements.

MATHEMATICAL SUPPLEMENT

Consider a rigid bar of mass m, and of any shape, such as the rod of Figure 2. An axis of rotation is provided at x, horizontal, and perpendicular to the plane of the diagram. The distance from x to the center of mass of the bar will be designated by L, and the angle between the direction of L and the vertical by θ. Inclination to the right indicates a positive θ.

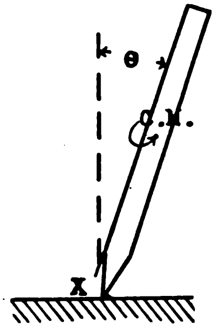

Fig. 2

Diagram for development of mathematical theory.

Let the axis undergo a horizontal acceleration, a, toward the right. This will produce a counter-clockwise inertial torque. Considering also the clockwise gravitational torque we have as the resultant (positive when counter-clockwise) torque

$$Q = m\,a\,L \cos\theta - m\,g\,L \sin\theta \qquad (1)$$

Restricting θ to small values this may be written

$$Q = m\,a\,L - m\,g\,L\,\theta \qquad (2)$$

Seismic accelerations are oscillatory in character, and more or less similar to the accelerations of simple harmonic motion. We shall investigate the behavior of the bar when subjected to horizontal simple harmonic motion in the plane of the diagram, of period T, and amplitude r. The acceleration of such motion is

$$\frac{4\,\pi^2 r}{T^2} \sin\left(\alpha + \frac{2\,\pi t}{T}\right)$$

Equation (2) then becomes

$$Q = \frac{4\,\pi^2\,m\,L\,r}{T^2} \sin\left(\alpha + \frac{2\,\pi t}{T}\right) - m g\,L\,\theta \qquad (3)$$

Equating this torque to the rate of change of angular momentum we obtain the differential equation of motion of the bar

$$\frac{d^2\theta}{dt^2} - \frac{m g\,L\,\theta}{I} = \frac{-\,4\,\pi^2\,m\,L\,r}{I\,T^2} \sin\left(\alpha + \frac{2\,\pi t}{T}\right) \qquad (4)$$

Introducing the constants A, B, and K, this becomes

$$\frac{d^2\theta}{dt^2} - A\,\theta + B \sin(\alpha + K\,t) = 0$$

The general solution of Equation (4) is found to be

$$\theta = C_1 e^{-\sqrt{A}t} + C_2 e^{\sqrt{A}t} + \frac{B}{K^2 + A} \sin(\alpha + K\,t) \qquad (5)$$

The case which is of interest in this investigation is that in which the bar is initially at rest at some particular angle, θ', at the instant when $t = 0$. These conditions make possible the determination of the constants in equation (5). They are

$$C_1 = \frac{B}{2\,(K^2 + A)}\left(\frac{K}{\sqrt{A}}\cos\alpha - \sin\alpha\right) + \frac{\theta'}{2}$$

$$C_2 = \frac{-B}{2\,(K^2 + A)}\left(\frac{K}{\sqrt{A}}\cos\alpha + \sin\alpha\right) + \frac{\theta'}{2}$$

Inserting these values we obtain

$$\left.\begin{aligned} \theta = {} & \left[\frac{B}{2\,(K^2 + A)}\left(\frac{K}{\sqrt{A}}\cos\alpha - \sin\alpha\right) + \frac{\theta'}{2}\right] e^{-\sqrt{A}t} \\ & + \left[\frac{-B}{2\,(K^2 + A)}\left(\frac{K}{\sqrt{A}}\cos\alpha + \sin\alpha\right) + \frac{\theta'}{2}\right] e^{-\sqrt{A}t} \\ & + \frac{B}{K^2 + A}\sin(\alpha + Kt) \end{aligned}\right\} \qquad (6)$$

It will be found convenient to expand equation (6) and recombine it into the following form

$$\left.\begin{aligned} \frac{K^2 + A}{B}\,\theta = {} & \left(\frac{K^2 + A}{B}\,\theta' - \sin\alpha\right)\cosh\sqrt{A}t \\ & - \frac{K}{\sqrt{A}}\cos\alpha\sinh\sqrt{A}t + \sin(\alpha + Kt) \end{aligned}\right\} \qquad (7)$$

The conditions described above are physically realized by allowing the upper end of the bar to rest against a stop, the position of which may be varied if it is desired to change θ'. If this system be subjected to horizontal simple harmonic motion as described above, the bar will not rotate or rise from its stop unless Q attains positive values at some time in the cycle, that is, unless the acceleration at some time exceeds $g \tan \theta'$, or for small angles simple $g\,\theta'$. If the maximum acceleration slightly exceeds this threshold value, the bar will rise from its supporting stop but will drop back as the acceleration decreases again, without reaching its vertical position. If the acceleration be still further increased, however, the bar will eventually be carried over the vertical position and will fall down on the other side. The critical acceleration sufficient to do this is a function of θ', of T and r, and of the constants of the bar. We proceed to find its value.

The downfall of the bar means that θ passes permanently to negative values. Since the sine term in equation (7) merely oscillates about zero it is required that the sum of the first two terms of the right member shall remain negative for all large values of t. The hyperbolic functions approach equality, so that after t has become large these first two terms may be replaced by

$$\left[\frac{K^2+A}{B}\theta' - \sin\alpha - \frac{K}{\sqrt{A}}\cos\alpha\right]\sinh\sqrt{A}t$$

Since the hyperbolic sine is always positive its coefficient must become negative if the bar is to upset, the threshold of instability being defined by the condition

$$\frac{K^2+A}{B}\theta' - \sin\alpha - \frac{K}{\sqrt{A}}\cos\alpha = 0 \tag{8}$$

It has been stated that the bar first begins to rise when the acceleration attains the value $g\,\theta'$. This is stated thus:

$$\frac{4\pi^2 r}{T^2}\sin\left(\alpha + \frac{2\pi t}{T}\right) = g\,\theta'$$

But at this instant $t = 0$, so we have a as defining equation for sin α

$$\frac{4\pi^2 r}{T^2}\sin\alpha = g\,\theta'$$

or

$$\sin\alpha = \frac{g\,T^2\theta'}{4\pi^2 r}$$

and

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1 - \frac{g^2\,T^4\,\theta'^2}{16\,\pi^4 r^2}} \tag{9}$$

Inserting these values in (8), replacing the values of A, B, and K, and simplifying

$$\theta' = \frac{4\pi^2 r}{g}\sqrt{\frac{m\,l}{m\,l\,T^4 + 4\pi^2 I\,T^2/g}} \tag{10}$$

or

$$\frac{4\pi^2 r}{T^2} = \theta'\sqrt{g^2 + \frac{4\pi^2 I g}{m\,L\,T^2}} \tag{11}$$

Therefore maximum acceleration, where R is the radius of gyration, is

$$\theta'\sqrt{g^2 + \frac{4\pi^2 I g}{m\,L\,T^2}} = g\,\theta\sqrt{1 + \frac{4\pi^2 R^2}{g\,L\,T^2}}$$

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SECTION OF METEOROLOGY

SYMPOSIUM ON NEEDS AND POSSIBILITIES OF MEASUREMENTS OF ULTRA-VIOLET LIGHT IN SOLAR SPECTRUM AND OF THE OZONE-CONTENT OF THE HIGH ATMOSPHERE

DESIRABILITY OF SUCH MEASUREMENTS

C. F. Marvin

In opening this discussion upon the subject of ultra-violet radiation, it is unnecessary for me to elaborate upon the desirability of securing quantitative observations of such radiation and of pursuing investigations thereof. During the past fifty years great progress has been made in collecting exact observations of the total radiation received from the Sun, and the time is now at hand when we need to know more about the *quality and composition* of this radiation. How much of it is infra-red, how much visible and luminous, and how much ultra-violet, and so forth. What are the effects of rays of particular classes on human affairs? How do they vary from season to season, year to year, and place to place? These are questions of importance, not alone to the meteorologist but to the biologist, the botanist, and the plant physiologist; to the medical profession; to those interested in hygiene, sanitation, and public health; and to the industries as well.

The Weather Bureau is drawn into the question, both by its own personal interest and also as a public agency to which all those interested apply for observational data and statistics. As a matter of fact, sufficiently simple and dependable instruments are not yet available by which reliable and comparable observations can be secured. The methods and apparatus for making such observations are still in the laboratory stage of evolution.

One purpose of this symposium is to bring out a full discussion of what we may call the present laboratory state of the science in order that the instrument-makers and designers may go to work intelligently to produce instruments which can be safely put into the hands of the average observers with a reasonable assurance of accurate, comparable observations.

One of several proposals that have been made is that certain standard dyes contained in quartz tubes and exposed to sunshine for definite times will serve to show ultra-violet intensities by the amount of fading of the dye. The question of purity, permanence, and comparability comes in here to such a high degree as to make simple methods of this kind too uncertain to be acceptable.

Great interest will be awakened by the valuable papers of this symposium and the important contributions they will make to the theoretical side of the question. The practical problem back of all is a standard type of instrument capable of analyzing solar radiation into certain major parts which can be measured in appropriate standard units of some kind and by means of which uniform observations can be secured from many widely distributed stations.

U. S. Weather Bureau,
Washington, D. C.

OZONE IN THE UPPER AIR

W. J. Humphreys

It appears to be well established, and now generally accepted, that there is an appreciable amount of ozone in the upper air and practically none at all in the lower. The whole of this ozone, seemingly variable in quantity, would make a layer only three to four millimeters thick at standard temperature and pressure. That is, it amounts to only one part, roughly, by weight, in two million of the whole atmosphere. With the exception of xenon it is the rarest of all aerial gases, yet one of the most important. Its importance is owing to its power of absorption of radiation and to its selectivity in respect to the wave-lengths absorbed.

It is well known that diatomic oxygen rather strongly absorbs radiation of wave-length round about 1800 Ångström units, and that some of this oxygen is thereby transformed into the triatomic variety, or ozone. Now, ozone has a broad band of metallic-like absorption centered at about wave-length 2500, and extending more and more feebly several hundred units on either side. In consequence of this absorption the ozone is reverted to ordinary or diatomic oxygen. Presumably, therefore, the relative amounts of ordinary oxygen and ozones sixty and seventy kilometers above the surface are, to some extent, at least, determined by the relative intensities of these two portions of the solar spectrum, the ozonizing and the deozonizing. At the base of the stratosphere, however, or, rather, at the top of the troposphere, the ozone presumably quickly reverts to ordinary oxygen owing to the presence there of an appreciable amount of water vapor. Apparently, then, we may assume, as a crude first approximation, that oxygen is being converted into ozone and ozone converted to oxygen at some definite level, say sixty kilometers above the surface of the Earth; that all the ozone that reaches the top of the troposphere is there quickly reverted to oxygen, and that ozone is neither formed nor destroyed, as such, to any appreciable extent, at any other levels. To the extent that this assumption is true there must be a continuous circulation of oxygen and ozone between these two levels, oxygen up and ozone down, with the excess of production over destruction of ozone at the upper level equal to its destruction at the lower. Furthermore, since the total amount of the ozone present at any time between its upper and lower levels is very small in comparison with the other gases present, say one part in 500,000, it follows that the oxygen-ozone circulation must be very slow, even approaching ordinary diffusion, and the density distribution of the ozone roughly that which it would be if determined by gravity alone.

From these considerations one would infer that when the base of the stratosphere is highest, that is, when the center of an anticyclone is passing, there will be the least amount of ozone overhead; and the most when the stratosphere is lowest, or when a cyclone is passing. Besides, as a lifting of this base by only three kilometers, or, say, from nine to twelve kilometers above the surface, reduces the pressure at that base by fully one-third, and as the lifting of this base from cyclone to anticyclone occasionally is even more than here assumed, it follows that we may expect the ozone to be much greater over cyclones than over anticyclones. And that is just what observations show. We should also expect that the amount of ozone would be less in the upper air during summer and autumn, when the base of the stratosphere averages high, than during winter and spring, when, on the average, it is lower. This, too, is supported by observations. Finally, we should expect the ozone over middle and higher latitudes to be at least twice as great as it is near the equator, owing to the corresponding five or six kilometers difference in the level of the base of the stratosphere. This inference has not yet been checked by observations.

The above assumption that the ozone is distributed in the stratosphere substantially as determined by gravity, and therefore mainly in the lower portion of this region, is not in accord with certain observations that indicate its level to be forty-five to fifty kilometers above the Earth. However, these observations involve such uncertainties as might well call for a suspension of judgment. Besides, if the ozone is mainly at these high levels it is not obvious how the quantity of it could be affected by the height of the barometer or time of the year, relations that appear to be easier to determine, and perhaps therefore more reliable, than the height at which the ozone chiefly occurs. At any rate, the assumption of a gravity distribution of the ozone in the lower fifty kilometers, or thereabouts, of the stratosphere, is still allowable, it is believed, and especially so since it suggests definite test-observations—measurements at various latitudes, and in conjunction with air-soundings that give the height of the base of the stratosphere.

It is practically certain that the ultra-violet limit of the solar spectrum as observed at the surface of the Earth is fixed by the ozone of the stratosphere. If there were no ozone in the atmosphere, the flood of ultra-violet radiation, at least down to 2,000 Ångström units, that would get through to the surface of the Earth would be destructive to many organisms, as now constituted. On the other hand, if the great ultra-violet absorption-band of ozone were shifted a little, say 500 Ångströms, towards the visible, seeing, so long as there were eyes to see, would nowise be affected, but it might well be that under these conditions, the total absence of all anti-ricketic radiation, the human race, and many others, would soon disappear.

In still another, but less important way, the ozone of the upper air affects the surface of the Earth. It has powerful absorption-bands in the far ultra-red, where Earth-radiation is strong. It therefore keeps the surface of the Earth at a slightly higher temperature than it otherwise would have.

Clearly, therefore, the ozone of the atmosphere, though small in amount, is very important, and its accurate measurement at various latitudes and under different conditions a matter of considerable scientific interest. Such measurements, already begun in a few places, are made with a suitable spectroscope, and a hopeful approach to accuracy and standardization appears to have been attained.

It is reasonably certain, of course, that the amount of ozone in the upper air reflects in some measure the quantity, and quality, of solar radiation, but it is so very sensitive to barometric changes (really, we believe, to the height of the base of the stratosphere) that it will be very difficult to separate with certainty the purely solar effects, at least those of brief duration, from those due to other causes. However, this is no excuse for minimizing efforts, but rather a good reason for making the measurements of the ozone in the upper atmosphere as highly accurate as possible, and as abundant as practicable, consistent with this indispensable precision.

LITERATURE

There is a considerable literature on the absorption of radiation by ozone, and its amount and distribution in the atmosphere. The following few papers, however, with the references they in turn cite, cover the subject fairly well:

LADENBURG AND LEHMANN. Ann. der Phys., 21:305 (1906).

CHARLES FABRY. Proc. Lond. Phys. Soc., 39:1 (1926).

Dobson, Harrison, and Lawrence. Proc. Roy. Soc., 114: 521 (1927).

Cabannes and Dufay. J. Physique, 8:125 (1927).

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Washington, D. C.

ULTRA-VIOLET SOLAR RADIATION

Edison Pettit

The work of Abbot and his colleagues[1] has shown that the radiation from the Sun varies and that the variation is most pronounced in the violet end of the spectrum. Whether we regard the variation as due to a change in temperature, a variable transmission of the outer solar envelope or a variable emissivity, this effect on the ultra-violet is to be expected. Dobson[2] in England using a photographic method, measured the ultra-violet solar radiation transmitted by silver-films on thirty-four days in 1922, and found variations as great as five hundred per cent, and a standard daily range of thirty per cent outside the atmosphere. It occurred to the writer that the uncertainties of photography could be eliminated by measuring the radiation directly with a thermo-couple; that disturbances of the atmosphere could be made differential by standardizing the ultra-violet with the green region of the spectrum which varies only slightly; that the accuracy could be increased by making the apparatus automatic, which would permit an increase in the number of observations, and that the combination of this procedure with the favorable atmospheric conditions on Mount Wilson might lead to results of some interest.

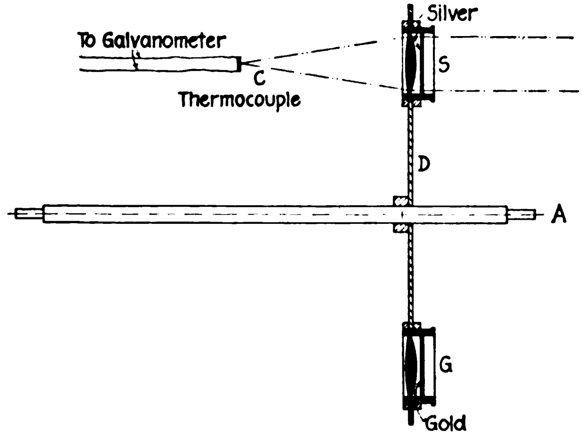

Fig. 1

Diagram of optical system of the ultra-violet solar radiometer.

This was accomplished by an apparatus constructed as follows: In Figure 1, *A* is a spindle carrying a disk, *D,* having two cells set diametrically opposite one another. Cell *S* contains a quartz lens and plate, both silvered chemically on their inner surfaces with two coats of silver. Cell *G* contains a similar quartz lens and plate, both coated on the inner surfaces with gold by sputtering. These are crossed lenses, giving mini-

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[1] Ann. Astroph. Obs., Smithsonian Inst.. 4: 206 (1922).

[2] Proc. R. Soc., London, 104: 252 (1923).

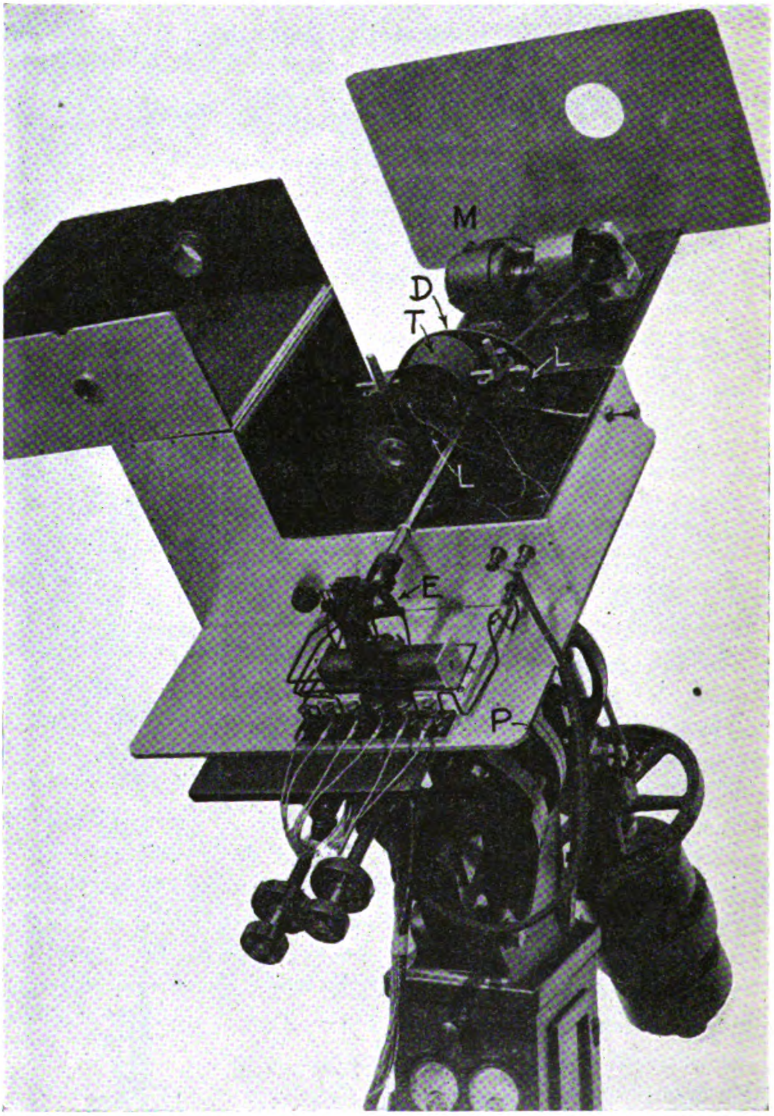

Fig. 2

Ultra-violet solar radiometer at Tucson.

mum spherical aberration, and are of one-inch aperture and two inches focal length. The separators are formed to fit the surfaces producing practically air-tight contacts. Diaphragms in contact with the plates are used to adjust the galvanometer-deflections in the apparatus on Mount Wilson, but not in the Tucson apparatus, about to be described. Exposed to the Sun, the lens forms an image of it, one-half millimeter in diameter, in ultra-violet light of wave-length 0.32 μ, on one junction of the compensated thermo-couple *C*. An escapement attached to the spindle *A*, operated electrically by a contact-clock, causes the image to fall first on one, then on the other junction of the compensated thermo-couple, then permits the disk to rotate a half revolution, and repeats the process substituting the gilded lens which transmits green radiation in wave-length 0.50 μ. The deflections thus produced in the galvanometer attached to the thermocouple by the ultra-violet and green radiations from the Sun are recorded photographically on a moving plate and determine the ratio ultra-violet to green radiation every four minutes.

Figure 2 shows a radiometer recently installed at the Desert Sanitarium of Southern Arizona at Tucson. In this instrument the thermocouple is in an air-cell having a large heat-capacity, and not in a vacuum as on Mount Wilson. This makes it possible to compare the galvanometer-deflections themselves, since the sensitivity is constant, there being no secular change due to state of vacuum. The motor *M*, working through a friction-disk exerts a continuous torque on the spindle which is permitted to rotate only at intervals of one minute when the escapement *E* is released by the magnets. An air-cell thermo-couple *T* having quartz windows and receivers one mm square is connected to a D'Arsonval galvanometer in the room below. *D* is the disk carrying the lenses *L*. A wire running over a pulley *P* on the polar axis has the moving plate attached to the other end upon which are registered photographically the deflections of the galvanometer.

Figure 3, upper part, shows reproductions of the galvanometer records for October 19 and 20, 1926. The inside rows of dots are produced in each case by the radiation transmitted by silver, $\lambda = 0.32\ \mu$; the outside rows are due to the radiation transmitted by gold, $\lambda = 0.50\ \mu$. The heavy lines are hour-angle marks, put on by the contact-clock, and indicate local solar time. It will be seen that the ultra-violet light is zero for some time after sunrise, reaching a maximum at noon, whereas the green radiation has a sensible value at sunrise and rapidly approaches its maximum. The lower part of Figure 3 shows a graphical solution for each of the plates. The logarithm of the ratio, ultra-violet to green radiation, is plotted against secant *Z* obtained from the hour-angle.

This plot is a straight line from which two values are taken, that for the zenith, secant $Z = 1$, and that for no atmosphere.

Figure 4 shows march of the monthly means of the ratio, ultra-violet to green, since May, 1924, compared with Dobson's and Harrison's measurements of ozone (below), Abbot's measurements of the solar constant, and the Mount Wilson daily sunspot group-numbers (above). Considerable correspondence will be noted in the general trend of the last three curves. The ultra-violet values are for no atmosphere, but practically the same curve would result if the values for one atmosphere were used, so nearly are the atmospheric disturbances eliminated.

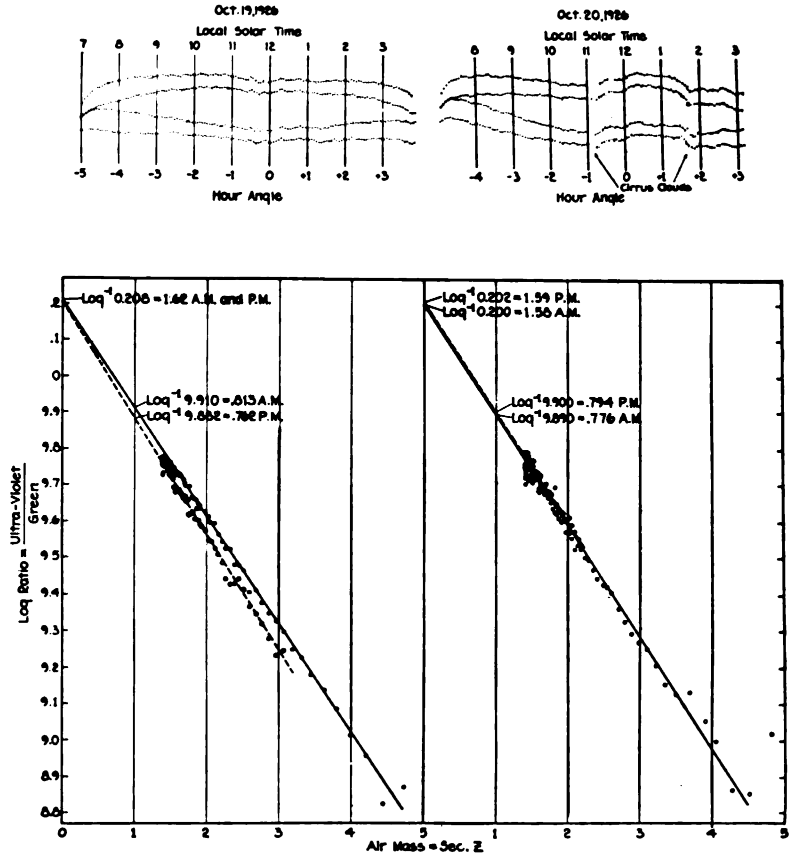

FIG. 3

Radiometer records and plots for October 19 and 20, 1926.
(At left with a typical day with afternoon run affected by haze and at right a day with no haze in the afternoon.)

The ozone-curve seems to run counter to the ultra-violet curve. The ozone-curve represents a seasonal variation, although it is difficult to see how the obvious explanation of changing solar declination can apply,

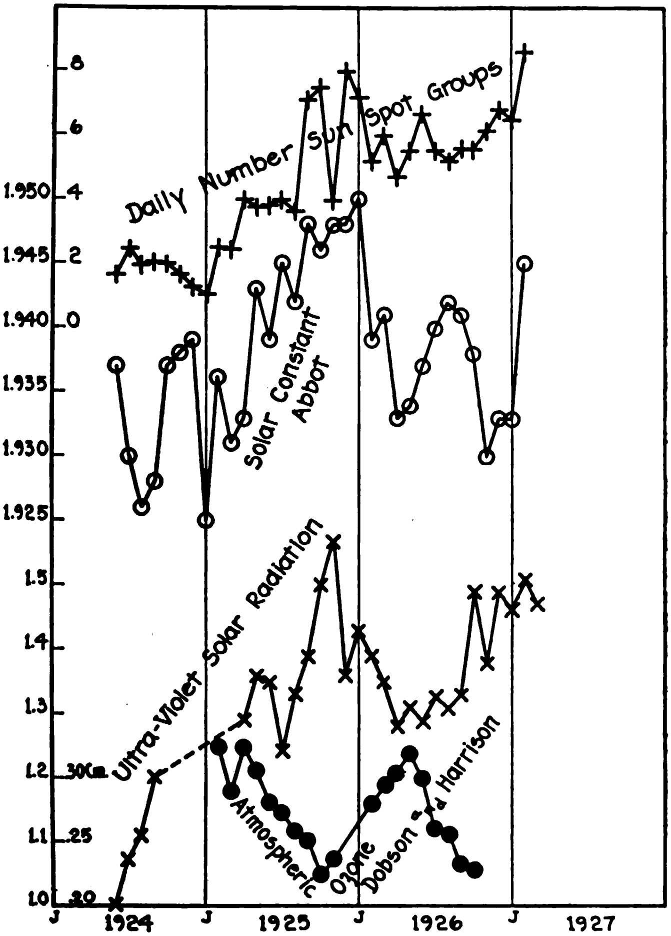

FIG. 4

March of the monthly means of atmospheric ozone, ultra-violet solar radiation, solar-constant, and sunspot group-numbers.

since the time of maximum ozone lags behind the time of maximum insolation, rather than the contrary, which we might expect. Probably it is too early to draw conclusions in this connection.

That variations in the ozone-content of the atmosphere do not sensibly affect these measurements of ultra-violet radiation was shown by placing a tube of ozone in the optical path of the instrument and varying the pressure in the tube. In this way it was shown that an increase of one hundred per cent in ozone decreases the ultra-violet measurements only five per cent, while complete elimination of ozone increases them only three per cent. That the transmissions of the filters are not affected by exposure to sunlight was shown by increasing the area of the diaphragms placed in front of the lenses, thus exposing previously unexposed portions of the filters after the apparatus had been in operation more than two years. The deflections increased in proportion to the increased area. In January of the present year the cells were taken apart and examined. No trace of tarnish or other defect was observed in the films. A careful determination of the transmission of both plates was made, and this will be repeated toward the end of the year to check further the stability of the metallic films.

TABLE 1

Station	H(cm)	E/E_0 Obs.	$E/E_0 = e^{-kH}$ Comp.		Air mass $=B/B_0$	Log E/E_0 Obs.
			Rayleigh	Cabannes Dufay		
Mt. Whitney (Smithsonian)...	350000	0.615	0.592	0.569	0.653	9.7889
Mt. Wilson.......	174200	0.480	0.519	0.489	0.812	9.6812
Tucson..........	76000	0.370	0.479	0.452	0.916	9.5682
Pasadena........	25800	0.325	0.466	0.440	0.969	9.5199
Oxford (Dobson)..	6400	0.315	0.452	0.427	0.993	9.4983
Sea-level.........	0	....	0.448	0.422	1.000	....

$$k = \frac{32\,\pi^3\,(u-1)^2}{3\,N\,\lambda^4} \quad \text{(Rayleigh)} = \frac{8\,\pi^3\,(u^2-1)^2}{3\,N\,\lambda^4}\left(\frac{6+3\,\rho}{6-7\,\rho}\right)$$

(Cabannes and Dufay)

Table 1 shows a comparison between the observed and computed atmospheric transmissions for several elevations for ultra-violet light of wave-length 0.32 μ. A very considerable disparity is noted as sea-level is approached. This is shown graphically in Figure 5. Here the straight lines are the theoretical vertical transmissions computed from Rayleigh's law of scattering and the modified formula of Cabannes and Dufay[3] which takes polarization into account. It will be noted that for air-masses less than 0.8 (which is equivalent to an altitude of 6,000 feet) the transmission is greater than that called for by the theory, and at lower altitudes it is much less, being only 0.70 as great near sea-level.

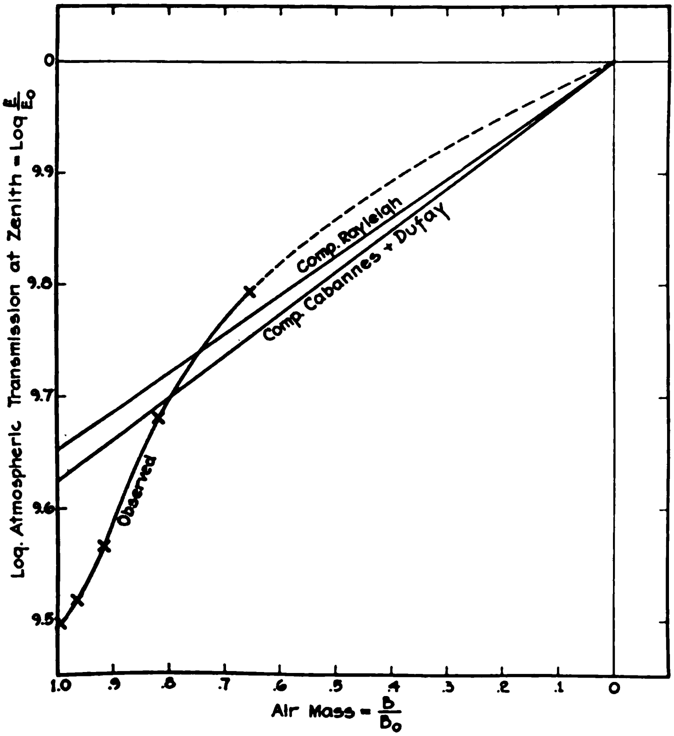

FIG. 5

Comparison of the observed and computed atmospheric transmissions for ultra-violet light for $\lambda = 0.32\,\mu$.

A direct comparison between Pasadena and Mount Wilson gave sixty-five per cent on a fairly clear day, which is in fair agreement with the value sixty-eight per cent obtained from the transmission in Table 1.

Measurements of sky-radiation effective on a horizontal surface determined photographically in wave-length 0.32 μ show, for air mass 1.3, one-twentieth that of direct sunlight on Mount Wilson, and one-fourth that of direct sunlight in Pasadena on a fairly clear day. Under these conditions the radiation in this wave-length, falling on a horizontal plane surface from the Sun and sky combined in Pasadena is 0.8 that on Mount Wilson, the greater radiation from the sky making up for the deficiency from direct sunlight to some extent.

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[3] J. physique et le radium, 7: 257 (1926).

A peculiar meteorological phenomenon has been noticed both in the Mount Wilson and Tucson measurements. When the sky is perfectly clear so that no whiteness can be seen up to the limb of the Sun from sunrise to sunset the morning and afternoon observations agree in giving nearly the same results, i. e., the intensity of the ultra-violet light is the same for a given hour-angle east and west of the meridian. Ordinary clouds and haze do not seem to affect the ratio ultra-violet to green remarkably, but haze of another kind does. Estimates of the radius of the corona about the Sun due to sky whiteness are made at intervals during each day's run and are recorded on a scale of zero to ten. Generally, this whiteness is nearly zero until 11 A. M., then it begins to rise rapidly, attaining a value of six or seven by 1 P. M., after which it gradually increases to eight or nine by sunset. This haze has the appearance of smoke and in the afternoon may be seen from Mount Wilson, filling the valley below and rising in clouds considerably above the mountain. Figure 3 illustrates, at the right a nearly perfect day, and at the left a typical day affected by haze in the afternoon. On account of this effect, only morning observations have been used to study the solar variation. It is possible that this phenomenon is connected with atmospheric ionization, and the afternoon runs may furnish material for its study.

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ANNUAL VARIATION OF THE SUN'S SPECTRUM IN THE REGION 3200 to 2900 ÅNGSTRÖM UNITS

GEORGE WINCHESTER

The intensity or variation of the spectrum in the violet and ultra-violet region is ordinarily measured by means of a thermo-couple or bolometer-arrangement. The physical, chemical, and physiological properties of the ultra-violet end of the spectrum are evidently not closely related to heat, and obviously any measurement based on the phenomenon of heat is unsatisfactory. The physiological properties of the region 3400 to 2900 *Å* have recently attracted considerable attention and some dependable method of measurement other than one based on the phenomenon of heat would be very acceptable. At the present time the threshold values of certain metals seems to point in a promising direction. Einstein's photo-electric equation between the energy absorbed and that retained by an electron freed from a metal by ultra-violet light is

$$\frac{1}{2} mv^2 = hv - w \quad \text{where } w = hv_0$$

v_0 being the low frequency limit for that particular metal.

Now suppose several metals with long-wave limits between 3400 and 2900 Å units are mounted in photo-electric cells; each cell will give us an indication of the intensity of a definite part of this region. The cells may be calibrated against each other; for example, by means of an ultra-violet monochromator. And thus by the method of subtraction the intensity of certain rather narrow bands in this direction may be measured photo-electrically.

If the variation in intensity is being determined a certain check might be obtained by means of Schumann plates read by means of a micro-photometer.

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PHOTOGRAPHIC SPECTROPHOTOMETRY IN THE ULTRA-VIOLET REGION*

LOYD A. JONES

It is assumed that the audience to which this paper is presented is interested chiefly in the application of photographic spectrophotometry to the measurement of energy distribution in solar radiation. The author feels some hesitancy in presenting a paper on this subject since he has not had any actual experience in solar-radiation measurements. A large amount of work has been done in our laboratory, however, on the application of photographic methods to the determination of the ultra-violet absorption characteristics of dyes and other selectively absorbing media and to the measurement of the spectral distribution of sensitivity in photographic material.[1] Although the application of such methods to the measurement of solar radiation undoubtedly presents certain problems not involved in the work in which we have had most experience, yet many of the fundamental principles are common to both types of work. It is hoped that a discussion of some of the methods which we have found useful and of the manner in which photographic materials should be used in order to eliminate some of the errors frequently encountered may be useful to those who are interested particularly in the measurement of solar radiation.

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* Communication No. 314 from the Research Laboratory of the Eastman Kodak Company.

[1] L. A. JONES AND O. SANDVIK. J. O. S. A. 12: (4) 401, April, 1926.

The photographic plate has been used for many years in spectrophotometric work and owing to its sensitivity to radiation of frequencies outside the visible spectrum has become practically indispensable where it is necessary to work in regions of low or complete invisibility. Unfortunately, many methods of using the photographic plate for this type of work have been devised without a complete knowledge or due consideration of the inherent characteristics of these materials, with the consequences that results obtained have been more or less disappointing from the standpoint of repeatability and precision. Failures to obtain the required precision, which are usually traceable to improper usage or lack of knowledge of the fundamental characteristics of photographic materials, should not be considered as sufficient evidence for unqualified condemnation. The author feels very strongly that with proper experimental technique and intelligent interpretation of results the photographic method is capable of giving spectrophotometric data in the ultra-violet region equal or superior in precision to that obtained by any radiometric method. These remarks should not be interpreted to convey the impression that no satisfactory methods have been developed, for such is not the case. Some very good instruments have been devised and results of satisfactory quality have been obtained by their use. The literature of photographic spectrophotometry is voluminous and no attempt will be made in this paper to review the work that has been done in this field or to present a complete bibliography of the subject.

Before proceeding with a discussion of the instruments and method it may be advisable to state briefly some of the peculiarities of photographic materials which may give rise to errors when these materials are used for the measurement of radiation-intensities.

Intermittency effect. The photographic plate does not in general integrate correctly an intermittent exposure. An exposure (exposure, E = intensity, I, times time, t) of definite magnitude given as a series of intermittent flashes generally yields, upon development a lesser density than the same exposure applied continuously. Hence only under certain conditions will correct values of intensity be obtained if the exposing mechanism is of the intermittent type. This intermittency effect has been studied by Abney,[2] Schwarzschild,[3] Sheppard and Mees,[4] and many others. The most recent work is that by Davis[5] in which he shows that the difference between the density produced by equal continuous and intermittent exposure may be either positive or negative depending on whether the intensity factor of exposure, I, is greater or less than the optimal intensity, I_0, of the photographic material in question. For precision work in photographic photometry it is therefore undesirable to use intermittent exposing methods.

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[2] W. de W. Abney. Phot. J. 18: 56 (1893).

[3] K. Schwarzschild. Astroph. J. 11: 92 (1900).

[4] Sheppard and Mees. Investigations on the Theory of the Photographic Process, p. 222, 1907.

[5] Raymond Davis. Bur. Standards Sci. Paper no. 528, 1926.

Failure of the reciprocity law. The density produced by a given exposure, E, is dependent not only upon the value of the exposure, $E = I \cdot t$, but upon the magnitudes of intensity, I, and exposure time, t. The relation between the values of intensity and time-factors in exposure and the magnitude of the resultant density has been studied by many workers in this field and the literature of the subject is very extensive. Perhaps the best known of this work is that of Schwartzschild[6] who formulated the law generally known by his name,

$$E = I \cdot t^p$$

where p is a constant. Kron[7] in 1913 carried out a very extensive research in this subject and from his results obtained the following equation

$$E = t \cdot I \cdot 10^{-a\sqrt{\log\frac{I}{I_0}+1}}$$

This subject has been given a great deal of attention in this laboratory during the past ten years and several papers reporting our results have been published.[8] It has been conclusively proven that the Schwartzschild equation is not correct and that the exponent p is not a constant. Our results which are based on many thousands of observations do not fit well the equation proposed by Kron as given above. An alternative equation which was suggested by Kron, but which he found did not fit his results satisfactorily has been found by us to fit the observed values more precisely than any other analytical expression thus far tested. The curve obtained by plotting the data in the form of "constant density curves" is of the catenary type and is expressed by the equation

$$E = \frac{I_0 t_0}{2}\left[\left(\frac{I}{I_0}\right)^a + \left(\frac{I}{I_0}\right)^{-a}\right]$$

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[6] K. Schwarzschild. Phot. Corr. p. 171, 1899; Astro. Phys. 11: 89 (1900); Beitr. z. Phot. Photen. d. Gestirne.

[7] Kron. Eder's Jahrbuch, p. 6, 1914.

[8] Loyd A. Jones and Emery Huse. J. O. S. A., 7: 12, 1079, Dec. 1923; 11: (4) 319, Oct. 1925; Loyd A. Jones, Emery Huse, and Vincent C. Hall. J. O. S. A., 12: (4) 321, Apr. 1926; Loyd A. Jones and Vincent C. Hall. J. O. S. A., 13: (4) 443, Oct. 1926; L. A. Jones, V. C. Hall, and R. M. Briggs. J. O. S A., 14: (3) 223, March, 1927.

in which I_0 (optimal intensity) and a are constants for any photographic material.

Relation between gamma and wave-length. If density, D, which is defined by the equation

$$D = \log_{10} \frac{1}{T}$$

where T is the transmission-coefficient of the silver deposit in question, is plotted as a function of log exposure a curve such as is shown at A in Figure 1 is obtained. A part of this curve is represented to within the limits of experimental error by a straight line. The slope (λ) of this line with respect to the x-axis is defined by tan a where a is the angle which the straight line makes the x-axis. The curve B in the same figure is the D-log E characteristic obtained with the same material but using a longer development time than was used in case of curve A. It is evident therefore that the slope of the straight line, $dD/d\log E$, is dependent upon the extent to which development is carried.

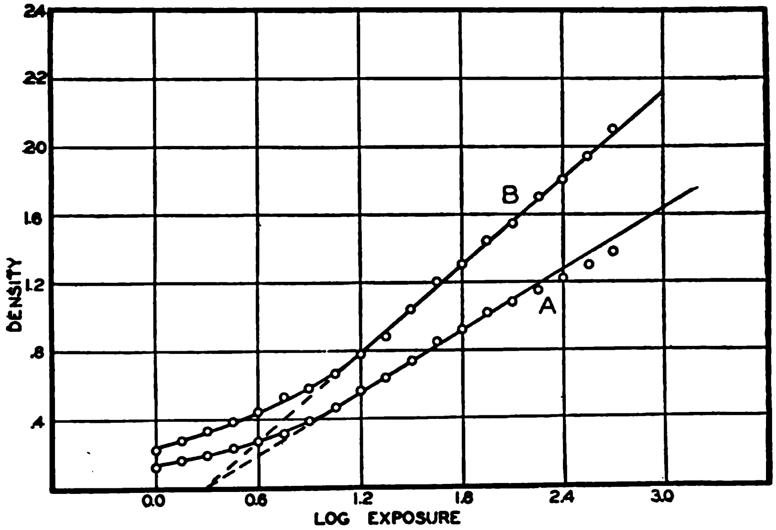

FIG. 1

Density-log E characteristic of a photographic plate.

Moreover, for a fixed time of development, γ is a function of the wave-length of the radiation used in making the exposure. This dependence of slope on wave-length is shown graphically in Figure 2 which shows gamma plotted as a function of wave-length for various development-times, these curves applying to Eastman D. C. ortho plates. It will be noted that when the development time is short, low contrast, γ is fairly constant in value in the region from 3,000 to 4,000 Ångströms but when development is increased the γ obtained at 3,000 is very much less than that at 4,000. This relation between γ and wave-length of the exposing radiation is different for different photographic materials and in photographic spectrophotometry this relationship must be determined unless some method is adopted which automatically eliminates its effect. In a paper dealing with the sensitometric characteristics of photographic materials in the ultra-violet, Harrison[9] has published curves showing the relation between γ and wave-length for several different brands of photographic plates. His results indicate that for some plates γ is practically independent of wave-length in the region between 2,500 and 3,500 Ångströms, while for other materials there is an enormous variation within the same wave-length range. The author feels that it would

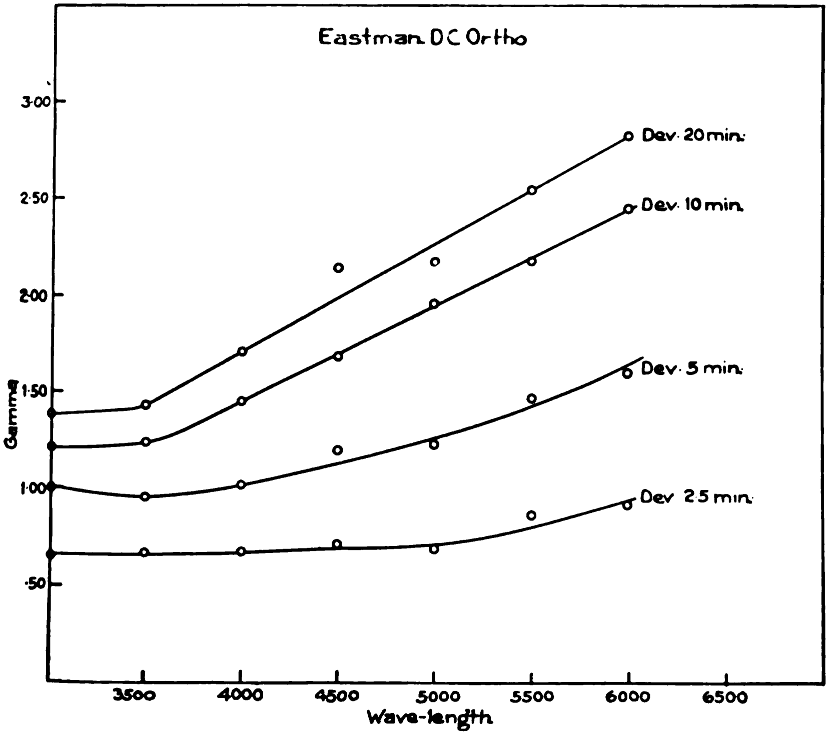

FIG. 2

Relation between wave-length and gamma for different times of development.

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[9] HARRISON. J. O. S. A., 11: 341 (1925).

be unwise to assume constancy of γ with wave-length even for materials of the same kind as used by Harrison without making a careful check on the particular emulsion-number being used. It is well known that different batches of emulsion may show slight variations in characteristics. While these variations of quality are not important for the practical purposes they may easily be sufficiently large to cause undue error in photographic spectrophotometry.

Non-uniformity of effective sensitivity. The most carefully made photographic materials may show measurable differences in sensitivity at various points, even when the area used is relatively small. These variations in sensitivity may be due to such factors as inequality in the thickness of coating, variation in the specific sensitivity of the emulsion, or slight differences in the rate at which development takes place at different points on the plate. While these inequalities in general are too small to be of consequence in the practical work to which these materials are usually applied they may become of great importance and of sufficient magnitude to introduce serious errors when an attempt is made to use these materials for the purposes of quantitative measurements.

Errors in development. It is very difficult to develop two samples of photographic material to exactly the same extent. This is due to many factors such as the variation in the rate of circulation of the developing solution over the surface of the plates or films, variation in temperature and time of development, etc. Of all the methods thus far devised for obtaining uniform development over a relatively large area it is probable that the so-called *brush* method proposed and described by Bloch[10] and Clark[11] is the best. Measurements made in this laboratory show that the errors attributable to lack of uniformity of development occurring when plates are developed in a tank or in a tray, are appreciably diminished by the use of the brush method.

Criteria of intensity equality. From a consideration of these characteristics the conditions under which two radiation-intensities are equal may be rigidly specified. Let two immediately adjacent small areas, A and B, on the surface of a photographic material be exposed to the two intensities to be compared. Let this exposed material be developed in such a way that the two areas received precisely the same treatment in developing, fixing, washing, drying, etc., and the resultant density be measured under identical conditions of illumination and observation. Let the various factors involved be designated for convenience as follows:

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[10] O. Bloch. Phot. J. 61: 425 (1921).

[11] W. Clark. Phot. J., 65: 76 (1925).

Area	A	B
Intensity of incident radiation	I_1	I_2
Wave-length of incident radiation	λ_1	λ_2
Exposure time	t_1	t_2
Density	D_1	D_2

In general I_1 is equal to I_2 only when both exposures were made non-intermittently and when

$$t_1 = t_2,$$
$$\lambda_1 = \lambda_2, \text{ and}$$
$$D_1 = D_2$$

In case the two radiations being compared are not monochromatic the condition that $\gamma_1 = \gamma_2$ must be replaced by the requirement that the two heterogeneous radiations shall be of identical spectral composition.

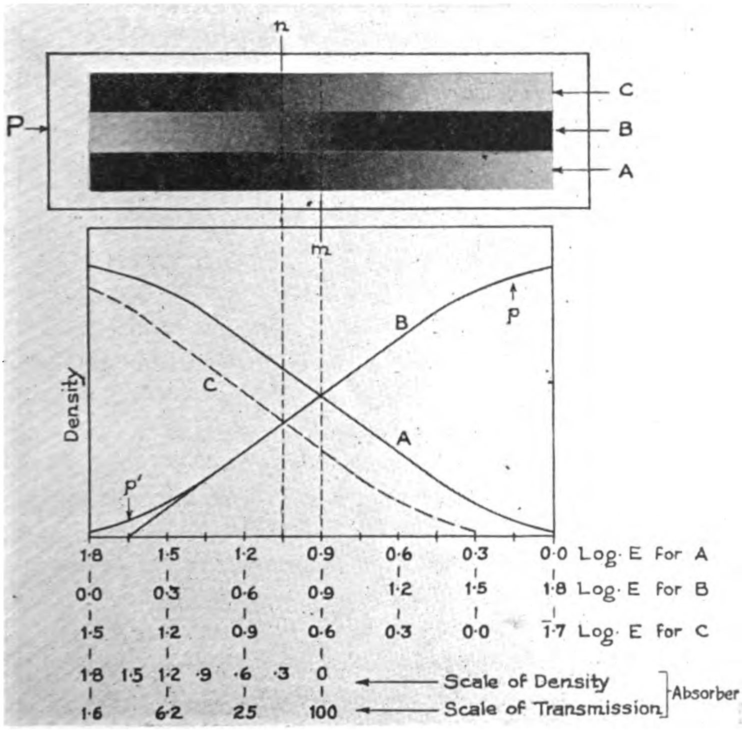
n
P
C
B
A
m
B
p
Density
C
A
p'
1·8 1·5 1·2 0·9 0·6 0·3 0·0 Log·E For A
0·0 0·3 0·6 0·9 1·2 1·5 1·8 Log·E For B
1·5 1·2 0·9 0·6 0·3 0·0 1·7 Log·E For C
1·8 1·5 1·2 ·9 ·6 ·3 0 Scale of Density
1·6 6·2 25 100 Scale of Transmission
Absorber

Fig. 3

Diagram illustrating method of measuring radiation-intensity.

Any photographic spectrophotometric method to be entirely free from criticism must meet the requirements stated above. The validity of results obtained by a method not conforming to these conditions must be subject to doubt until it has been definitely proved that the error introduced by failure to meet the theoretical requirements in any particular case is within the limits of tolerances as established by the precision required.

A method for making spectral absorption-measurements in the visual region[12] has been developed in this laboratory which meets all of the requirements stated above. It is thought that this method can be extended with advantage to the measurement of solar radiation in the ultra-violet. The principle involved can best be explained by reference to Figure 3. The narrow rectangular area A represents the density produced by subjecting the photographic material to an exposure of which the time-factor, t, is constant and the intensity-factor, I, increases logarithmically from $\log E = 0$ at the right-hand end to $\log E = 1.8$ at the left-hand end. The relation between density and log exposure in this area is represented by curve A. The narrow strip B represents an exposure on the same material to the same light-source increasing from $\log E = 0$ at the left-hand end to $\log E = 1.8$ at the right-hand end. This density-distribution corresponds to curve B. It is evident that the two curves will intersect at a point midway between the zero points of the two strips, this intersection corresponding to the point at which the density on one is equal to the density on the other. Now if a third strip, C, is exposed under exactly the same conditions as A with the exception that a light filter of unknown absorbing power is placed between the light source and photographic plate the density illustrated by a strip C will be obtained corresponding to curve C. The point at which density-balance between strips C and B occurs is given by the intersection of the curve C with B and the lateral displacement of this with respect to the intersection point of curve A and B will be a direct measure of the absorbing-power of the unknown filter. In extending this to the measurement of intensity of radiation it is only necessary to make the exposure of strip C to a source of unknown intensity but of the same wave-length or spectral composition as that used in exposing strip B. The displacement of the point n, at which density-balance occurs, from the position of the point m is a direct measure of the intensity of the unknown source relative to that used in obtaining strip B. Emphasis is laid on the fact that the criterion of intensity-equality is that immediately adjacent positions of the same photographic material give equal densities for equal exposure-times.

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[12] LOYD A. JONES. J. O. S. A., 10: 561 (1925).

Obviously the development must be very nearly the same for the two areas and moreover from a consideration of the geometry of the figure the position of the point of balance is independent of the extent to which development is carried. Moreover it is not necessary that the density shall lie on the straight-line portion of the characteristic curve. It is obvious that the curves *C* and *B* must be of identical shape, assuming $\lambda_1 = \lambda_2$, hence the lateral displacement of curve *C* remains directly proportional to the relative intensity of the two sources regardless of whether or not the curves intersect on their straight-line portions.

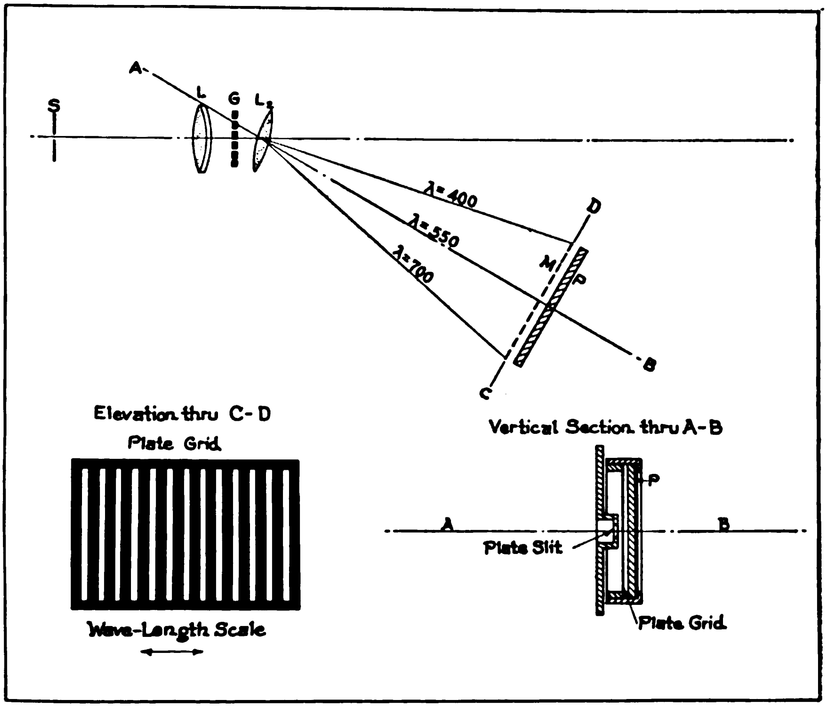

FIG. 4

Schematic drawing showing photographic spectrophotometer.

The method of applying this to work in the visual region is illustrated in Figure 4 in the upper part of which is shown a schematic horizontal section through the instrument. The slit *S* is illuminated by suitable light-source. The light is collimated by lens *L*, dispersed by the diffraction-grating *G*, and focused on the spectrum plane *CD* by the lens L_2, the photographic plate being placed in the spectrum plane. A parallel line grid-plate, lower left-hand corner of Figure 4, is placed in contact with the sensitive surface in the spectrum plane, the grid-strips lying vertically. The photographic plate and grid are mounted in a holder which can be moved up and down, perpendicular to the plane of the drawing behind a plate-slit which extends horizontally through the spectrum, that is in the plane of the drawing. Between the lenses L and L_2 is mounted an adjustable sector-diaphragm by means of which the intensity of radiation incident on the photographic plate can be controlled.

The exposures are made as follows. The photographic plate with the shielding grid is moved vertically at a *constant linear velocity* while at the same time the sector-diaphragm is closed in such a manner as to cause the intensity on the photographic plate to *decrease logarithmically.* In this way an exposure corresponding to the strip B in Figure 3 is obtained. The grid-plate is now moved laterally by a distance equal to the opaque strip, thus protecting from further exposure the areas which have been exposed and uncovering those areas which have not been exposed. Now the absorbing material to be examined is inserted, the plate moved vertically in the opposite direction at a constant uniform velocity, and the diaphragm opened in such a manner as to give an exposure *increasing* logarithmically thus giving an exposure through the filter as

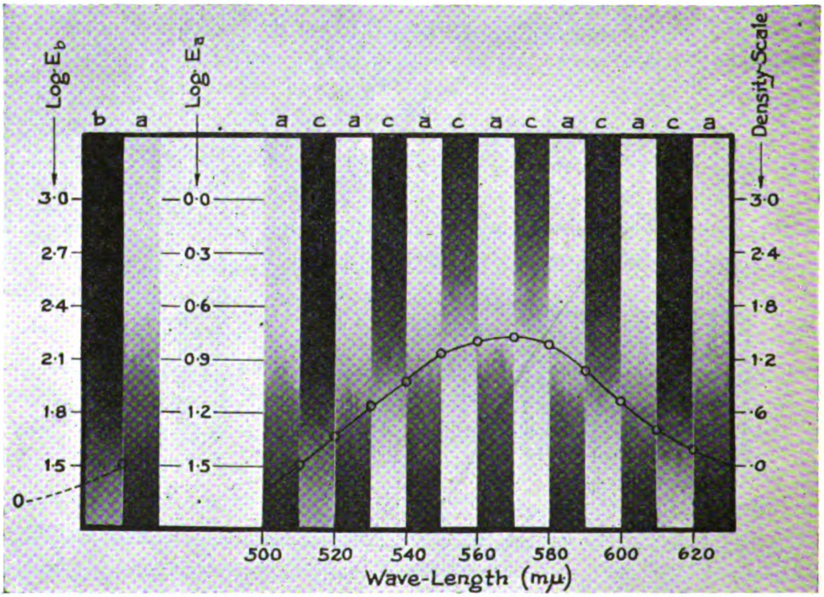

Fig. 5

Record obtained with the photographic spectrophotometer.

shown in strip *C*. For work in the visual a diffraction-grating is used giving normal dispersion. Hence a constant grid-spacing corresponds to constant wave-length difference. The grid-strip used is of a width corresponding to a wave-length band of five μ width. The appearance of the resulting plate after development is shown in Figure 5. A curve connecting the points of equal density gives directly the wave-length density-curve of the absorber. In Figure 6 the schematic drawing shows the method used for moving the photographic plate up and down at a constant linear velocity and for opening and closing the sector-diaphragm to obtain a logarithmic increase and decrease of intensity. The cam plates 15 and 14 are driven at constant linear velocity by means of a synchronous motor and carefully cut lead-screw.

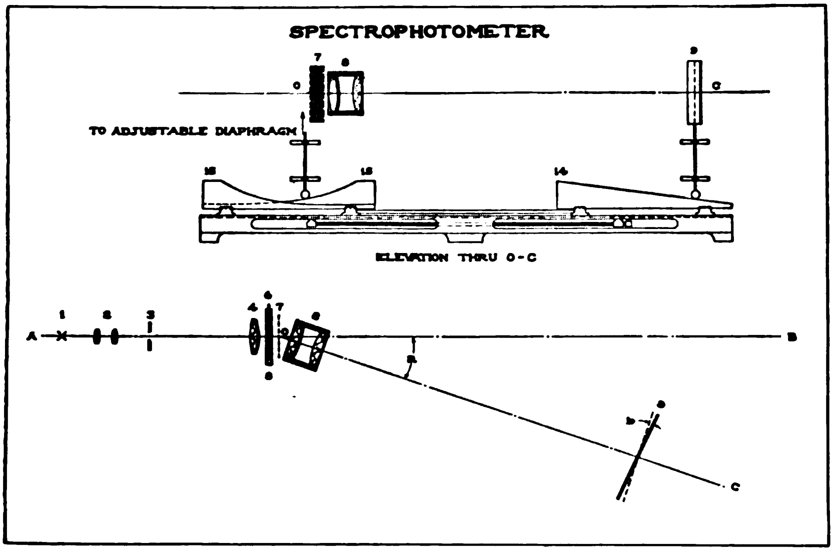

Fig. 6

Diagram illustrating method of moving photographic plate and actuating the intensity-control.

In order to utilize the full width of the photographic plate it is advantageous in practice to displace the balance-point *O* so that it lies near one edge of the plate. Then a decrease in the intensity due to the absorber will give balance-points lying between *O* and the upper edge of the plate, Figure 5.

The determination of the point at which density-equality between adjacent strips and curves occurs can be made with great precision by means of a microdensity-comparator designed to isolate a small area on each strip near the dividing line. It is unnecessary to make absolute measurements of density since only the position of *density-equality* is of importance. The microdensity-comparator may be either a visual or physical photometer type.

In modifying this method to meet the requirements of solar-radiation-measurement it will probably be necessary to replace the diffraction-grating with a quartz prism and obviously all of the lenses must be made of quartz. Furthermore a standard source must be used in making the comparison-exposure. By setting up the standard source in a suitable position off of the optical axis and by using a total reflecting prism in front of the entrance slit, which can be moved in and out at will, the first series of exposures can be made to the solar radiation and the second to the radiation from the standard source. It is probable that for the region between 450 and 350 μ the grid-plate method as described previously could be used to advantage thus obtaining relative energy determinations throughout an appreciable wave-length band from one exposure. In the region of wave-lengths shorter than 350 where the rate of change of solar intensity with wave-lengths is great it is probable that scattered light will become serious if an attempt is made to determine radiation-intensity over a wave-length band of appreciable extent. Fabry and Buisson [13] used two spectroscopes in tandem in order to obtain monochromatic radiation free from impurities due to scatter in the optical system. This is undoubtedly the best method for obtaining high spectral purity although a fair approximation to satisfactory purity can be had by using extremely narrow monochromatic filters transmitting radiation of the wave-length being measured. In our work (loc. cit.) on the sensitivity of photographic materials in the ultra-violet region we found it quite impossible to obtain sufficient purity by any method except the double monochromatic illuminator. The use of monochromatic filters in this work was almost prohibitive due to the necessity of determining the transmission of the filter itself for the various wave-lengths. In measuring solar radiation, however, if the filter is interposed during exposure both to the sunlight and the comparison-source it will be unnecessary to know the transmission of the filter for the wave-length in question.

By using a monochromatic illuminator to furnish radiation for the spectrophotometer as shown in Figure 6 high spectral purity could be obtained. It would be necessary to dispense with the grid and to make a pair of graded exposures, one using solar radiation and one using the standard source, at each wave-length. In this case the plate-slit, Figure

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[13] CHARLES FABRY AND H. BUISSON. Astroph. J. 54: 297 (1921).

4, would be replaced by a small square opening having a dimension in the direction of dispersion equivalent to a very narrow wave-length band. The result obtained in this way would consist of a single pair of density-strips graded in opposite directions, the position of the balance-point being used to determine the relative intensity of the solar radiation and the comparison source.

The mechanical construction of the instrument necessary for the application of this method could be appreciably simplified by adopting a step by step exposure-method in lieu of the continuous one described. Density measurements made on these would give a sensitometric curve for the solar radiation and for the comparison-source radiation at each wave-length. The point of intersection of the two curves, corresponding to the point of intersection of curves *B* and *C* in Figure 3, would then determine the relative intensity of the two radiations. The photographic plate could be moved by hand or by an automatic mechanism corresponding to the discontinuous changes in the diaphragm-setting. A series of fixed diaphragms of the radial sector-type would probably be better than a set of the variable diameter type since the former would be independent of any zonal aberration present in the optical system. The intensity could with advantage be controlled by using screens made by depositing platinum cathodically on thin plates of quartz. This material is relatively nonselective in the region between 250 and 450 μ as shown by the spectrophotometric density-curve, Figure 7. Such screens can be calibrated by any suitable radiometric or photographic method. Since they are nondiffusing their calibration is not as troublesome as screens made by using developed photographic images. In the case of the latter materials it is essential to calibrate them in the position in which they are to be used in order to be certain that the value of density obtained is equivalent to their effective density-value.

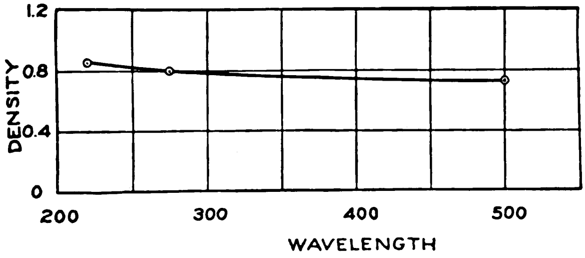

FIG. 7

Spectrophotometric density-curve.

Double monochromatic quartz illuminators are now available commercially, an excellent one being manufactured by Carl Zeiss, Berlin. By the addition to one of these instruments of a suitable set of absorbing diaphragms and a plate-holder at the exit slit of the instrument it should be possible to apply the principle of the method outlined above. This consists essentially of making a series of sensitometric exposures to the radiation of unknown intensity and an exactly similar set to a radiation of known intensity, this being done at each wave-length. The sensitometric exposure may be either continuously graduated as in the instrument designed for work in the visual, or discontinuous (step by step) if such is more convenient. In case of the continuous variation in exposure the relative energy will be determined by finding an intensity-balance between the two sensitometric strips, thus eliminating the necessity of making density measurements. In case of the step-by-step method it will be necessary to make density measurements and to plot the resultant density-log *E* characteristic. The point of intersection between the two characteristic curves thus obtained will determine the required relative intensity-value.

As a comparison-source it is probable that a tungsten-ribbon filament mounted in a quartz bulb is best. Such a source supplied with current from a storage battery and controlled by a potentiometer method should be constant over long periods of time with very high precision. The relative intensity of radiation emitted at various wave-lengths by a tungsten filament at a known temperature can be computed by the well known radiation formula. The constants for tungsten have been determined with high precision. A tungsten filament operating at a temperature of 2,960° K emits sufficient radiation in the region between 290 and 400 μ to serve very satisfactorily as a standard source with which to compare the solar radiation. The tungsten-ribbon filament mounted in a quartz bulb has been used in this laboratory as a standard of radiation in this region in connection with our work on the measurement of photographic sensivity. Using a double monochromatic illuminator of relatively high aperture and ordinary commercial photographic materials exposures down to 300 can be obtained in reasonably short time.

Sensitivity of photographic materials—The sensitivity of photographic materials extends into the far ultra-violet and a little difficulty is experienced in spectrophotometric work in this region in obtaining sufficient sensitivity for practical purposes. As stated previously the gamma in most cases is relatively low at wave-lengths between 2,500 and 3,000 Ångströms, thus resulting in maximum densities which are relatively low. The paper previously mentioned by Harrison[9] gives information relative to the distribution of sensitivity for several well known photographic materials. The same author has also published a paper[14] dealing with the effect of various fluorescent oils on increasing the sensitivity of photographic materials to ultra-violet radiation. This procedure may be advantageous in working in the extreme ultra-violet, wave-length shorter than 2,500 μ, but it is not probable that such procedure will be found of any particular advantage in solar radiation-measurements.

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[14] GEORGE R. HARRISON. J. O. S. A., 11: 113 (1925).

In conclusion, it is desired to emphasize again that certain definite conditions should be kept in mind when using photographic materials for making measurements of radiation-intensity. These may be summed up briefly in the statement that in general two radiation-intensities are equal only when for equal times of exposures equal densities are produced, it being assumed that both exposures are non-intermitttent, that the two radiations are of the same wave-length or if heterogeneous are identical in spectral composition, and that the two densities have been produced by identical processing, that is development, fixing, washing, drying, etc. It is preferable that the two densities compared lie immediately adjacent to each other on the same photographic plate in order to minimize possible local variations in sensitivity of the materials.

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BLEACHING OF METHYLENE-BLUE-ACETONE-WATER SOLUTIONS BY ULTRA-VIOLET RADIATION

HERBERT H. KIMBALL AND IRVING F. HAND

In response to many requests, the Weather Bureau has made some tests of the reliability of the bleaching of a methylene-blue-acetone-water solution, when exposed to solar radiation, as a measure of the intensity of the ultra-violet component in sunlight. It is understood that this method is employed in England and other countries, and details were obtained from a paper by Webster and others, in the *Lancet* for April 12, 1924.

We have used a stock-solution consisting of 5.8 cc. of a 0.1 per cent methylene-blue stock added to thirty cc. of acetone and made up to one hundred cc. by the addition of distilled water.

Ten tubes of clear fused quartz about six inches long with thin walls and an internal diameter of three mm are filled with even fractional portions of the stock-solution from ten per cent to one hundred per cent to serve as comparison, or color-match tubes. To prevent deterioration of their contents they are sealed and left in a dark-room, as are also the stock-solution and all apparatus used in these tests.

For measuring ultra-violet light, tubes similar to the comparison tubes are filled with the stock-solution, sealed with beeswax, and exposed in a horizontal position above the roof of the building. The axis of the tube points north and south.

The first exposures showed surprisingly large bleaching effects. So, also, did the stock-solution when exposed in a tube of ordinary glass, which is not supposed to transmit ultra-violet light. This indicated either that the solution was sensitive to light of wave-lengths greater than 3,500 Ångströms, or that the chemicals were impure.

Through the courtesy of Dr. Garby and Mr. Pinck of the Fixed Nitrogen Research Laboratory, Bureau of Soils, our supposedly pure acetone and methylene-blue were submitted to repurifying processes, the former through fractional distillation, the latter first by crystallization in an aqueous salt solution and afterwards dissolved in alcohol, filtered and precipitated by the addition of ether.

It is difficult to obtain pure methylene-blue, as it is not entirely dried out at 105° C, and it begins to decompose at 110° C. For this reason, in any series of measurements the same stock-solution should be used throughout if comparable results are desired.

Using these purified chemicals, the bleaching through the walls of ordinary glass became negligible, while through quartz on most clear days it is a measurable quantity. The measurement consists in matching the color of the solution in the tube that has been exposed to sunlight with the comparison tubes described above. This permits comparison of the bleaching effect on one day with that on another, but does not give measurements of the intensity on any standard scale.

It was pointed out by Webster, in the *Lancet* previously quoted, that the bleaching effect is 0.6 greater with the solution at a temperature of 25° than at a temperature of 15° C. That high temperature accelerates the bleaching has been confirmed by exposing two tubes in a similar manner except that one was kept at a temperature of about 98° F by placing a heating unit under it with a sheet of dark colored asbestos between, while the other was at 50° F, the temperature of the air. Two tests of this kind indicate a bleaching of sixty and eighty per cent in the warmed tubes as compared with forty and fifty per cent in the tubes at air temperature.

The following average values for December 1926 to March 1927, inclusive, show a decided seasonal effect due not only to increased receipt of ultra-violet radiation with increased solar altitude, but also to a temperature-effect. Expressed in percentages, the bleaching for December averaged thirty-eight; for January twenty; February forty-eight, and March, twenty-six. In each month the daily exposure was about six hours, commencing at 8 A. M. and was confined to clear days. A few exposures on days with dense clouds showed only a trace of bleaching. The temperature on the days when the tubes were exposed averaged about the same for December and January but were much higher in February and still higher in March. The lowest monthly average, twenty per cent in January, is partially accounted for by average wind-velocities of twenty-six miles per hour on the days when the tubes were exposed. No means are given for April, but during this month complete bleaching frequently takes place between 8 A. M. and noon.

As would be expected, the hours of maximum bleaching occur between 10 A. M. and 2 P. M.

On April 8 the tubes were thoroughly cleaned by heating in an oxygen-hydrogen flame and then carefully refilled. The bleaching on this date, however, was a negative quantity, that is, the solution actually turned a slightly darker blue in color. The reason for this remains unexplained although it is probably due to an impurity of some kind in the solution.

Our results indicate that this method of determining the intensity of ultra-violet in sunlight gives only approximate comparisons from day to day, since they are dependent upon so many factors, such as purity of chemicals, and their temperature, which latter is partly dependent upon wind.

U. S. Weather Bureau,
Washington, D. C.

PRELIMINARY NOTE ON CONCLUSIONS REGARDING THE CONSTITUTION OF THE UPPER ATMOSPHERE INDICATED BY DATA OF RADIO

E. O. HULBURT

The phenomena of the waves of wireless telegraphy have led to very definite information about the ionization in the upper atmosphere. Assuming that the pressures in the upper atmosphere are those of the classical isothermal conductive equilibrium hypothesis of Humphreys, Jeans, etc., with negligible amounts of oxygen, and that the ionization is due to the absorption of the short ultra-violet light of the Sun, the ionization is calculated explicitly and is found to be that required by radio for the day-time. At night, however, due to recombination, calculated in a way apparently correct, the ionization decreases in a manner which seems quite contradictory to the facts of radio. This same conclusion is reached whether the upper atmosphere contains hydrogen or not.

Calculation of the types of ionization due to α-particles and β-particles from the Sun and to penetrating radiation are made. Radio evidence seems (at the present moment) to be noncommital as to the extent of the influence of these agencies of ionization.

If the upper atmosphere contains oxygen, all the calculations will be profoundly modified. Cario's recent paper which attributes the auroral line to oxygen, Lindemann's rather vehement answer (Astrophysical Journal, March 1927) to Sparrow's criticism of his meteor-theory, and other fields of evidence, indicate that the constitution and pressures of the upper atmosphere are perhaps not those of the classical calculation.

The attempt is now being made to determine to what other conclusions, if any, the data of radio point.

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SECTION OF TERRESTRIAL MAGNETISM AND ELECTRICITY

(Washington, D. C., April 28, 1927)

SYMPOSIUM ON CORRELATIONS OF VARIOUS RADIO PHENOMENA WITH SOLAR AND TERRESTRIAL MAGNETIC AND ELECTRIC ACTIVITIES

PURPOSE OF SYMPOSIUM

N. H. HECK

A good case could be made for considering radio transmission itself as a geophysical subject and certainly its very evident relations with terrestrial magnetism and atmospheric electricity are an appropriate subject for the American Geophysical Union. It is, therefore, not merely a matter of interest to the members but a duty to consider the progress made in the investigation.

The recent rapid growth of interest in the correlations of various radio phenomena with solar and terrestrial magnetic and electric activities is in part to be ascribed to the fact that perfection of transmitting and receiving apparatus makes it reasonable to assume that interference is to be ascribed to outside phenomena and not to the apparatus. An irreducible minimum is being reached beyond which new methods of attack are necessary and complete understanding of the phenomena involved. Improved apparatus for recording signal-intensity is becoming an important factor in this situation.

The practical importance of the investigation from the commercial and military viewpoint and from that of the public interested in receiving radio programs from distant stations is too obvious to need discussion. There is, however, a special noteworthy example in the world radio longitude-determinations of 1926. Since the apparatus used in this work recorded automatically the distant signals, the interference from various causes was of the greatest importance and added greatly to the time required and the consequent cost of the work.

Since representatives are here from all the organizations interested in this work, I have no intention of reviewing the history, feeling sure that each speaker will introduce such history as may be necessary. It is not inappropriate for me to mention, however, that the Coast and Geodetic Survey has recognized its importance and that for several years the observer in charge of the magnetic observatory at Sitka, Mr. Franklin P. Ulrich, has been studying with the aid of the Signal Corps, U. S. Army, and the U. S. Navy, the relations between aurora, magnetic storms, and difficulties in electrical transmission and reception. In the case of radio, his work has been handicapped by variations in transmission and his results have been somewhat inconclusive, but it is certain that in the case of the more marked disturbances his work will prove of value when he has made a sufficiently large accumulation.

The Coast and Geodetic Survey is also furnishing magnetic-observatory results to the various investigators of the subject and it stands ready to do the same for all those interested. Attention is called to the fact that it has observatories in the north, in the tropics, and in the intermediate latitudes. It would be of very great value to carry on atmospheric-electric observations and automatic recording of electric signals at one or more of the observatories.

In conclusion, I wish to quote the closing remarks of Dr. Michael Pupin in his address as retiring president of the American Association for the Advancement of Science at Philadelphia in December, 1926:

"When we speak of electrical communications we usually mean exchange of messages between man and man. But a careful study of the static, of the fading of radio communications, and of the so-called earth-currents in cables will undoubtedly reveal that they are not, as usually believed, disturbances which only annoy the clumsy methods of human operators. They are messages which are far more important to mankind than mere communications between mortals. For instance, I can not help feeling that the so-called earth-currents in cables are messages telling us many secrets which are going on in the Sun, the central power-station which supplies the moving power to all our organic and human activities."

U. S. Coast and Geodetic Survey,
Washington, D. C.

SUNSPOTS AND SOLAR RADIATION

C. G. Abbot

(Abstract[1])

Pending a thorough revision of all solar-constant observations made at the Smithsonian station at Montezuma, Chile, certain small errors of scale have come to light. Corresponding corrections have been applied to the list of provisional monthly mean observations, 1918 to 1926. A fairly close correlation exists between the variation of the "solar constant of radiation" and the Wolfer sunspot numbers. Higher solar radiation

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[1] Complete paper is published as Smithsonian Misc. Collect., vol. 80, no. 2, 1927.

attends larger sunspot numbers. The relation is not linear. More rapid changes of radiation accompany changes at times of low absolute numbers. At high numbers the change tends to be asymptotic.

Regular periodicities of the solar radiation of 25⅔, 15, and 11 months, and higher harmonics of these periods are found. Of these the 25⅔ month period is much the strongest. It has also long been known in weather phenomena.

Daily observations of solar radiation are going on at Table Mountain, California, Montezuma, Chile, and Mount Brukkaros, Southwest Africa, under the Smithsonian direction. It is hoped that a comparison of their results will yield, within a few years, definite evidence as to the controverted question as to whether or not sunspots at their crossing of the Sun's central meridian tend to be accompanied by a temporary diminution of the solar constant.

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SUNSPOTS AND MAGNETIC AND ELECTRIC DISTURBANCES

Louis A. Bauer

The continued study of the correlations between solar activity and disturbances in the Earth's magnetic and electric condition is of special interest during the present period of increasing sunspottedness. Neither the number, area, nor position of spots on the Sun's visible disk may be taken, however, as a safe index for the prediction of the occurrence of magnetic storms, nor of the production of the electric currents in the Earth's crust which are responsible for interruptions in telegraphy. There have been notable magnetic storms when there were no spots on the Sun's visible disk. The Earth's magnetic activity does not cease during a year of minimum or of almost no sunspottedness.

The desire has therefore arisen for other measures of solar activity which would supplement the information derived from spots and prominences. Possibly continued measurements of the magnetic fields of disturbed areas of the Sun and the information gained with the aid of the spectroheliosscope and the spectrograph will, in the course of time, yield data on which to base new measures. But it may happen also in the end that changes in the Earth's magnetism and electricity, radio transmission and ultra-violet radiation of the Sun, will be our most definite measures of those solar agencies with which we are particularly concerned.

It will be of interest to record here that a cooperative plan is now in progress between the Mount Wilson Observatory and the Department of Terrestrial Magnetism for the study of solar and magnetic disturbances. A magnetic variometer giving both visual and photographic indications of changes in horizontal intensity has been supplied by the Department and has been in operation at Mount Wilson since last August. The purpose is to indicate to the Mount Wilson observers the times for special intensive study of solar phenomena, with all the means available at the Observatory.

We must give attention to the various measures of magnetic activity before fruitful correlative studies of terrestrial and solar phenomena can be undertaken. Taking, for example, as measures, the well-known magnetic-character numbers, or using numbers indicating the frequencies of magnetic disturbances, it is found that both of these measures show a tendency towards a double periodicity during the year—maxima near the equinoctial months and minima near the solstitial months. The measure known as the inter-diurnal variability of the horizontal intensity, tends to show a similar double periodicity during the year. Generally, the sunspot measures show no such periodicity. Thus the magnetic measures contain effects, due in some manner to the Earth's orbital motion, which require elimination before the correlation with solar phenomena, month to month, week to week, day to day, may be attempted.

Magnetic measures based on the total change in any particular magnetic element during a given period, a day for example, likewise exhibit annual periodicities which, in general, do not correspond with the double periodicity just described. So here again effects of a terrestrial character, unless we are dealing with very severe magnetic storms, must first be eliminated. Time will not permit us to enter into the various methods for eliminating, as far as possible, terrestrial effects.

The year 1920 is of particular interest in connection with the remarks just made. It so happened during that year that the Wolfer monthly mean sunspot-numbers exhibited to some degree a double periodicity, with maxima in March (70.2) and in October (49.6). According to the magnetic-character numbers the maximum disturbance activity apparently occurred in September. Quantitative measures of magnetic disturbance would show, on the other hand, a maximum in March—the month of highest sunspottedness.

In March, 1920, there occurred two notable magnetic storms, namely, March 4 to 5, and March 22 to 23, the second being two and more times severer than the first. Owing to the limited scale, the magnetic-character number assigned to both of these cases is, however, the same, namely, 2.

We thus see why quantitative and qualitative measures may not agree in their indications and the precautions which must be used in correlative studies.

The correlation between sunspottedness and terrestial magnetism is most pronounced for the periodic magnetic variations. Thus the correlation-coefficient between the annual mean diurnal range and the annual mean sunspot-number (or area) may reach 0.9. For a group of seventy-eight principal magnetic storms registered at the Cheltenham Magnetic Observatory 1913 to 1922, the correlation-coefficient ranged from 0.35 to 0.64, according to the measure of solar activity used. The solar measure, which gave the highest correlation was a quantity derived from the expression, S_p $(1-\rho)$, in which S_p is the Greenwich projected sunspot-area, and is the mean distance of a spot, or group, from the center of the Sun's disk. That is, on the average, the spots near the Sun's center appear to be the most effective, but there are a number of noteworthy exceptions to this general rule.

The correlation-coefficient between sunspottedness and polar lights and that between sunspottedness and earth-current changes are found to be high.

Although the electrical phenomena, of which we have cognizance during brilliant displays of polar lights in high altitudes, show a high correlation with magnetic storms, the same cannot be said with regard to the electrical phenomena in the lower regions of the atmosphere. No pronounced disturbances, in general, occur in atmospheric electricity during severe magnetic storms. The disturbances in atmospheric electricity arising from thunderstorms, for example, are far more definite and pronounced.

If we take, however, only the fine weather days, then we find the probability high that the atmospheric potential-gradient and its diurnal and annual ranges, as also the air-earth current, are subject to sunspot influence. The completion of another cycle of observations at widely-distributed stations, not subject to local disturbing influence, will probably have to be awaited, however, before all the questions as to the precise nature of any solar-activity influence on atmospheric electricity may be definitely settled.

We have as yet insufficient reason to conclude that the high correlation between solar activity and the Earth's magnetic activity indicates a direct association, as cause and effect, of sunspots and magnetic storms. It seems safest at present to assume that both sets of phenomena are effects of one, as yet undiscovered cause, which may simultaneously affect the condition of the entire Sun, or at least of a large portion of it. Thus more or less permanent centers of activity on the Sun, which are the true source of the Earth's magnetic disturbances, may also be the cause of sunspot formations in other regions of the Sun.

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Carnegie Institution of Washington,
Washington, D. C.

MAGNETIC OBSERVATIONS AT THE MOUNT WILSON OBSERVATORY

Seth B. Nicholson[1]

In August, 1926, a magnetograph was installed at the Mount Wilson Observatory in cooperation with the Department of Terrestrial Magnetism of the Carnegie Institution of Washington. The magnetic variometer and recording device with all the auxiliary equipment were furnished by that Department and its installation was personally supervised by Mr. Fleming.

Only the intensity of the horizontal component is recorded, as the plan is to compare the solar observations made each day with the magnetic records, looking for the type of solar activity which may be directly responsible for particular magnetic disturbances. No statistical study of the number of sunspots and their relation to the mean daily range in H is contemplated. In the thirty-foot pit of the sixty-foot tower telescope another similar magnetic variometer is installed, which can be read at the time of each solar observation. The observer is then aware of the state of the Earth's magnetic field and can make more frequent solar photographs if a magnetic storm is in progress.

The routine solar observations consist of direct photographs of the Sun with a 175-mm image; spectroheliograms, with a 53-mm image of the Sun's disk, using the H_α-line of hydrogen and the $K2$-line of calcium; and prominence spectroheliograms with the K-line of calcium. These routine photographs are supplemented by H_α-spectroheliograms with the 175-mm image showing in detail the regions around active spot groups. A spectrohelioscope is being constructed with which to make still more detailed visual observations of the motions of the hydrogen gas in the neighborhood of spots.

At the 150-foot tower telescope, where the image of the sun is 430 mm in diameter, visual observations of the magnetic polarity and field-strength of each spot are made daily. These observations and the heliographic positions of the spots are recorded on a full-sized tracing of the Sun's disk.

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[1] Read by W. S. Adams.

In addition to these observations of sunspots, daily measures are made of the intensity of the ultra-violet radiation from the Sun.

It is hoped that these solar observations, when compared with the daily records of terrestrial magnetic activity, may help to solve the problem of how solar activity is responsible for changes in the Earth's magnetic field.

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THE CORRELATION OF RADIO RECEPTION WITH SOLAR ACTIVITY AND TERRESTRIAL MAGNETISM—II

Greenleaf W. Pickard

This paper is a continuation of an earlier one[1] on the same subject, in which day-reception at 15-25 kilocycles, night-reception at 1330 kilocycles and night-reception at 8-9 megacycles were compared with sunspot-numbers and magnetic measures. Although in the former paper definite relations were shown, there were also discrepancies, particularly in the relation of sunspots to reception and magnetism. When these three elements were graphically compared over any considerable period, the curves swung alternately in and out of step in an irregular-seeming manner, although preserving in a general way a 27-day period.

It must be confessed at the outset that the relation between individual sunspots, reception, and magnetism is still far from definite. In general it appears that terrestrial elements are most disturbed when the sunspots are large and numerous, and particularly when they face most nearly earthward. But often large spots cross the meridian without either accompanying magnetic disturbances or changes in reception, severe magnetic storms sometimes occur when the only sunspots visible are near the Sun's limbs and reception disturbances are not always accompanied by visible happenings on the solar disc.

Dr. Bauer has said[2] that none of the present measures of the Sun's activity is adequate to explain the occurrence and magnitude of disturbances in the Earth's magnetism; that neither the number, area, nor position of spots on the Sun's visible disc may be taken at present as a safe index for the prediction of the occurrence of magnetic storms. As the result of an examination of radio-reception records extending over nearly twenty years, the author finds this statement also applies to individual cases of reception disturbances. But if a sufficient number of cases are examined statistically, very definite general relations appear, and at the present time it is not an over-statement to say that the major cause of reception disturbance is of solar origin.

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[1] The correlation of radio reception with solar activity and terrestrial magnetism. Proc. Inst. Radio Eng., 15: (2) February, 1927.

[2] Studies concerning the relation between the activity of the Sun and of the Earth's magnetism. Terr. Mag., 30: (4) December, 1925.

Undoubtedly both magnetism and reception are also subject to disturbances which are either not of solar origin, or which at least do not follow solar rotation-periods or the longer swing of the sunspot cycle. Thus, diurnal changes are found in both reception and terrestrial magnetism, and seasonal changes are also well known. A solar eclipse distinctly affects radio reception,[3] and less definitely appears to disturb the Earth's magnetism. There also appear to be relations between certain meteorological elements and reception which are not paralleled by magnetic changes, probably because reception and weather are rather local matters, while terrestrial magnetism is quite literally a world-wide affair.

While reception disturbances of non-solar periods are important and fertile fields for investigation, the present paper must confine itself principally to those magnetic and reception changes associated with the 27-day solar rotation-period, and so these other effects must be here eliminated rather than examined. This has been done by taking a sufficiently long series of observations, dividing this into 27-day periods, and finding the daily means. This operation averages out anything which does not recur in each solar rotation, and as will appear from the figures gives a clearer picture of the interrelation than that presented in the former paper.

Although there is no reason to suppose that the solar centers responsible for terrestrial disturbances are necessarily the sunspots themselves, the evidence is now strong that these centers are at least confined within the same solar zones as the sunspots themselves. If it be assumed that these centers, like the accompanying sunspots, are confined within definite areas for long periods, but shift about within these areas from month to month, some of the discrepancies previously found can be quite simply explained. And even if, as at present appears, the wanderings of the active centers are random affairs, the mean of a number of solar rotations should show definite terrestrial-disturbance maxima corresponding with the meridian crossing of the centers of the active zones, instead of the somewhat irregularly spaced maxima accompanying the transit of the individual centers.

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[3] The effect of the solar eclipse of January 24, 1925, on radio reception. Proc. Inst. Radio Eng., 13: (5) October, 1925.

Before periodic means of sunspot-numbers, magnetic measures or reception observations are made, it is often useful to reduce the numerical values to such a basis that periods of high absolute values will not dominate the means of the entire series. In the present paper this is done by converting the daily values into ratios with a moving mean of the period under investigation, thus obtaining a measure of activity within the period.

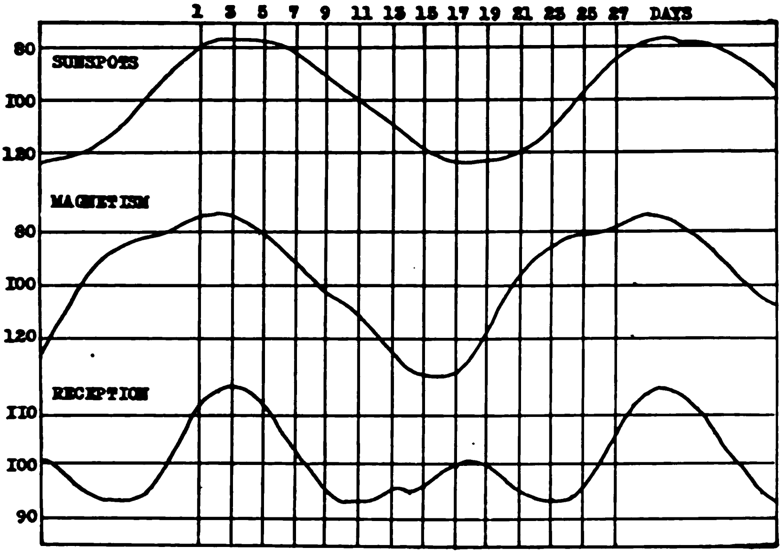

FIG. 1

Mean of twelve solar rotations, October 25, 1908 to September 17, 1909. (Sunspots, magnetic character of day, and maximum night-ranges of "DF" working.)

By taking a sufficient number of periods, even very rough reception-measurements can be made to show definite relations to sunspots and terrestrial magnetism. As an illustration of this, Figure 1 shows a comparison of the maximum nightly ranges of two-way working between station *DF* at Manhattan Beach, New York, and ships at sea,[4] Wolfer final sunspot-numbers and van Dijk's magnetic character-of-day numbers.

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[4] ROBERT H. MARRIOTT. Radio range variations. Proc. Inst. Radio Eng., 2: (1) March, 1914.

The interval taken is from October 25, 1908 to September 17, 1909, corresponding to twelve solar rotations of 27.3 days. The sunspot-curve shows only the fundamental, but the magnetic and to a greater extent the reception curves show the existence of shorter periods. Owing to the extreme roughness of the reception data, no attempt has been made to investigate these shorter periods, although this has been done for some of the later and more accurate reception-measurements. The relation of reception to the other elements is inverse, which appears characteristic of night-reception save at ultra-high frequencies. It is of interest to note that this transmission was all at frequencies within the present broadcasting band of 500-1500 kilocycles, so that it may be compared with present-day measurements, allowance being made for the fact that in 1908 to 1909 spark-transmitters of relatively high decrement were used.

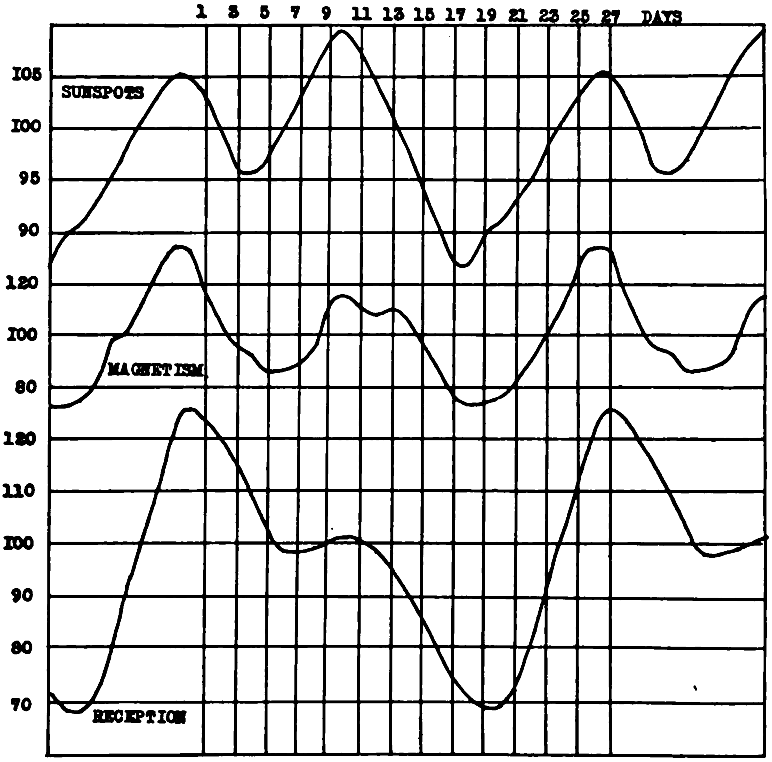

FIG. 2

Mean of five solar rotations, January 14 to May 30, 1917.
(Sunspots, magnetic character of day, and Washington day-reception of "POZ.")

During 1917, a sunspot maximum year, the U. S. Naval Radiotelegraphic Laboratory made audibility meter measurements at Washington of day-reception from station *POZ* at Nauen, Germany. Owing to war conditions, Dr. Austin considers these the least accurate of his long series of measurements of this station; nevertheless Figure 2 clearly shows correlation between this reception, sunspots, and terrestrial magnetism. Unlike the 1908 to 1909 night-reception of the first figure, the day-reception here is directly related to sunspots and magnetism, which seems to be the normal relation of day-reception at low frequency; the 1926 exception to this rule will be discussed later.

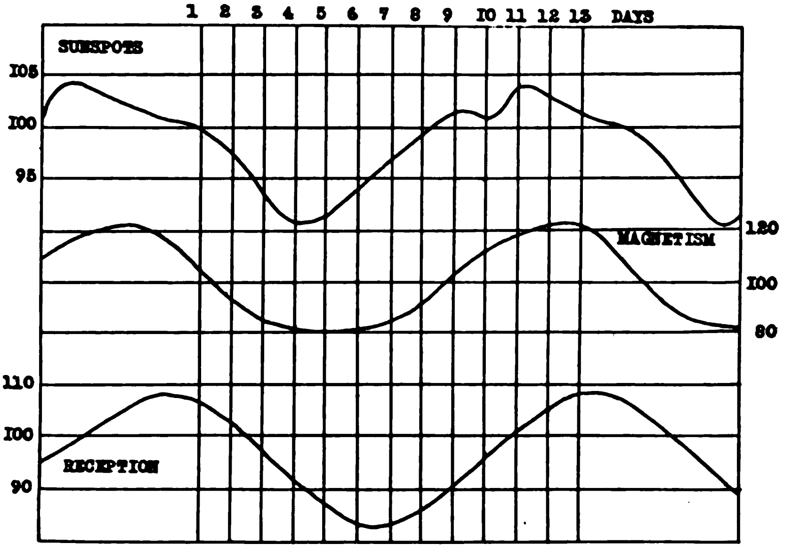

FIG. 3

Mean within five solar rotations of ten 13.64-day periods, January 14 to May 30, 1917.

(Sunspots, magnetic character of day, and Washington day-reception of "POZ.")

All three curves of Figure 2 show shorter periods of considerable amplitude, and despite the roughness of the measurements, an analysis has been attempted for the second harmonic of the solar rotation-period, or 13.6 days. The result is shown in Figure 3, as three nearly sinusoidal and parallel curves. The low amplitude of the sunspot-curve is due to the nature of the sunspot-numbers, which are taken over the entire visible solar disc, or over nearly 180° in solar longitude. This process leaves very little of the shorter or harmonic periods in the sunspot-numbers, although it will be shown later that these are prominent in spot-numbers taken in restricted zones, instead of over nearly a hemisphere.

In Figure 4 is shown a comparison of sunspots, terrestrial magnetism, and low-frequency day-reception in 1923, a sunspot-minimum year. The

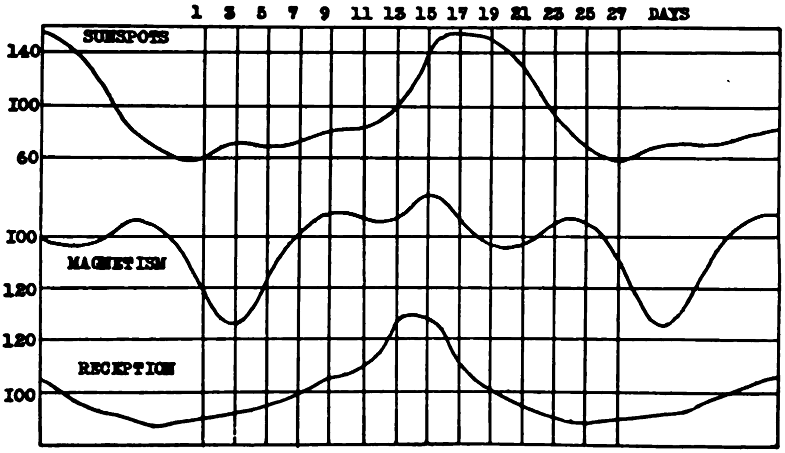

FIG. 4

Mean of seven solar rotations, May 14 to November 21, 1923. (Sunspots, magnetic character of day, and Washington day-reception of "POZ.")

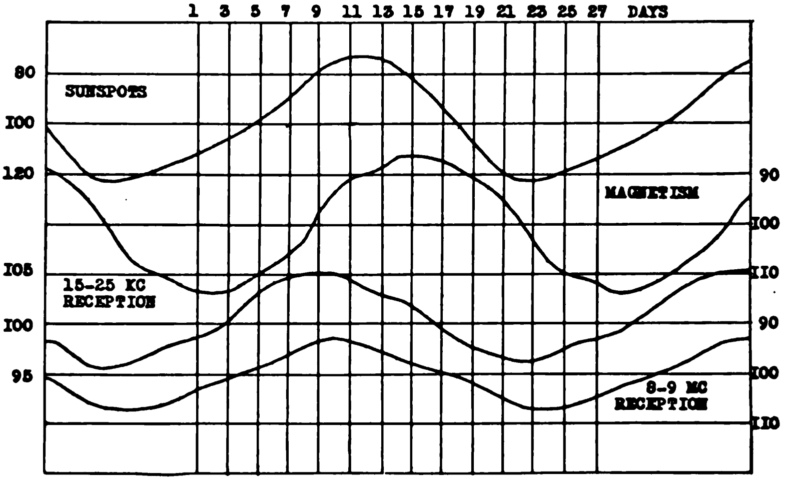

FIG. 5

Mean of eight solar rotations, January 25 to August 31, 1926. (Sunspots, diurnal range of horizontal intensity at Cheltenham, 15-25 kilocycle day-reception at Washington, and 8-9 megacycle night-reception at Washington.)

irregularity of the sunspot-curve is in part due to the fact that forty-seven per cent of the days in this interval had a spot-number of zero. Again a direct relation is found between day-reception, solar activity, and magnetism. Very similar results are obtained for day-reception in 1922, 1924, and the first part of 1925. But in the fall of 1925 an inversion commenced, and for at least 1926 day-reception from Nauen was found to be inversely correlated with sunspots within the solar rotation-period.

This 1926 inversion for low-frequency day-reception is not peculiar to the Nauen station. In Figure 5 is shown the 1926 relation between sunspots, magnetism, day-reception for a group of nine stations in the 15-25 kilocycle band, and night-reception for seven stations in the 8-9 megacycle band. A close agreement between the two reception-curves is evident, but the sunspot and magnetic curves lag four and six days, respectively, behind reception. The correlation between sunspots, magnetism, and 8-9 megacycle night-reception is direct, but is inverted for the 15-25 kilocycle day-reception.

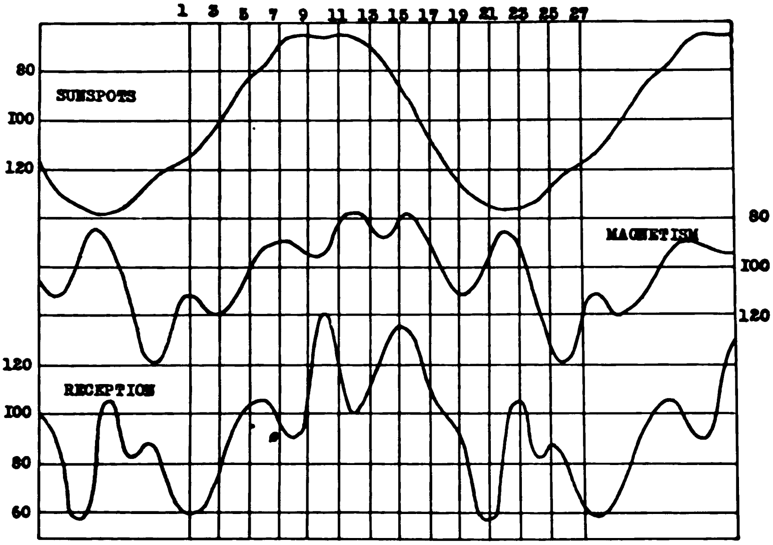

FIG. 6

Mean of eight solar rotations, January 25 to August 31, 1926.
(Sunspots, diurnal range of horizontal intensity at Cheltenham, and night-reception at Newton Centre from "WBBM" at Chicago.)

As set forth in the former paper, it appears that reception in the frequency band of 500-1500 kilocycles shows higher correlation with solar and magnetic disturbances than does any other investigated portion of the radio spectrum, and for that reason the author's measurements have been largely confined to night broadcast reception. In Figure 6 are given the means within a 27.3-day period of the Wolfer provisional sunspot-numbers, the diurnal range of the Earth's horizontal magnetic field as measured at Cheltenham Observatory, and the mean night-field at Newton Centre, Massachusetts, from station *WBBM* at Chicago, transmitting at 1330 kilocycles over the interval January 25 to August 31, 1926. While the sunspot-curve is nearly sinusoidal, indicating the absence in the Wolfer numbers of any large amplitude periods shorter than the fundamental, both the magnetic and reception curves obviously contain shorter-period elements. The relations of the fundamental solar rotation-period is brought out in Figure 7 by smoothing with a moving 13-day mean, which largely eliminates the shorter periods.

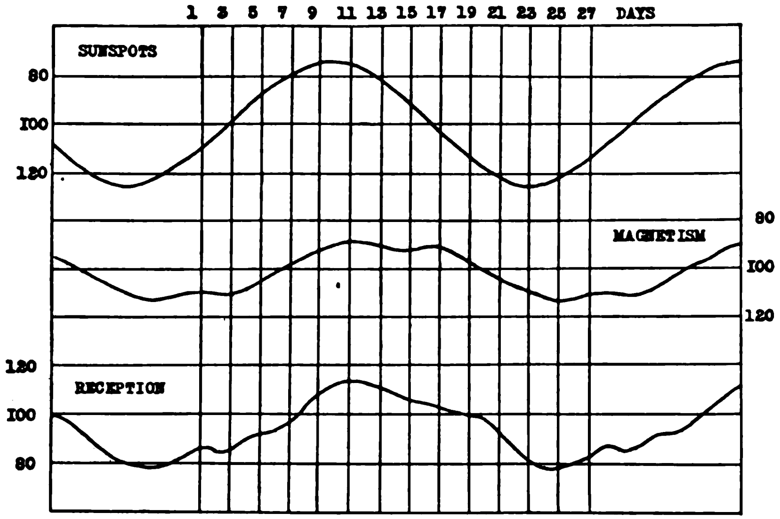

FIG. 7

Mean of eight solar rotations, January 25 to August 31, 1926, smoothed by thirteen-day moving mean.
(Sunspots, diurnal range of horizontal intensity at Cheltenham, and night-reception at Newton Centre from "WBBM" at Chicago.)

In Figure 8 periodic daily means are taken within the second harmonic of the solar rotation-period, or 13.6 days. As might be expected, this greatly reduces the amplitude of the sunspot-curve, but reception and magnetism still show shorter period elements. In Figure 9 the third harmonic of 9.1 days is shown, with very small amplitude in the sunspot-curve, but with a marked distortion of the reception-curve, indicating a still shorter-period element. Finally, in Figure 10 the fourth harmonic of 6.8 days is shown, with the sunspot-curve reduced to an amplitude of one per cent, and the reception and magnetic curves nearly sinusoidal, indicating that they contain no important shorter-period components.

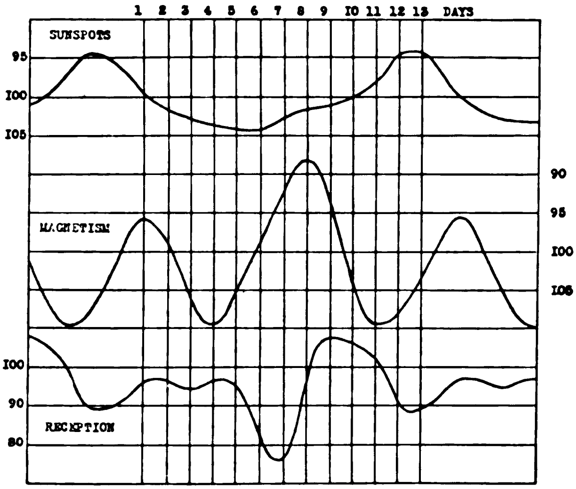

Fig. 8

Mean within eight solar rotations of sixteen 13-64-day periods, January 25 to August 31, 1926. (Sunspots, diurnal range of horizontal intensity at Cheltenham, and night-reception at Newton Centre of "WBBM".)

As indicated above, the Wolfer sunspot-numbers cannot be effectively used in either the investigation of periods shorter than 27 days, or the effect of sunspot-position. Through the kindness of the U. S. Naval Observatory a list has been given of the times of meridian-passage of the larger sunspots of 1926, and a comparison of this with reception shows very clearly that depressions of night-reception are in general coincident with the meridian-transit of the spots, while the appearance of large spots and spot groups near the Sun's limbs have little or no correlation with reception. Through the courtesy of Mr. H. H. Clayton, who has taken daily sunspot-observations for the past several years, sunspot-numbers for a zone ten degrees on each side of the central solar meridian have been furnished for each day of 1926. Using these sunspot-numbers and night-reception from *WBBM*, the relations within the fundamental 27.3-day period, and its second, third, and fourth harmonics are given in Figures 11 and 12. The correlation between the sunspot-numbers for the twenty-degree central zone and reception is very high, and the amplitude of the sunspot-curve for even the fourth harmonic of 6.8 days is quite large. Not only are the sunspot and reception-curves closely alike in form, but they agree in phase to the nearest day, which confirms the conclusion reached from the study of the U. S. Naval Observatory data, that reception-disturbances generally coincide with the earthward presentation of the sunspots.

Measurements of night-static were also taken at 1330 kilocycles, and periodic means made over fourteen solar rotations, from January 25,

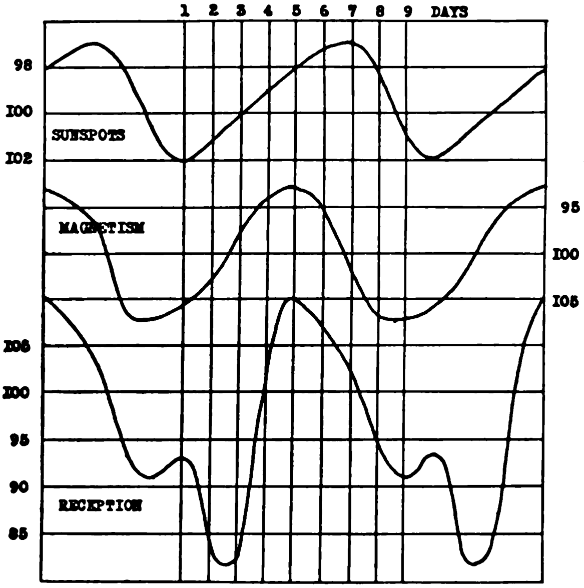

Fig. 9

Mean within eight solar rotations of twenty-four 9.1-day periods. January 25 to August 31, 1926. (Sunspots, diurnal range of horizontal intensity at Cheltenham, and night-reception at Newton Centre of "WBBM".)

1926 to February 11, 1927. The result, as compared with sunspots, is shown in Figure 13. Unlike the night-signal at the same frequency, the relation is direct. A similar analysis of night-static in 1908 to 1909[1] at approximately the same frequency gives the same direct relation, which is again the inverse of the night-signal. On the other hand, an investigation of day-static at low frequency over a period of several years shows an inverse relation to sunspots, although the correlation is by no means as definite as that shown in Figure 13 for night-static. At present there is no satisfactory explanation for this interrelation of sunspots, signal, and day and night-static, either on a basis of static-wave transmission, or the assumption of variability in static sources.

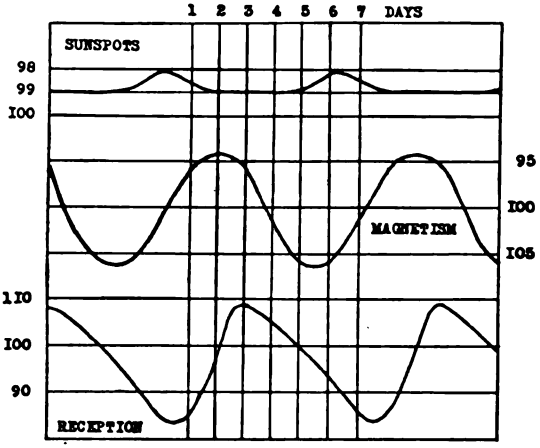

Fig. 10

Mean within eight solar rotations of thirty-two 6.8-day periods, January 25 to August 31, 1926. (Sunspots, diurnal range of horizontal intensity at Cheltenham, and night-reception at Newton Centre of "WBBM".)

In Figure 14 is an example of a double

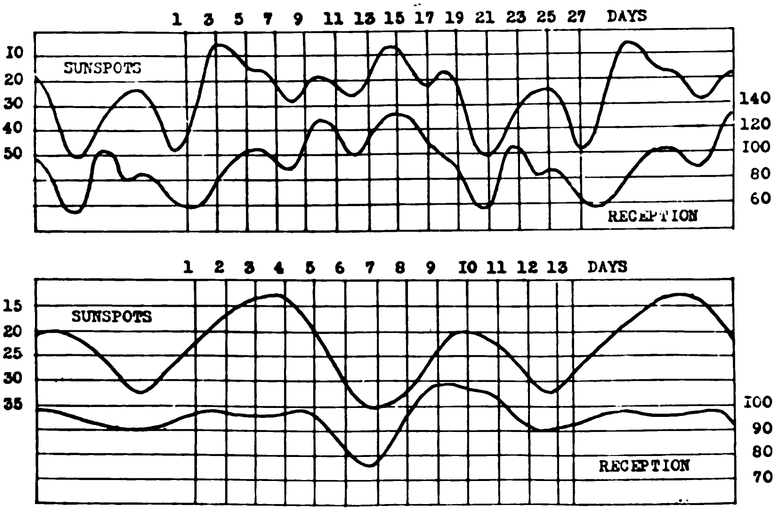

FIG. 11

Mean of eight solar rotations, January 25 to August 31, 1926. (Sunspots within ten degrees of solar meridian and night-reception of "WBBM" for fundamental and second harmonics of solar rotation.)

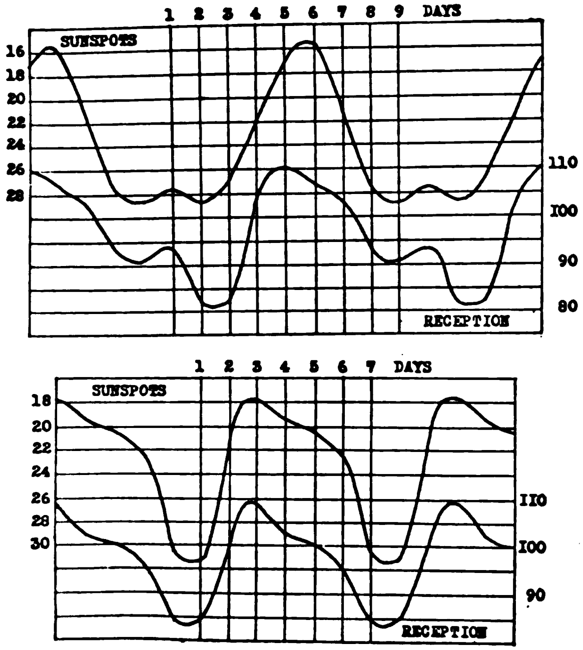

FIG. 12

Mean of eight solar rotations, January 25 to August 31, 1926.

(Sunspots within ten degrees of solar meridian and night-reception of "WBBM" for third and fourth harmonics of solar rotation.)

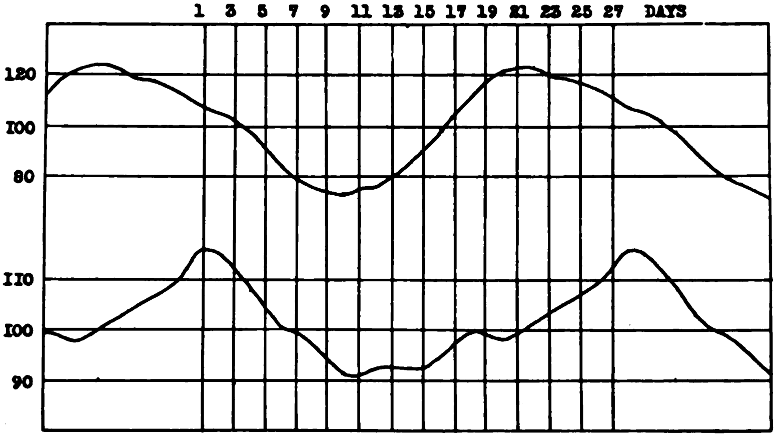

FIG. 13

Mean of fourteen solar rotations, January 25, 1926 to February 11, 1927. (Sunspots and night-static at 1330 kilocycles.)

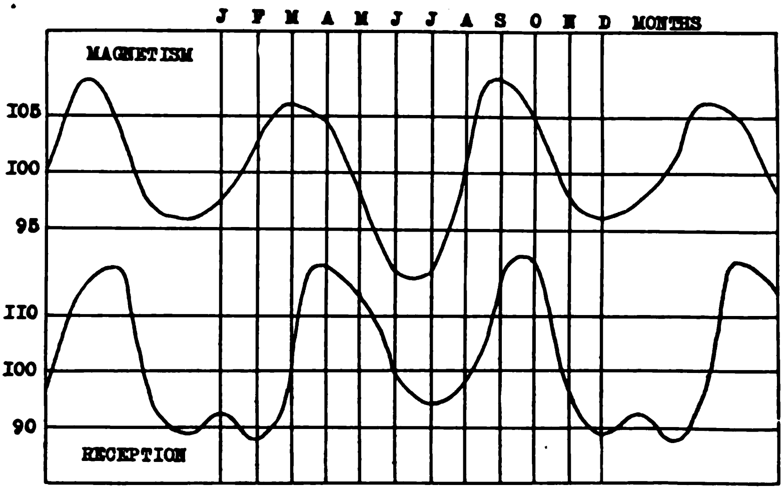

FIG. 14

Mean of eight years, 1916 to 1924, showing monthly averages of magnetic character of day and Washington day-reception from "POZ."

frequency annual period in reception, closely paralleling that already recognized in terrestrial magnetism. Here periodic monthly means have been taken over interval of eight years, from 1916 to 1924, and well-defined maxima show at or near the vernal and autumnal equinoxes. As this is day-reception, the relation with magnetism is direct; if sufficient data were available to construct a similar graph for night-reception minima would probably be found at the equinoxes. A summation of night spark-reception in 1908 to 1909, scattered observations of broadcast reception in 1923 to 1924, and *WBBM* measurements of 1926 to 1927 does, however, show distinct minima in the spring and fall.

Mr. Clayton has recently found an interesting relation between night-reception of *WBBM* and the number of sunspots in a central zone of 13.3° (the width of a day's travel in solar rotation). When the number of spots in this zone was under ten, the relation between spot-number and reception was inverse; when the spot-number was over twelve, a direct relation was found. This is in accord with the predominant inverse relation which has heretofore been found for night-reception, for in two-thirds of the cases examined for 1926, the sunspot number for the zone was under ten. The relation between sunspot-number in a limited central zone, day- and night-reception, and magnetic measures is now under investigation, and will form the subject matter of a third paper in this series.

Wireless Specialty Apparatus Co.,
Boston, Mass.

RADIO ATMOSPHERIC DISTURBANCES AND SOLAR ACTIVITIES[1]

L. W. Austin

The suggestion that atmospheric disturbances might be due to a bombardment of the Earth's atmosphere by electrified particles from the Sun was first made by De Groot[2] in 1917. A number of years later the U. S. Navy began experiments looking toward the establishment of a connection between these disturbances and solar activity. It was thought that there might be such a connection in the case of the type of atmospherics which sometimes produces simultaneous disturbances in the receiving apparatus at widely separated points as in Honolulu and San Francisco or even in Honolulu or San Francisco and Berlin.[3] There seemed to be some evidence that these simultaneous disturbances took place when large sunspots were in the center of the Sun's disk facing the Earth. The observations were made in San Francisco and in Washington and were continued intermittently for more than a year but without leading to any definite conclusion.

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[1] Published by permission of the Director of the United States Bureau of Standards.

[2] C. De Groot, Proc. Inst. Radio Eng., 5: 75 (1917).

[3] Baumler, Jahrb. d. drahtlosen Teleg., 19: 325 (1922); Proc. Inst. Radio Eng., 14: 765 (1926).

During the past year an examination has been made of the observational material on daylight-disturbances between 9,000 and 20,000 meters wave-length collected by this laboratory since 1918. In the earlier measurements the strength of the atmospherics was determined by the shunt across the telephone at which three crashes could be heard in ten seconds. This was subject to all the general inaccuracies of the shunted telephone method, including the effect of changing observers and possible changes in the sensitiveness of the telephones from time to time. The present method which was adopted in 1924 rates the strength of the disturbances in terms of the measured strength of an artificial signal which is just readable through them. The changes in methods of measurement do not permit comparisons of the earlier and later observations which are of much quantitative value. It seems certain, however, that the daylight-atmospherics at 12,000 meters between 1920 and 1924 were considerably stronger than during 1918 and 1919 and during 1925 and 1926. This would indicate a general negative correlation with sunspots. This is shown in Figure 1. The values before 1924 are dotted to indicate their comparatively low accuracy. Figure 2 shows the deviations of the individual monthly averages from the three-year monthly averages of the 20,000-meter atmospheric disturbances and of the sunspot-numbers from 1924 to 1926. The deviations are used, rather than the monthly averages themselves, in order to eliminate the large seasonal variations of the atmospherics. The increase in the sunspot-numbers with the advancing eleven-year cycle is evident in the figure. The atmospheric-disturbance curve, however, is rather noncommittal, but with some evidence of a negative correlation. The indistinctness of this relationship can be compared with the much more definite correlation of sunspots and signal-intensity, as shown in Figure 3 where the mean values of the daylight signal-strength of several long-wave stations are compared with the sunspot-numbers by the same methods and over the same period of time.

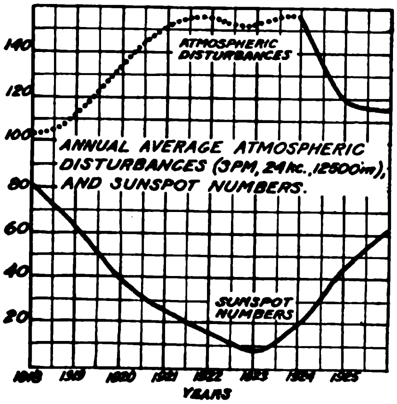

FIG. 1
Annual average atmospheric disturbances and sunspot-numbers.

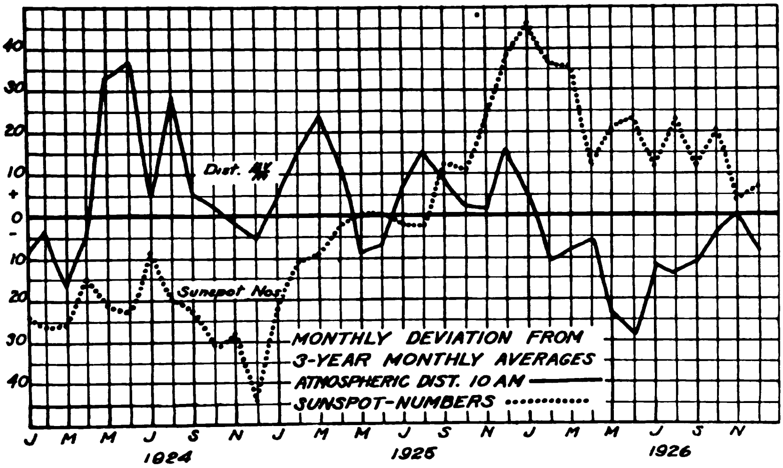

FIG. 2

Monthly deviation from three-year monthly averages for atmospheric disturbances at 10 A. M. and sunspot-numbers.

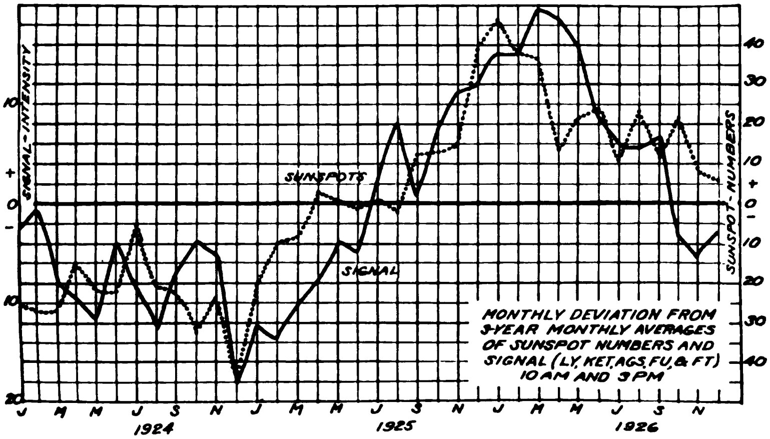

FIG. 3

Monthly deviation from three-year monthly averages of sunspot-numbers and signal-intensity at 10 A. M. and 3 P. M. for stations "LY," "KET," "AGS," "FU," and "FT."

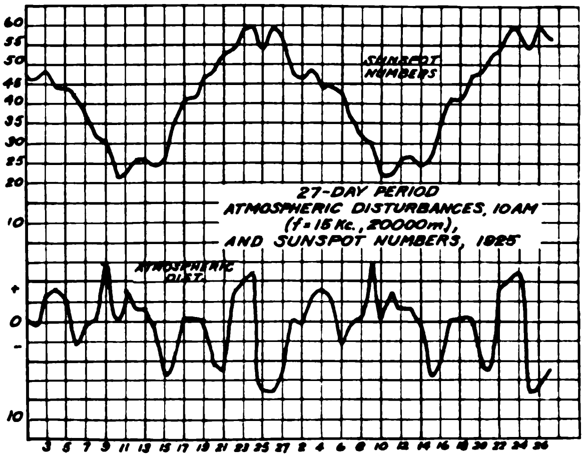

FIG. 4

Twenty-seven-day period of atmospheric disturbances at 10 A. M. (f = 15 Kc., 20,000 m.) and sunspot-numbers for 1925.

If the atmospheric disturbances are dependent upon sunspots, it would seem probable that there would be regular changes in disturbance-intensity in the period of the Sun's rotation. Figure 4 shows the twenty-seven-day periodic averages of the 20,000-meter atmospheric disturbances and sunspot-numbers in 1925, the points being repeated during a second twenty-seven-day period for clarity. The disturbance curve is somewhat smoothed by three-day moving averages. It is seen that this curve, while very irregular, shows perhaps some slight evidence of being in nearly opposite phase to the sunspots.

Figure 5 shows a comparison of ten-day averages of 20,000-meter atmospheric disturbances and sunspots from January 1 to December 31, 1926. Here the prevailing correlation appears to be also on the whole negative.

For a comparison with these rather indefinite evidences of the solar influence on atmospherics, Figures 6 and 7 show their much more definite connection with terrestrial phenomena. Figure 6 represents the quite close relationship between 20,000-meter atmospherics and local temperature averaged in ten-day periods. Here the well-known parallel variations with seasonal temperature are seen, and in addition there is a remarkable correspondence even in the small peaks and troughs of the two curves. Figure 7 represents the relationship of atmospherics and the number of thunderstorms recorded in a region within approximately two hundred miles of Washington. Here again, as in the case of the temperature, we find a close relationship. The connections shown in Figures 6 and 7 evidently have to do with atmospherics of comparatively local origin. Many of the disturbances are, of course, known to

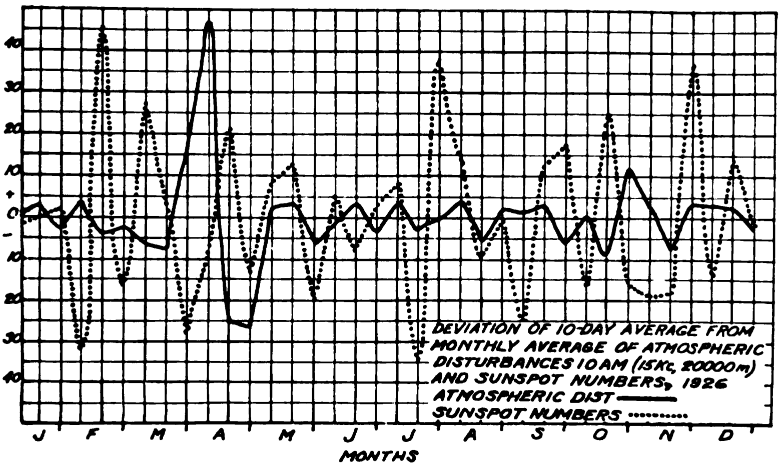

Fig. 5

Deviation of ten-day average from monthly average of atmospheric disturbances at 10 A. M. (f = 15 Kc., 20,000 m.) and sunspot-numbers for 1926

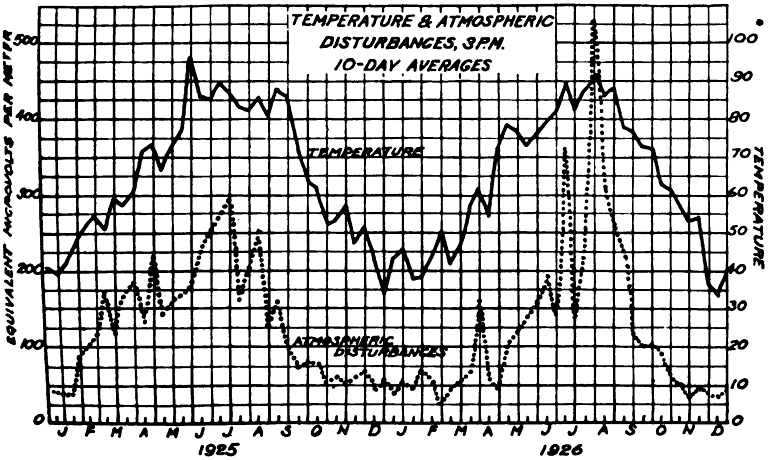

Fig. 6

Ten-day averages of temperature and atmospheric disturbances at 3 P. M. for the years 1925 and 1926.

come from great distances, but these two figures show how prominent a part relatively local atmospherics play in the difficulties of radio reception in Washington.

In conclusion, although there seems to be some evidence of solar influences on long-wave daylight atmospheric disturbances, at present, the proof seems insufficient to establish this with certainty. It may be that the influence of solar activity on the weather and that of the weather on the atmospherics may be the indirect path by which the connection must be traced.

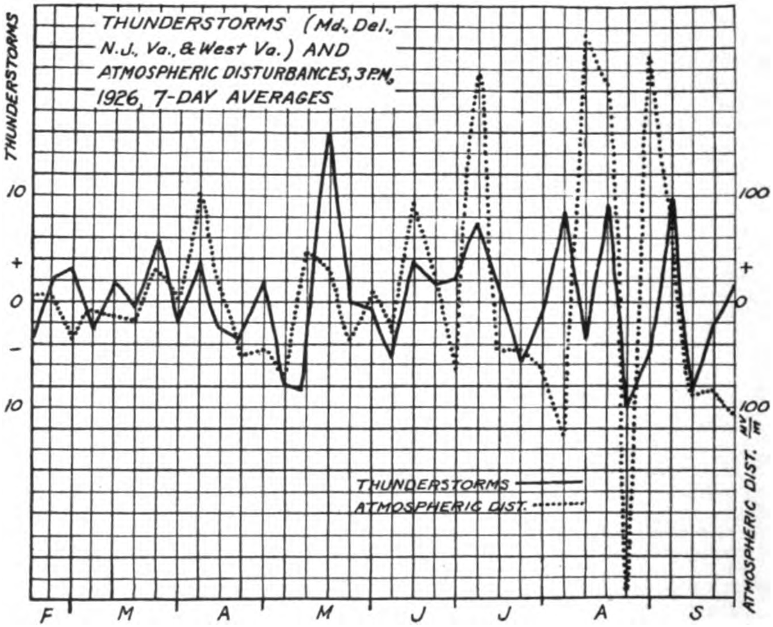

FIG. 7

Seven-day averages of thunderstorms (Maryland, Delaware, New Jersey, Virginia, and West Virginia) and atmospheric disturbances at 3 P. M. for 1926.

U. S. Bureau of Standards,
Washington, D. C.

A SUGGESTION OF A CONNECTION BETWEEN RADIO FADING AND SMALL FLUCTUATIONS IN THE EARTH'S MAGNETIC FIELD

G. BREIT

The fading of radio signals has been referred to time and again as an interference effect. The suggestion that the upper atmosphere may be an important factor in enabling electromagnetic waves to propagate over appreciable distances has been the origin and support of this view. The wave traveling along the ground and the wave arriving by reflection have been supposed to interfere. Within the last few years, however, it has become obvious that more than simple interference is involved. Thus short-wave transmission, i. e., transmission on wave-lengths below 100 meters in length, showed in the hands of Taylor and Hulburt[1] that beyond a certain range signals are received entirely by an overhead route. In spite of this fact, the signals show fading. Part of this is explained as a general shift in the height of the reflecting layer. However, this can be by no means all, because much of the fading is rapid and does not correspond to the disappearance of the signal in the vicinity of the receiver such as occurs at the skip distance. In fact, the experiments by M. A. Tuve and the writer[2] on reflection of waves from the upper atmosphere showed that fading exists for the reflected waves quite independently of interference with the ground-wave Two suggestions have been made at the time as to a possible cause of the fading. One of them supposed the down-coming wave to be a result of the interference of two or more waves. During a discussion of the subject held at the Department of Terrestrial Magnetism, Dr. A. H. Taylor suggested that the Earth's magnetic field might be the cause of some of the fading. On examining the matter, the writer found that since the upper atmosphere must show magnetic double refraction and since, therefore, there must be in general at least two possible rays between the transmitter and receiver, a change in the interference conditions of these two rays must take place whenever the Earth's magnetic field changes. The change in the relative phase of the two rays turns out to be large even if the Earth's field changes slightly. Several gammas (0.00001 Gauss) may turn the plane of polarization of the ray through 30°. As a result, the vertical component of the electric intensity may be changed from 0 to its maximum value in a field of fifteen gammas. The interference effects here discussed are due to the passage of the ray through an appreciable thickness of ionized air. If the rays are reflected, the interference effects cannot be nearly as pronounced. A study of the magnitude of fading in connection with small fluctuations in the Earth's field may lead, therefore, to an understanding of the manner in which the ionization increases with the height.

Some simple calculations are given below in order to show how the interference conditions may change. At the same time the effect of a change of frequency and of the ionization are estimated. The calculations do not pretend to be exact. They are intended to show the order of magnitude of the effects.

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[1] A. H. TAYLOR and E. O. HULBURT. Phys. Rev., 27; 189-215 (1926).

[2] G. BREIT and M. A. TUVE. Phys. Rev., 28: 554-575 (1926).

Before performing the calculations we shall restate the assumptions about the nature of propagation of radio waves which will be used. We suppose the upper atmosphere to be partly ionized into electrons as well as heavier ions. We neglect the effect of heavy ions and consider the effect of electrons alone. The density of free electrons is supposed to increase with the height up to a certain level. The magnetic field of the Earth affects the motion of electrons in the manner discussed by Appleton[3] as well as Nichols and Schelleng.[4] We calculate below the effects on the intensity of received signals due to change in the intensity of the Earth's magnetic field and due to small changes in the frequency of the emitted signals. We suppose that while either change takes place, the degree of ionization is constant. On account of certain mathematical difficulties, an exact solution of the problem does not seem to be worth while at this time. However, for some special limiting cases, exact solutions may be obtained and these can be applied probably with a reasonably small error to calculate the order of magnitude in general.

Letting

$$a = \frac{Ne^2}{\pi m}, \quad \nu_o = \frac{eH}{2\pi m c} \tag{1}$$

where N = number of electrons per cm^3, e = charge of the electron in electrostatic units, m = mass of the electron, H = intensity of the Earth's field. The two velocities of propagation of a beam traveling along H are given by

$$V(\nu) = c\left[1 + \frac{a}{\nu(\nu_o - \nu)}\right]^{-\frac{1}{2}} \tag{2}$$

$$V(\nu) = c\left[1 - \frac{a}{\nu(\nu_o + \nu)}\right]^{-\frac{1}{2}} \tag{3}$$

The first of these corresponds to circular polarization in the right-handed sense, while the second corresponds to circular polarization in the left-handed sense if the sense of propagation of the wave is the same as the sense of direction of H. We consider first an idealized condition in which the rays are approximately following the Earth's magnetic field. We may then use formulas (2) and (3) to calculate the refraction of the rays. Let first the ionization be proportional to the square of the height so that

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[3] E. V. Appleton. Proc. Phys. Soc., London, 37: 22D (1925).

[4] H. W. Nichols and J. C. Schelleng. Bell System Tech. J., 4: 215-234 (1925).

$$a = by^2$$

The two refractive indices corresponding to (2) and (3) are then

$$\left[1 + \frac{by^2}{\nu(\nu_0 - \nu)}\right]^{-\frac{1}{2}} \text{ and } \left[1 - \frac{by^2}{\nu(\nu_0 + \nu)}\right]^{-\frac{1}{2}}$$

Both are of the form $\sqrt{1 - a^2y^2}$

The differential equation of the path is

$$\frac{dy}{\pm\sqrt{\cos^2\theta_0 - a^2y^2}} = \frac{dx}{\sin\theta_0} \tag{4}$$

$$\left(a^2 = \frac{b}{\nu(\nu_0 + \nu)} \text{ or } \frac{b}{\nu(\nu - \nu_0)}\right)$$

leading to a solution of the type

$$y = \frac{\cos\theta_0}{a} \sin\frac{ax}{\sin\theta_0} \tag{5}$$

The phase of the wave at any point is $\varphi = \int \frac{2\pi\nu}{V} ds$, the integral being taken over the path followed by the ray. If φ changes by π, the interference of the ray with another one of constant phase is changed from constructive to destructive. We are interested, therefore, in the phase-difference $\Delta\varphi$ for the two rays if we are to determine the polarization of the wave at the receiving station. A change in the Earth's field δH produces a change $\delta\Delta\varphi$ in the phase-difference $\Delta\varphi$. If this change $\delta\Delta\varphi = \frac{\pi}{2}$, the direction of polarization of the resultant linear vibration is turned through 90°. We must obtain φ for either ray on the supposition that the position of the receiving and the transmitting stations is kept fixed. If the distance between them be l, formula (5) determines θ_0 by

$$\sin\theta_0 = \frac{al}{\pi}$$

Now, in general, if μ is the refractive index

$$\varphi = \frac{2\pi\nu}{c}\int \mu ds = \frac{2\pi\nu}{c}\int \sin\theta_0\left(1 + \left(\frac{dy}{dx}\right)^2\right) dx \tag{6}$$

provided the refractive index varies only with the height and provided it is unity at the surface of the ground. Using (5)

$$\varphi = \frac{2\pi\nu}{c} l \, \frac{1 + \sin^2 \theta_o}{2 \sin \theta_o} = 2\pi \frac{1 + \frac{a^2 l^2}{\pi^2}}{2\left(\frac{al}{\pi}\right)} \frac{l}{\lambda} \tag{7}$$

$$\left(\lambda = \frac{c}{\nu} = \text{wave-length}\right)$$

Keeping the range l and the frequency μ fixed, we suppose the Earth's field H to have undergone a variation δH. This changes a^2 on account of a change in ν_0 (see (4)). The change in φ due to δa is

$$\delta\varphi = \pi \frac{l}{\lambda} \frac{\frac{a^2 l^2}{\pi^2} - 1}{\frac{al}{\pi}} \frac{\delta a}{a} = \pi \frac{l}{\lambda} \frac{A^2 l^2 - 1}{A\, l} \frac{\delta A}{A} \tag{8}$$

Where $\pi A = a$, and

$$\delta\, \Delta\varphi = \delta\,(\varphi_2 - \varphi_1) = \pi \frac{l}{\lambda}\left[\frac{A_2 l^2 - 1}{A_2 l} \frac{\delta A_2}{A_2} - \frac{A_1 l^2 - 1}{A_1 l} \frac{\delta A_1}{A_1}\right]$$

It follows from (4) that

$$2\frac{\delta a_1}{a_1} = -\frac{\nu_o}{\nu + \nu_o} \frac{\delta \nu_o}{\nu_o}, \quad 2\frac{\delta a_2}{a_2} = \frac{\nu_o}{\nu - \nu_o} \frac{\delta \nu_o}{\nu_o}$$

Remembering also that $H^{-1}\, \delta H = v_o^{-1}\, \delta v_o$ we have

$$\delta\, \Delta\varphi = \frac{\pi}{2} \frac{l}{\lambda}\left[\frac{A^2{}_2 l^2 - 1}{A_2 l} \, \frac{\nu_o}{\nu - \nu_o} + \frac{A^2{}_1 l^2 - 1}{A_1 l} \frac{\nu_o}{\nu + \nu_o}\right] \frac{\delta H}{H} \tag{9}$$

Let $l = 0$, then

$$\delta\, \Delta\varphi = -\frac{\pi}{2} \frac{1}{\lambda}\left[\frac{1}{A_2} \frac{\nu_o}{\nu - \nu_o} + \frac{1}{A_1} \frac{\nu_o}{\nu + \nu_o}\right] \frac{\delta H}{H}$$

The quantity, A_2 may be obtained in this case from the height h to which the wave goes. It is easily shown to be for this case $A_2 = \frac{1}{\pi h}$ so that for zenith reflection

$$\delta\,(\Delta\varphi) = -\frac{\pi}{2} \cdot \frac{\pi h}{2} \frac{\nu_o}{\nu - \nu_o}\left[1 + \left(\frac{\nu - \nu_o}{\nu + \nu_o}\right)^{\frac{1}{2}}\right] \frac{\delta H}{H} \tag{10}$$

If h = 100 kilometers and $\lambda = 70$ meters, then

$$\frac{\nu_o}{\nu_o - \nu} \cong \frac{1}{2} \text{ and } \delta\,(\Delta\varphi) \cong -3.8 \times 10^3 \frac{\pi}{2} \frac{\delta H}{H}$$

Hence a change in H by $\frac{1}{3800}$ part of its whole amount would turn the plane of polarization of the downcoming beam through 90°. This would cause very severe fading even at a short distance from the transmitter for a change of 13γ (13×10^{-5} Gauss) in the Earth's field. A field one-third of that amount would cause appreciable fading. Variations of that order of magnitude in the Earth's field are very frequent and may, therefore, cause an appreciable amount of fading.

Although formula (10) applies strictly to the case of zenith reflection, it is a good approximation to (9) for considerable ranges because A_2l becomes equal to unity [for the numbers chosen above] if $l = 300$ kilometers. For one hundred and fifty kilometers (10) is still a reasonable approximation. This suggests a connection with the distance of maximum fading for broadcast wave-lengths. This may be determined in addition to the proportion of ground and reflected waves by the fact that for a certain l, $\delta\Delta\varphi$ in accordance with (9) must vanish. Since in formula (10) the phase-velocities (2), (3) have been used and since these velocities cannot be applied well for short-range transmission, we should expect the fading to be greatest some place between $l = 0$ and $l = 300$ kilometers.

Formula (7) gives also the change in the condition for interference if the magnetic field of the Earth stays constant and the frequency of the signal is changed. Using a^2 as given in (4) we get

$$2\frac{\delta a_2}{a_2} = -\frac{2\nu - \nu_o}{\nu - \nu_o}\frac{\delta\nu}{\nu}, \quad 2\frac{\delta a_1}{a_1} = -\frac{2\nu + \nu_o}{\nu + \nu_o}\frac{\delta\nu}{\nu}$$

and hence

$$\delta\varphi_2 = \frac{\pi l}{\lambda}\left[\frac{2}{A_2 l} - \frac{A_2^2 l^2 - 1}{2A_2 l}\frac{\nu_o}{\nu - \nu_o}\right]\frac{\delta\nu}{\nu}$$

$$\delta\varphi_1 = \frac{\pi l}{\lambda}\left[\frac{2}{A_1 l} + \frac{A_1^2 l^2 - 1}{2A_1 l}\frac{\nu_o}{\nu + \nu_o}\right]\frac{\delta\nu}{\nu} \qquad (11)$$

$$\delta\,(\varphi_2 - \varphi_1) = \frac{2\pi l}{\lambda}\left[\frac{1}{A_2 l} - \frac{1}{A_1 l} + \frac{1 - A_2^2 l^2}{4A_2 l}\frac{\nu_o}{\nu - \nu_o} + \frac{1 - A_1^2 l^2}{4A_1 l}\frac{\nu_o}{\nu + \nu_o}\right]\delta\nu$$

For short-range transmission this becomes

$$\delta\,(\varphi_2 - \varphi_1) = \frac{\pi}{2}\frac{\pi h}{\lambda}\left[-4\left(\sqrt{\frac{\nu + \nu_0}{\nu - \nu_0}} - 1\right) + \frac{\nu_0}{\nu - \nu_0}\left(1 + \sqrt{\frac{\nu - \nu_0}{\nu + \nu_0}}\right)\right]\frac{\delta\nu}{\nu} \tag{12}$$

With $\lambda = 70$ meters and $h = 100$ kilometers

$$\delta\,(\varphi_2 - \varphi_1) = \frac{\pi}{2} \times 1400\,\frac{\delta\nu}{\nu}$$

Hence if

$$\frac{\delta\nu}{\nu} = \frac{1}{1400}$$

the direction of polarization is rotated through 90°. This corresponds to a frequency change of 3,000 cycles per second. Hence even a frequency change of 1,000 should produce an appreciable effect on the intensity of the received signal.

Finally, we estimate for the above case the change in $\varphi_2 - \varphi_1$ which is produced by a change of ionization. We find in a way quite similar to that just used that

$$\delta\,(\varphi_2 - \varphi_1) = \frac{\pi}{2}\frac{l}{\lambda}\left[\frac{A_2^2 l^2 - 1}{A_2 l} - \frac{A_1^2 l^2 - 1}{A_1 l}\right]\frac{\delta N}{N} \tag{13}$$

where δN is the change in the number of electrons per unit volume. For short-range transmission this becomes

$$\delta\,(\varphi_2 - \varphi_1) = -\frac{\pi}{2}\cdot\frac{\pi h}{\lambda}\left(-1 + \sqrt{\frac{\nu + \nu_0}{\nu - \nu_0}}\right)\frac{\delta N}{N} \tag{14}$$

With the numbers used above, a rotation through 90° is produced if $\frac{\delta N}{N} = \frac{1}{1800}$. Since changes in the ionization are presumably responsible for changes in the Earth's field, it is reasonable to suppose that (9) and (14) should be combined in any actual case in order to give the resultant change.

In the above we have constantly referred to the short wave-length region ($\nu > \nu_0$). If the wave-length is long, i. e., if $\nu < \nu_0$, only the ray defined by (3) returns to the Earth. If this ray interferes with the ground, wave effects quite similar to those just discussed are produced. Formulas (9), (11), (13) must be modified, however, by omitting the terms in A_2. The order of magnitude of the effects is still the same.

Even though δ $(\varphi_2—\varphi_1)$ is the same for a considerable range according to (9), (11), (13), the fading produced may vary considerably over that range. Thus the value of $\varphi_2—\varphi_1$ is very important because it determines whether the electric vector of the downcoming wave is horizontal or not. The same fluctuation of H may produce, therefore, a re-enforcement of the signal in one place and a diminution in another.

Small frequency changes are, of course, known to produce an effect on the intensity of the received signal. It is also known that interrupted continuous waves show less fading on short-wave lengths than continuous waves.[5] It is somewhat difficult to point out definitely what has been happening in the experiments just mentioned. For short wave-lengths, however, it seems certain that the fading must be entirely due to the condition of the wave returned by the upper atmosphere. The high "resolving power" of the interference apparatus in that case demonstrates a behavior similar to the one discussed above. It cannot be interpreted on the basis of interference with the ground wave. It must be also remembered that experiments by Tuve and the writer (l. c.) showed that the reflected wave shows fading which is sufficiently great to explain an appreciable amount of the general fading.

The influence of magnetic storms on transmission has given opportunity for considerable discussion. However, it would be dangerous to maintain that the simple calculations given above explain the effects observed. Presumably magnetic storms are accompanied by large changes in the ionization of the reflecting layer. A complete explanation must, therefore, be based on formulas corresponding to both (9) and (13).

SUMMARY

The state of polarization of a radio wave returned by the reflecting layer is studied as a function of (a) small changes in the intensity of the Earth's magnetic field, (b) small changes in the frequency of the wave, and (c) small changes in the ionization of the atmosphere.

It is shown that for the special electron-distribution considered appreciable effects on the intensity of the signal are to be expected for fluctuations in the Earth's field of the order of five gammas, for changes in the frequency of the order of 1,000 cycles, and for changes in the ionization of the order of one part in 5,000. [The number applies to λ=seventy meters.]

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[5] R. BOWN, DEL. K. MARTIN, and R. K. POTTER. Bell System Tech. J., 5: 143-213 (1926); E. V. APPLETON and M. A. F. BARNETT, Proc. R. Soc., London, 109: 621-641 (1925); G. W. PICKARD, Proc. Inst. Radio Eng., 12: (2), 1924.

The dependence of fading on range indicates that there is a certain range of maximum fading which is of the general order of one hundred miles.

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CORRELATION OF STATIC WITH THE ATMOSPHERE

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The writer has had opportunity at sea to study static with relation to the atmosphere since the spring of 1924 on board the U. S. S. *Kittery*. During 1926, these observations were made with the assistance of the radio compass. The research originally started on board the U. S. S. *Kittery* is now being continued by the Hydrographic Office, Navy Department, in collaboration with the Bureau of Engineering, Naval Communications, and the Bureau of Aeronautics.

The purpose of this article is primarily to show that static has sufficiently definite relationship to the distribution of the atmosphere as plotted on the daily weather map to enable one by proper observations to make use of static in weather forecasting, and to make use of our present knowledge of atmospheric distribution and movement in static forecasting.

This problem must be considered from two different viewpoints. The first is by making use of instantaneous static intensities without reference to any directional properties that static may have. This might be called plotting the Highs and Lows of static such as we now plot Highs and Lows of barometric pressure. The first experiment conducted by the U. S. S. *Kittery* for the Navy Department before the introduction of the direction finder was on this principle.

Considering that a definite relationship is established between atmospheric structure or changing structure and static, facts ascertainable concerning static from one or more points will be of value in forecasting weather; and our knowledge concerning development and movement of atmospheric conditions will assist in making static forecasts. Add to this knowledge what directional properties static is found to have and we begin to realize the possibility of the value of a static observation to a condition remote from the station making the observation. It is this latter feature that will prove of greatest advantage to the mariner.

Two examples are given herewith of well-known meteorological conditions with contrasting types of pressure-distribution. One is a well-formed homogeneous High which covers the West Indies, the South Atlantic and East Gulf States, and the ocean-area adjacent thereto.

Figure 1 illustrates the static free and excellent communication-conditions attending this High. The ship at the time off Cape Haytien, Haiti, on March 16, 1927, with a two-kilowatt set had no difficulty in communicating with Port' au Prince, Haiti, Guantanamo, Cuba, San Juan, Porto Rico, Arlington, and New York.

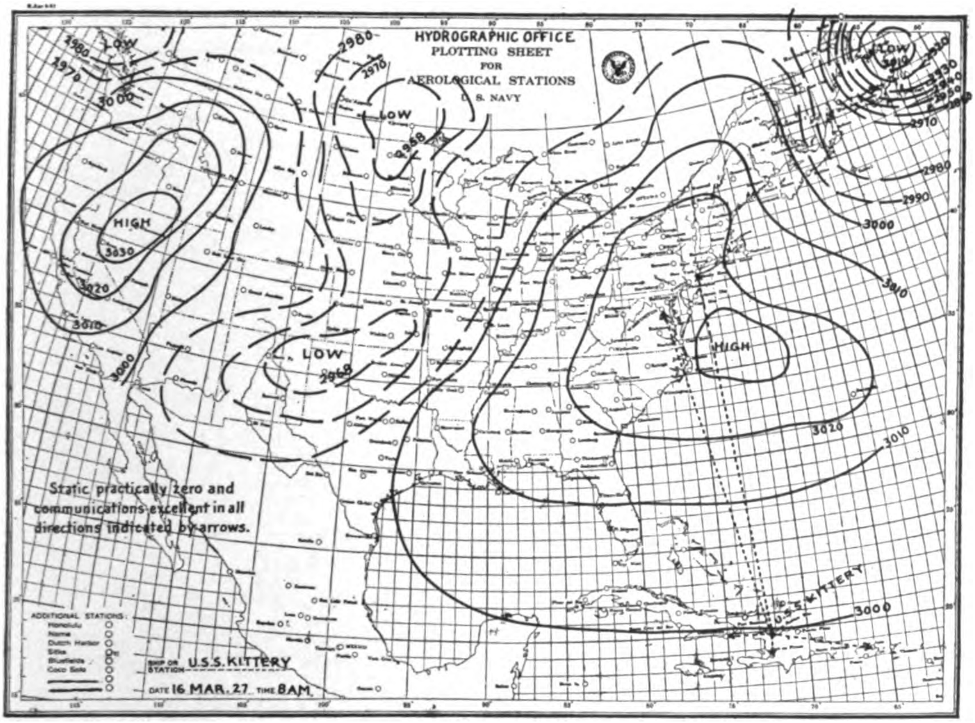


FIG. 1

Communication-conditions and isobars, March 16, 1927, at 8 A. M.

Figure 2, the other example, is of the same geographical area, but with a hurricane, the most intense type of Low (save tornadoes) occupying a part of the same area. The Low in this case is the historical Miami Hurricane shortly after it had passed over and devastated Turks Island. The position of the observer was the same during the observance of both state of weather and attendant radio reception-conditions, that is, off Cape Haytien, Haiti, in this case on the evening of September 16, 1926.

In the former, the High, static was practically zero and reception excellent in all directions over long ranges within the High. In the latter, the Low or Miami Hurricane, reception was impossible although attempts were made to communiciate with all stations indicated by arrows in Figure 2, namely, Arlington, San Juan, Guantanamo, Port au Prince, and even Cape Haytien, Haiti, which was only forty miles distant.

Our steering gear broke here as we were entering Tortuga Channel to obtain a better lee, and had an *SOS* been necessary, it would have been drowned out by the intensity of the static. Fortunately, the U. S. S. *Kittery* is a twin-screw ship, and we used our engines until the steering gear was repaired.

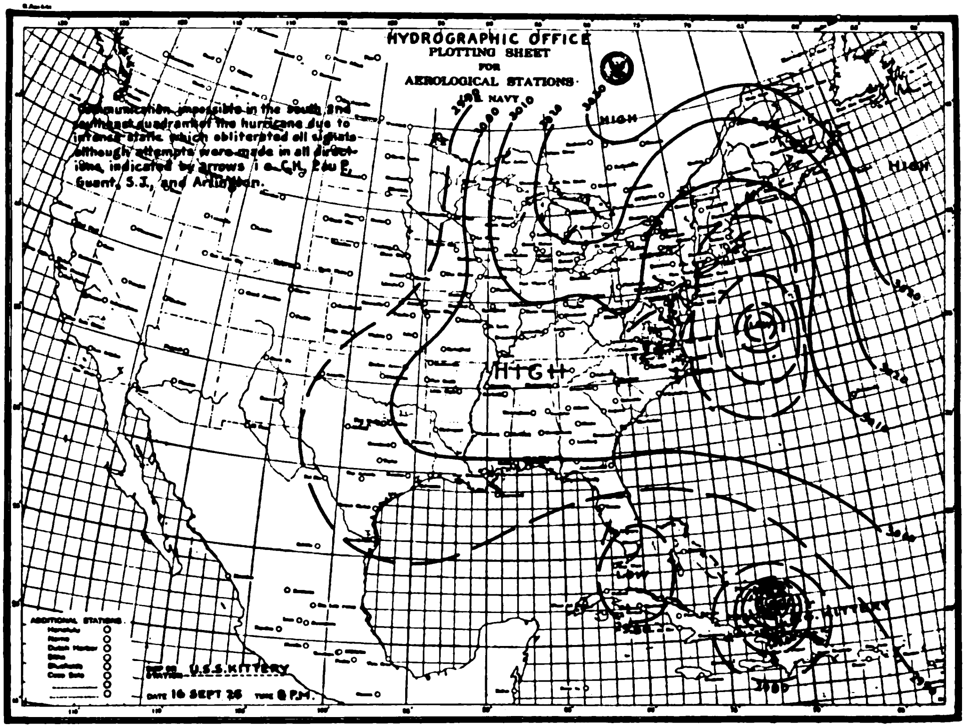

HYDROGRAPHIC OFFICE
PLOTTING SHEET
FOR
AEROLOGICAL STATIONS
NAVY
Communication impossible in the south and southeast quadrant of the hurricane due to intense static which obliterated all signals although attempts were made in all directions, indicated by arrows i.e. C.H., P au P, Guant., S.J., and Arlington.
HIGH
HIGH
HIGH
LOW
ADDITIONAL STATIONS
STATION U.S.S. KITTERY
DATE 16 SEPT 26 TIME 8 P.M.

FIG. 2

Communication-conditions and isobars, September 16, 1926, at 8 P. M.

Here we deal not with theory but with facts, namely, (1) that on one occasion static was terrific and the observer was within the 29.60-isobar southeast quadrant of a hurricane, and (2) on the other occasion the observer was within a homogeneous High and static was practically nil. The writer wishes to ask what this and similar phenomena would seem to indicate. To the observer in this case the same relative situations have recurred so often that he is of the opinion that Highs are relatively static free and Lows are attended with static somewhat proportional to the intensity of the disturbance. This seems to hold good from troughs of low pressure to hurricanes and from poorly defined Highs with mild static to great homogeneous Highs with practically no static. This High had traveled across the United States from the Pacific States where it was central on March 11. We have every reason to believe that the same relative good reception-condition had identical motion of translation. The rate of the motion of translation of Highs is shown in many publications of the U. S. Weather Bureau and authors of meteorological works (see Figure 3). With the same accuracy as Highs are now used in weather forecasting they can be used in forecasting the negative information concerning static conditions. The Low, or Miami Hurricane, had traveled at the rate of about three hundred miles per day from the doldrums well out to the eastward of the Lesser Antilles, possibly near the Cape Verde Islands. It is a certainty that this area of heavy static and impossible communication-conditions had traveled at the same relative position to the hurricane and at the same rate of three hundred miles per day.

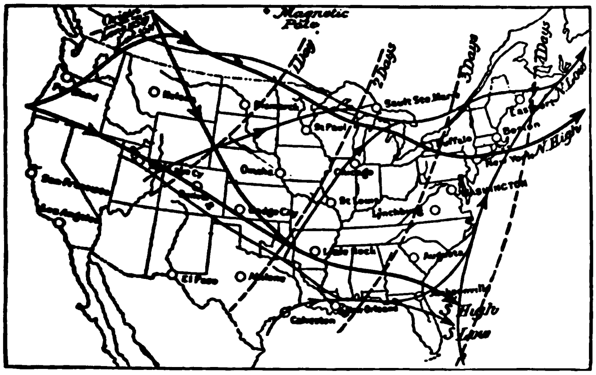


FIG. 3

The Bigelow system of storm-tracks across the United States.

Considering that these things are true and that Highs and Lows and our knowledge concerning their development, disintegration, and motion of translation is to be the basis for our static forecasting, we need next to study the structure of Highs and Lows with a view to locating that part of each which may under the most general conditions be expected to give the greater amount of static. We also want to know where within the High and Low to expect the minimum amount of static for that particular structure. In the Miami Hurricane, static was stronger in the southeast quadrant than in the west and southwest quadrants. In general, the eastern half of Lows will be more heavy in static than the western; the southeast, in general, than the northeast, and the southwest, than the northwest. On March 21 when the High mentioned above had moved to the eastward until the *Kittery's* position was in the southwest quadrant, static increased steadily.

In Highs it has been my observation that the center has the mildest static, the southeast quadrant less than the southwest, the northeast less than the northwest; especially is this true near the edge of the structure. Highs of different origin and hence different structure will have different static conditions associated with them. A Pacific High deep but of limited area will have a different static association from an Alberta High which is shallow but of great magnitude. And similarly Lows of different origin will have correspondingly different static attendant thereto.

The foregoing facts can be reconciled with results obtained by many who are studying static with the view of correlating it with the atmosphere. A few observations of others who have studied this subject abroad may be added here:

ABSTRACTS OF PAPERS ON THE METEOROLOGICAL RELATIONS OF ATMOSPHERICS

"In June, 1923, the Council received a letter from Mr. R. L. Latchem, of the cable steamer *John W. Mackay*, Nova Scotia, calling attention to the possibility of using, for the purpose of weather forecasting, observations of the character of those naturally occurring electro-magnetic waves, called atmospherics, which are so familiar to all wireless listeners. The Council appointed a small committee to consider whether observations on atmospherics might be of service in forecasting. As a preliminary to the work of the committee, the writer undertook to prepare the bibliography of the meteorological relations of atmospherics which is now reproduced here.

"The correlation between the occurrence of actual thunder or lightning and the radiation of atmospherics is so well known that many references to it have been omitted, but it is believed that no other well-authenticated data have been overlooked.

"It will be seen that very acute divergences of opinion exist as to possible relations between the intensity of atmospherics on the one hand and most of the phenomena definitely recognized as 'meteorological' on the other. There is an overwhelming consensus in favor of the obvious correlation with thunder, moderate agreement on the correlation with convective processes in the absence of reported thunder, but beyond this point the evidence is mutually contradictory. Reasons for this are not difficult to assign. It is established beyond doubt that the range of reception of atmospherics may reach one or two thousand miles; it is most probable that the range frequently, if not usually, attains the length of the Earth's semi-circumference. Comparisons between received atmospherics and local weather will therefore, in part be a comparison of the weather of a parish with the electrical phenomena of a hemisphere. Moreover it is still impossible to find a scale and a classification for the intensity and character of atmospherics which shall be generally acceptable, unambiguous, and capable of assessment by the average observer. The Beaufort of atmospherics is not yet.

"Nevertheless, the evidence that the atmospheric was well, if rashly, named is accumulating rapidly, and the summaries of the most recent work show that the location of 'cold fronts' by radiotelegraphic observations on atmospherics is an established possibility.

R. A. W. W."

Jackson, H. B.—London, Proc. R. Soc., 70 (A), 1902, p. 266.

After describing typical *X*-effects from lightning, remarks that *X*'s are more frequent in summer and autumn than in winter and spring, near high mountains than in open sea, in south than in north winds (in the Mediterranean), in the front of a cyclonic disturbance than in the rear, with falling barometer than with rising barometer.

Notes case of a heavy winter-gale, without observed lightning, which was preceded by *X*-disturbance.

Crawley, C. G.—London, Elect., 69, Jan. 31, 1913.

X's in Mediterranean are worst when pressure is low, temperature high, and humidity low. The sirocco forms an exception to this rule, as it always brings atmospherics.

Erskine-Murray, J.—London, Elect., 67, 1911, p. 219.

Describes observations showing that every observed lightning-flash produces an *X*, multiple flashes producing multiple *X*'s, and discusses the adequacy of the known thunderstorm distribution to account for all *X*-disturbance. The author proposes direction-finding for the location of *X*-sources.

Flajolet—Paris, C.-R. Acad. Sci., 154, 1912, p. 729.

Describes the recording of thunderstorms up to 500-kilometer distance by crystal detector and galvanometer.

Franck-Duboquier.—Nature, Paris, 41 (1), 1913, p. 218.

On the basis of one year's observations at Anche (Indre et Loire) the relations between *X*'s and weather are thus enunciated (the original French descriptions being retained to avoid ambiguity by loss of onomatopoeia).

(1) Violent "craquements" indicate a thunderstorm, approaching if the *X*'s become more and more frequent, receding if they become less frequent or less strong.

(2) A slight "sifflement" is produced by a heavy hailstorm passing near the receiver.

(3) Dry, isolated, and rather weak "claquements" generally precede a fall of temperature, a spring frost.

(4) If the wind is about to turn, the *X*'s are of short wave-length, and seem to come in strings.

(5) Numerous "crepitements," accompanied by fairly regular, strong and "fusant craquements," precede great barometric depressions and foretell gales.

Lutze.—Phys. Zeit., Leipzig, 14, 1913, p. 1194.

Finds that X's and changes of recorded potential-gradient go together. Balloon-observations show great increase of X's inside cloud, and a rapid decrease of X's with increasing height.

Delval.—Quoted in Perrett Maisonneuve's "La T. S. F. et la Loi" (1914).

Observations at Mt. St. Aubert, Tournai, May to September, 1912, led to the following conclusions: Absence of X's characterizes approach of fine weather; little precipitate cracklings—hail or heavy rain; distant cracklings—fine weather; numerous and prolonged cracklings—stormy weather; prolonged sound like water from a gutter—sudden change of weather; prolonged and singing sound seeming distant at first, approaching little by little till strong—violent thunderstorm or tempest.

Marriott, R. H.—Proc. Inst. Radio Eng., New York, 2, 1914, p. 37.

An outburst of X's follows a sudden change of wind on the North Atlantic coast of the United States, especially a change from south to north. X's also accompany a rise of temperature there.

Radiotelegraphic Investigations Committee of the British Association.—London, Rep. Brit. Ass., 1915.

X-storms coincide with convective weather, rapid fluctuations of pressure, and rapid movements of depressions.

In Malta, X's precede by several days the advent of convective weather.

In Australia, day-time rain is preceded by X's in 80 per cent of all cases.

In Ireland X's are bad with a northwest wind on the Atlantic coasts.

In Sierra Leone the periodic dry wind suppresses X's.

Goldschmidt, R.—"La Telegraphie sans Fils au Congo Belge" (Brussels, 1920).

From observations in the Belgian Congo, 1915 to 1917, the author believes that X's from lightning have an effective range not exceeding 100 or 200 kilometers.

In general a day of thunderstorm is preceded by several days of increased X-disturbance.

In general a day of thunderstorm is followed by several days of reduced X-disturbance. This law is more general than the former.

Wiedenhoff, S.—Jahrb. drahtl. Telegraph., Berlin, 18, 1921, p. 242.

Low X-intensity is found under rainless cloud, X's increase as cloud begins to break up or as rain begins. Diminishing convection-current goes with low X-intensity.

From a comparison of annual variation-curves the author summarises as follows: X-maxima go with low potential-gradient, but high convection-current and high numbers of ions per cc., and conversely for X-minima.

From diurnal-variation curves: X's are bad when the vapor-pressure is at a maximum at the surface and increasing above, when the conductivity at ground and aloft is increasing, and the surface wind decreasing. X-disturbance is at a minimum with maximum conductivity at a moderate height, with overcast sky, with maximum relative humidity, minimum temperature, and maximum air-pressure.

MARCHAND, H.—Rev. Gen. Sci., Paris, 32, 1921, p. 594.

Increased X-frequency indicates an imminent thunderstorm; if the intensity of X's is great the thunderstorm will be violent, with wind and rain; if the X-intensity remains low the thunderstorm is feeble. If the intensity augments the storm is approaching, and vice versa.

This diagnosis is valid at great distances, before the cloud associated with the storm has appeared at the receiving station.

ROTHE, E.—Paris, C.-R., Acad. Sci., 175, 1921, p. 840.

Infrequent but sustained X's with an intense high metallic note are due to more or less distant cloud. Visible lightning, and hail, give this type of X.

Lacoste (q. v.) is summarised.

LACOSTE, J.—Paris, C.-R., Acad. Sci., 175, 1921, p. 843.

Enunciates laws based on radiogoniometric observations at Strasbourg.

(1) If a distinct well-developed depression exists, with closed concentric isobars, the maximum X-disturbance proceeds from the south or southeast of the depression, the displacement in direction and variation of intensity allow of following the course of the depression.

(2) If the depression has a distant centre and less curvature, the X-maximum is still south or southeast of the periphery, but is less sharply marked.

(3) Secondary depressions, with "barometric pockets" and cols, correspond to storm-fronts, the direction of X-maximum is difficult to find, and X's are violent from all azimuths.

(4) A near thunderstorm gives violent X's from its azimuth.

Many X's received in clear weather are comparable in intensity to those of observed lightning. The author concludes that X's are due to more or less distant storm-phenomena, and are often produced in overcast regions.

The last few days' work in the period in question indicates that fog facilitates X-transmission.

ROTHE, E.—Ann. Phys., Paris, 17, 1921, p. 385.

X's come from definite azimuths when storm-clouds are distant, but are observed on all azimuths when the storm is very near.

Storm-clouds produce X's which gradually cease as soon as uniform rain starts, but X's persist to some extent when rain is violent.

Strato-cumulus most often furnish well-defined and oriented rumbling X's.

Re-summarises Lacoste (q. v.).

LACOSTE, J.—Paris, C. R. Acad. Sci., 176, 1922, p. 707.

Describes further work at Strasbourg in summer of 1922, confirming previous conclusions, and demonstrating possibility of forecasting new depressions. Remarks on the special value of the radiogoniometer in thunderstorm-forecasting.

Some thunderstorms give *X*'s which are numerous only on short-wave observations.

Watson Watt, R. A.—Nature, London, 1922, p. 680.

Preliminary results of directional observations on atmospherics during 1916 to 1918, by stations on British coasts, show that on an average five apparent sources of atmospherics per week were located. Of those checked against meteorological data, fifteen per cent agreed with reported thunderstorms, ten per cent with squall phenomena, sixty-nine per cent with rainfall within the twenty-four hours containing the time of observation, without reported thunderstorms or squalls. Thus ninety-four per cent of the apparent sources were associated with rainfall. Thirty-five per cent of the cases examined fell on the forward edge of a rain-area.

Wolf, F.—Jahrb. drahtl. Telegraph., Berlin, 19, 1922, p. 289.

Gives results of observations at Heidelberg, September, 1919, to August, 1921, on 2,000 and 12,500-metre wave-lengths. In the warm half-year *X*'s developed on the appearance of a new depression over northwest Europe, disappeared on its departure or filling up.

Stoye, K.—Jahrb. drahtl. Telegraph., Berlin, 20, 1922, p. 303.

Observations in the plains of Upper Rhineland, 1915 to 1917, on 600 to 2,000-metre wave-lengths, led to a classification of *X*'s in ten groups.

(1) Prevailing weather-type, a whistling sound decreasing in strength, ending almost in a whisper.

(2) Must type, weak hissing sound.

(3) Cloud break-up type, repeated groups in the form of explosive crashes. Specially marked on break-up of alto-cumulus.

(4) Cirrus type, "trrrrtasss" (palatal *r*), increasing as cirrus gathers, continuing till cirrus ceases to increase.

(5) Lightning type "rrrrrrssss" (palatal *r*), a more rapid *r* succession than in (4).

(6) Front type (squally wind), creak like a revolving clapper.

(7) Cumulus-development type, short single sharp cracks.

(8) Thunderstorm-cumulus type, sharp cracks of different strength, coming in groups.

(9) Sunrise type, violent rattling and scratching of longer duration, not in group form.

(10) Sunset type, superposition of (3), (9), and other types, partly due to surface cooling and condensation, partly to cloud break-up.

Every stage in the development of a thunderstorm released a definite type of disturbance. Groups 4 and 9 are superposed in thunderstorms.

The lightning-type was observed at night with perfectly clear sky.

The increase in disturbance-numbers started several hours before the onset of the thunderstorm, though there were exceptions to this rule.

In spite of unmistakable signs of thunderstorm the number of disturbances remain normal until a sudden release occurred.

Strong disturbances were found to come from strata of misty haze and also from forests with their well-known summer cloud-formation.

Direction-finding differentiated between sharply directed *X* and those showing little directivity. Localized thunderstorms and squalls could be located in azimuth.

Watson Watt, R. A.—London, Wireless World and Radio Rev., 12, 1923, p. 601.

Summarises work on *X*'s under the Meteorological Office and the Radio Research Board, gives examples of the location of thunderstorms up to 2,400 kilometers distant, by direction-finding on *X*'s, and discusses the relations of *X*'s, rainfall, and lightning.

Bureau, R., and others.—C.-R. Acad. Sci., 178, 1924, pp. 556, 1623; 179, 1924, p. 394; 180, 1925, pp. 529, 1122; Onde, Elec., 4, 1925, pp. 31, 58.

Atmospherics are the phenomenon that best gives evidence of the passing of meteorological disturbances in tropical regions; the other meteorological variables only indicate that passing in a much less regular way; moreover, they only give evidence of the meteorological disturbance when it has reached the observing station, while atmospherics announce it some hours in advance.

Lugeon, J.—Paris, C.-R. Acad. Sci., 180, 1925, p. 594.

Observations at Lausanne, 1919 to 1922; at Zurich, 1924.

Preliminary conclusions are stated thus:

"(1) Intensity, frequency, nature, and direction of *X*'s vary with altitude.

"(2) This variation is closely connected with variations of vertical temperature-gradient. Cooling augments, heating diminishes the phenomenon.

"(3) Lower layers are much more disturbed than higher.

"(4) Apart from *X*'s of proved distant origin, range of *X*'s originating in the lower layers does not seem to exceed 250 km., and remains almost always under 150 km. in mountainous regions."

Watson Watt, R. A.—London, Proc. Phys. Soc., 37, 1925, p. 23D; London, J. R. Aero. Soc., 29, 1925, p. 170.

Shows correlation between apparent sources of atmospherics and their meteorological environment. Of 490 cases examined, twenty-five per cent fell within 250 km. of reported thunder, a further twenty-eight per cent were associated with more distant thunderstorms or with squall phenomena, twenty-one per cent more with rain, thirteen per cent uncorrelated.

Reports the tracking of a cold front for 2,700 km. by automatic recorders of *X*'s.

Most of them seem to have correlated static with a local condition. I am trying to show how those local conditions are carried along with the general drift of the atmosphere and how static from them can be used in weather-forecasting.

I have listed the above facts in this particular manner in order to show how much can be forecast concerning static once its relationship to High and Low structure is well established. From our knowledge of the development and motion of translation of Highs and Lows we have the basis of a static forecast which will be enhanced by an instrument which will make use of the directional properties of static.

TRACKING THE MIAMI HURRICANE

Figure 4 shows the Miami Hurricane as it was passing over the Florida Peninsula. Here it will be seen that from Guantanamo where the U. S. S. *Kittery* was then moored the heaviest static was in the general direction of the southeast quadrant. The center of the storm was six hundred miles distant.

Figure 5 shows the location of the centers of the Miami Hurricane from the time it was first observed to the eastward of the Lesser Antilles to the time it filled up and dissipated over Mobile, Alabama. Did not this static belt drift along with the hurricane? How far will a direction-finder be able

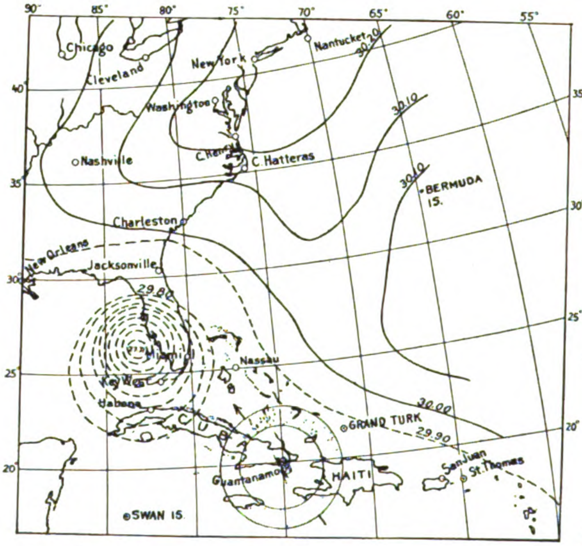


FIG. 4

Miami Hurricane passing over the Florida Peninsula.

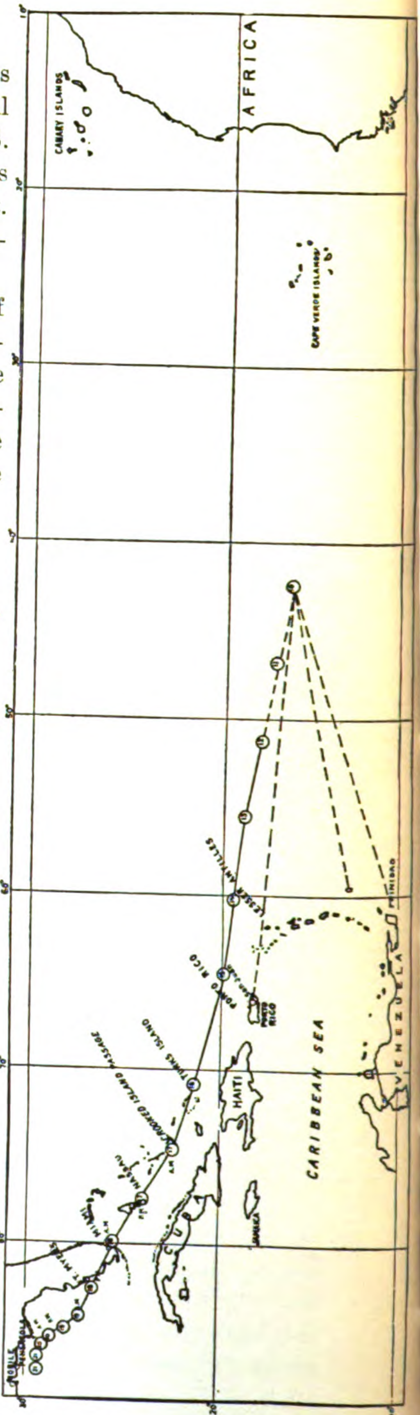


FIG. 5—*Right*

Diagrammatic history of Miami Hurricane including illustration of proposed use of radiocompass in locating and tracing such storms.

(Known path of Miami Hurricane extended through doldrums to easternmost breeding-place of hurricanes with known part of path shown by solid line.)

to reach out with sufficient precision to make its use practical in habitually tracking such storms. The dotted lines show where the system may enable us to reach out in the region to the eastward of the Lesser Antilles, where barometer-readings are scarce, and to triangulate such a static center in hurricane season from St. Thomas, Barbados, and Trinidad.

THE DOLDRUMS, BREEDING-PLACE OF HURRICANES

Figure 6 shows the ideal atmospheric distribution for summer from which the hurricane-belt, the doldrums, may be studied. In addition to those hurricanes that form to the eastward of the Lesser Antilles, we see the breeding-place for the early spring and late fall type that form around the Isthmus of Panama between the North Atlantic or States High and the South Pacific High. The selection of compass-stations here as elsewhere will depend upon the proved range of accuracy of the radio compass for this kind of work.

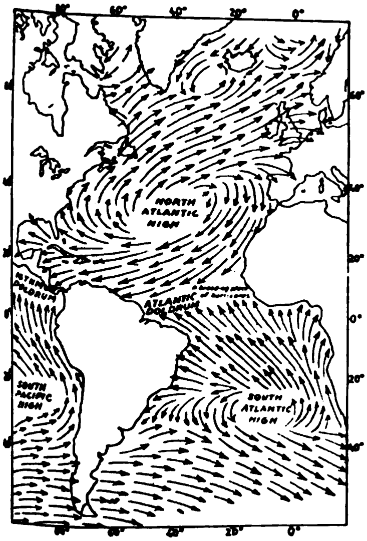


FIG. 6

Breeding-places of hurricanes—doldrums.
(After Bartholomew's "Physical Atlas," volume 3, plate 14.)

Anyone working a field as new and undeveloped as this naturally finds comfort in another independent source wherein his conclusions have been verified. Mr. Friis of the Bell Telephone Research Laboratory has taken observations with an instrument developed at that laboratory over a period extending from last August up to the present date. The laboratory very generously cooperated with the Hydrographic Office in this work and I show you here the results of a part of their observations which substantiate work previously done by me.

Figure 7 shows the High central at Cliffwood, New Jersey, November 25, 1926. Static was zero during the passage of this homogeneous High.

Following its passage, the Low shown in Figure 8 in its motion of translation to the eastward passed over the same station. Static rose in intensity from one to over four hundred microvolts per meter during

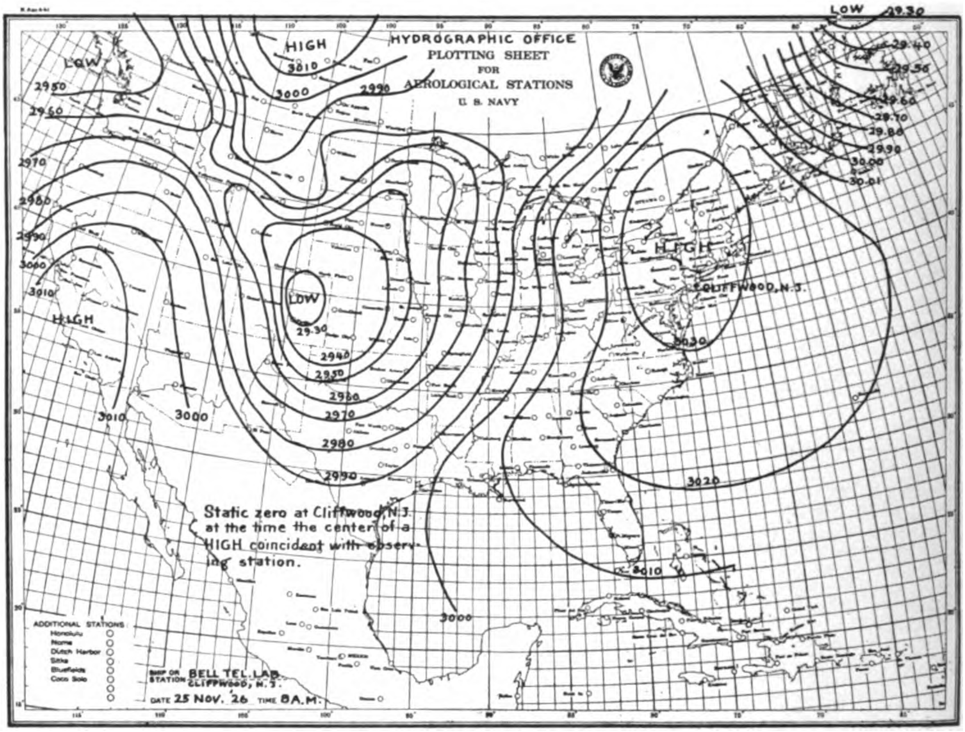


Fig. 7

Communication-conditions and isobars, November 25, 1926, at 8 A. M.

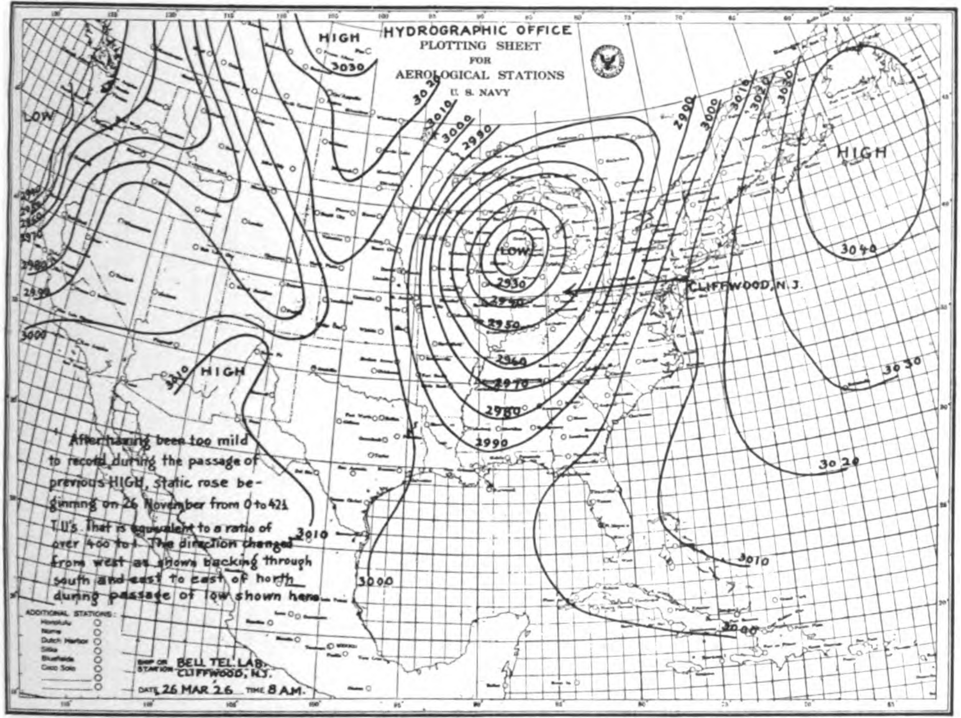
its passage. The direction shifted consistently from west as shown to north of east during the passage.

The graph of direction and intensity of static shown in Figure 9 was received by their machine.

Some idea of the effect of this static intensity on the received signal is gained when the reader realizes that the Bell Telephone System try to maintain for trans-Atlantic work a received signal-strength for code of five microvolts per meter and for telephony of fifteen to twenty microvolts per meter.

The Hydrographic Office is pleased to see its desire of a year ago so quickly fulfilled by commercial enterprise. We said then that an instrument that would record both direction and intensity of static would be necessary to the successful accomplishment of this work. Dr. Austin at the Bureau of Standards has also developed a fine automatic static recorder and there are still others.

The recorder we used is shown in Figures 10 and 11. It is slightly



After having been too mild to record during the passage of previous HIGH, static rose beginning on 26 November from 0 to 42½ T.U's. That is equivalent to a ratio of over 400 to 1. The direction changed from west as shown backing through south and east to east of north during passage of low shown here.

Fig. 8

Communication-conditions and isobars, March 26, 1926, at 8 A. M.

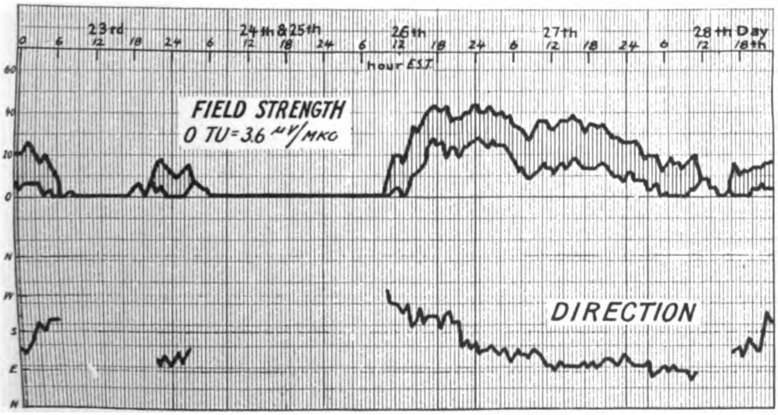
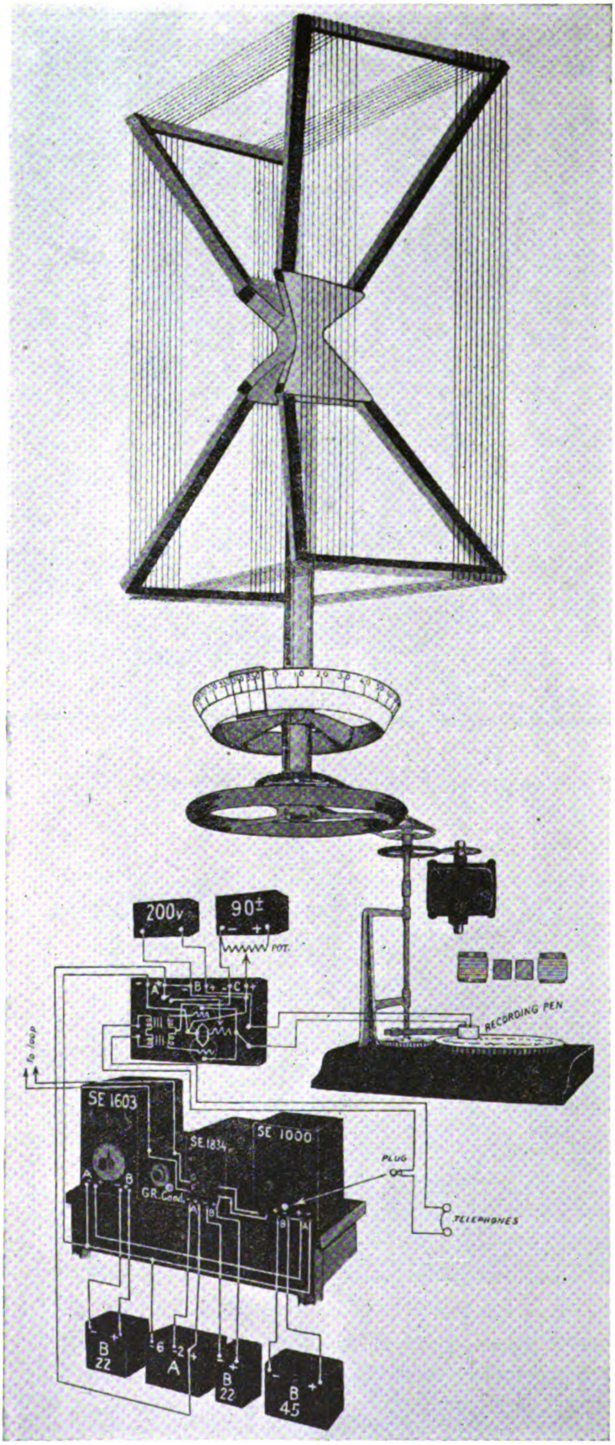


Fig. 9

Intensity and direction of static, November 23 to 28, 1926, as recorded at Bell Telephone Research Laboratory at Cliffwood, New Jersey.



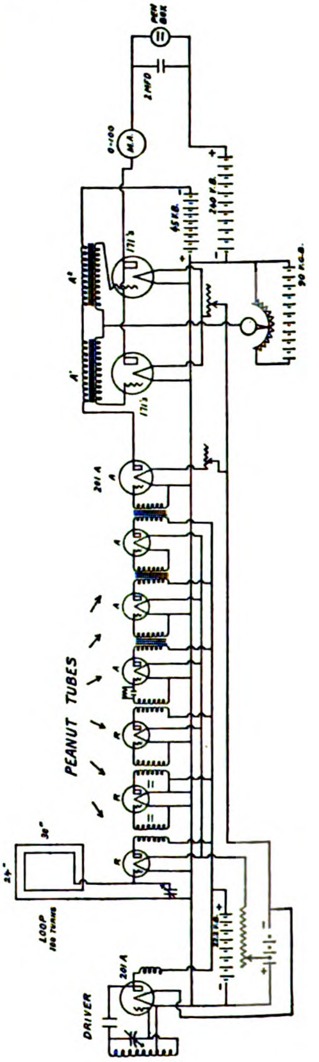


Fig. 10—*Left*

Kincaid static-recorder.

Fig. 11—*Above*

Schematic diagram of Kincaid static-recorder. (U. S. S. *Kittery* installation.)

different from the hook-up we used at Bellevue Laboratories which was as follows:

The loop, an *SE*-515*A*, was six feet high, four feet three inches wide, and consisted of ninety turns, spaced over an axial length of four feet three inches. The loop center was grounded.

The terminals were connected to an (*SE*-1934) amplifier and tuned by a 0.01 m. f. d. variable condenser (*SE*-1830).

The amplifier had three radio frequency-stages, a detector, and two audio frequency-stages of amplification.

An additional audio frequency-stage was used, the output of the latter being connected in parallel and the negative grid-bias being adjusted for zero plate-current, at zero signal, by means of a potentiometer. The plate-current of the two tubes actuated a pen which was shunted by a one m. f. d. condenser.

The instrumental combination described above results in depression of the pen for each static crash.

The pen records on a card, which rotates very slowly in synchronism with the compass-loop, two maxima and two minima being indicated for 360° of rotation (a bilateral loop-operation). Some unilateral experiments were conducted at Bellevue.

The motor was so geared as to be able to make one revolution in six to ten minutes. From this it can be seen that each of the inclosed static graphs contains from three to ten complete revolutions of the compass and consumes from twenty minutes to one and one-half hours.

The recorder used at Cliffwood by the Bell Telephone Laboratories is shown in the schematic diagram, Figure 12. It is an ordinary double detection-set that requires altogether ten tubes, of which the last low-frequency amplifier-tube must be able to handle ten watts in order to prevent overloading. The power-supply may be rectified alternating-current.

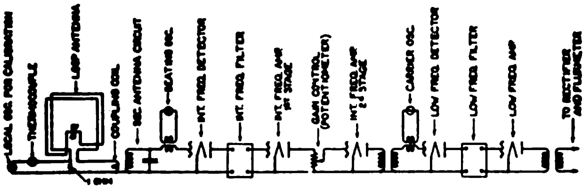


FIG. 12

Schematic circuit-diagram of receiving set.

The gain control is inserted in the first intermediate frequency-amplifier in order to be sure that no tubes are overloaded. The local oscillator shown is used for amplification-calibration of the set and requires no special shielding as its input voltage induced into the loop is comparatively large.

The selectivity of the set is determined by three separate units, viz., the antenna-circuits, the intermediate-frequency filter and the low-frequency filter, each of which has a specific use. Carson and Zobel[1] have made the following statement.

"In filters designed to select a band of frequencies of width *W*, the ratio of energy transmitted through the network by the signal and by random interference is inversely proportional to the band-width and increases inappreciably when the number of sections is increased beyond two."

The main purpose of the filters is therefore not to define the frequency-band of the set in so far as static is concerned, but to exclude continuous wave-interference. It is hoped that five hundred cycles wide frequency-bands[2] can be maintained free of c. w. interference for static measurements, and the simplest way to obtain such a band in the receiver is to make the low-frequency filter an efficient low-pass filter that cuts off every frequency above six hundred cycles. More than two coupled circuits are hardly required in the antenna-circuits, but the intermediate-frequency filter ought to have sharper cut-off points than two coupled circuits will give. The selection of filters naturally depends upon the c. w. interference and it may in some cases be possible to reduce the number of filters and thereby make the recorder cheaper. The records shown later correspond to a frequency band of 2,000 cycles—between 57.5 and 59.5 k. c.—but it will probably not be long before c. w. interference makes it necessary to reduce this band-width. It is desirable to have a loud speaker connected to the output of the set and occasionally listen for c. w. interference.

The constant output control-apparatus is shown in Figure 13. The fluxmeter is seen in the upper right corner. Full deflection corresponds to 2×10^{-4} coulomb. The needle is normally free to move except when the cam *Z* presses the needle down until its point touches the scale *OS*. The shaft carrying the cam *Z* and the disc *N* is rotated one complete turn in fifteen seconds by a clock-motor. The different elements are explained in the figure and the whole action may be understood by studying this carefully. However, it is probably worth while to go through a complete fifteen-second period and explain in detail the purpose of each part.

Time in seconds, 0 to 10:

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[1] JOHN R. CARSON AND OTTO J. ZOBEL, "Transient Oscillators in Electric Wave-Filters," Bell System Technical Journal, 2: (3) 27.

[2] Bands at 15, 30, 60, 120 kilocycles would probably be satisfactory.

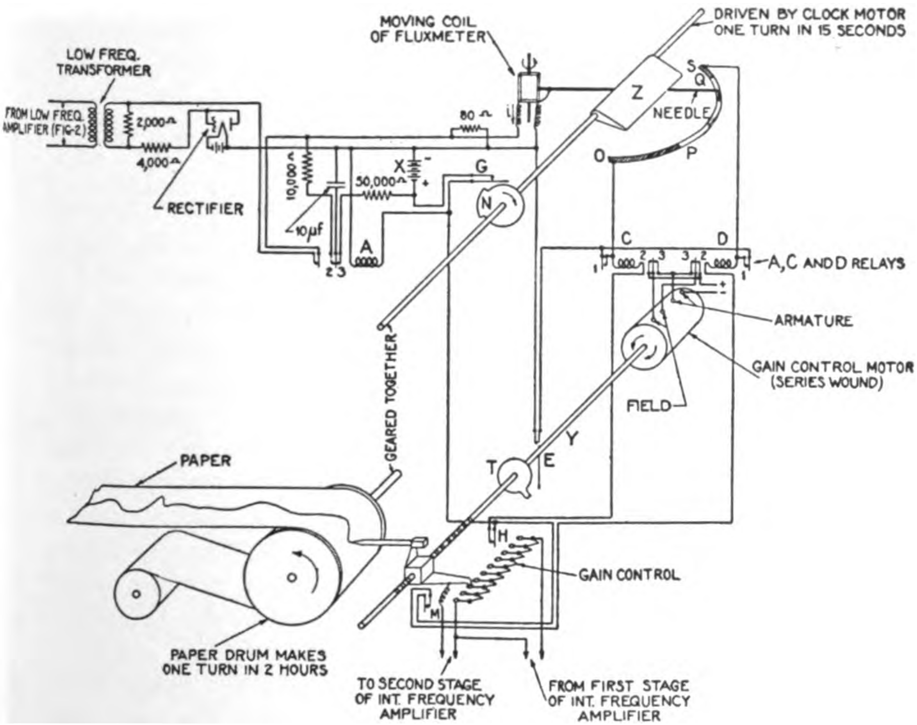
MOVING COIL OF FLUXMETER
DRIVEN BY CLOCK MOTOR ONE TURN IN 15 SECONDS
LOW FREQ. TRANSFORMER
FROM LOW FREQ. AMPLIFIER (FIG-2)
2,000Ω
4,000Ω
RECTIFIER
80Ω
10,000Ω
10μf
50,000Ω
X
G
N
A
2 3
Z
S
Q
NEEDLE
O
P
C
D
1 2 3 3 2 1
A, C AND D RELAYS
ARMATURE
GAIN CONTROL MOTOR (SERIES WOUND)
FIELD
GEARED TOGETHER
PAPER
Y
T
E
H
GAIN CONTROL
M
PAPER DRUM MAKES ONE TURN IN 2 HOURS
TO SECOND STAGE OF INT. FREQUENCY AMPLIFIER
FROM FIRST STAGE OF INT. FREQUENCY AMPLIFIER

Fig. 13

Schematic circuit of constant-output control apparatus.

Switch *G* is open, therefore relay *A* is open and no current can pass through the windings of relay *C* and *D*. (These relays start the gain control-motor, which is therefore shut off.)

Contact 1 of relay *A* is closed and closes the circuit consisting of the secondary winding of the low-frequency output transformer, the rectifier for the static currents and the fluxmeter. The 2,000 and 4,000-ohm resistances in this circuit insure distortionless input voltage to the rectifier. The fluxmeter is damped by an eighty-ohm shunt. The needle, which was initially at zero, will therefore move, its deflection being proportional to ***idt***.

Time in seconds, 10 to 14:

Switch *G* is closed by the cam on the revolvng disc *N* and locks relay *A*.

Contact 1 of relay *A* is opened and opens the rectifier fluxmeter-circuit, thereby bringing the fluxmeter-needle to a stop.

Contact 3 of relay *A* is closed and makes the battery *X* charge the ten-μf condenser through the 50,000-ohm resistance.

Time in seconds, 11 to 14:

The cam *Z* presses the needle-point down on the scale *OS*. Now, one of three things will happen.

(1) Static has decreased since the last period, so that the needle point will make contact with the metal strip *OP* and close the following circuit: Battery *X*, needle of fluxmeter, winding of relay *C*, switch *H*, and switch *G* to battery *X*. Relay *C* is therefore closed and its closed contact 2, together with contact 3 of the open relay *D* will start the gain control-motor. After approximately half a turn of the gain-control or motor shaft *Y* the needle point is lifted from *OP* by the rotation of the cam *Z*, but relay *C* stays closed due to the fact that it is self-locking through its contact 1, so that the shaft *Y* continues turning until the switch *E* is opened by the disc *T*. This opens the self-locking circuit of relay *C*. Relay *C* therefore opens and the gain control-motor stops after the shaft *Y* has made exactly one complete turn and increased the gain of the set one step (4 transmission units or 1.58 times). Notice that the opening of the needle-point contact does not break any current, due to the use of self-locking relays. This preserves the needle-point contact.

(2) Static has not changed since the last period. The needle point will now touch the insulating strip *PQ* and nothing else will happen, i. e., the gain of the set remains unchanged.

(3) Static has increased since the last period so that the needle-point now will make contact with the metal strip *QS* and close relay *D* and as in case 1 the motor will start and turn the shaft *Y* one turn, but this time in the opposite direction, i. e., the gain of the set is decreased one step.

Time in seconds, 14:

Switch *G* is opened again by the revolving disc *N* and opens relay *A*. Contact 2 of relay *A* is closed and will discharge the 10-μf condenser through the fluxmeter, thereby bringing the needle back to zero. (Notice that the time-constant of this discharge-circuit is $10{,}000 \times 10 \times 10^{-6} = 1.10$ second.)

Time in seconds, 15:

A new period has started.

The purpose of the switches *M* and *H* is to stop the motor when the gain control-switch arm has reached the end of the scale.

The recorder is of such recent development that no comprehensive data are yet available.

Figure 14 shows part of an actual record of the static received on a set tuned to 57.5 to 59.5 kilocycles. The ordinates represent the attenuation of the gain control of the set, and it is to be remembered that the gain of the rest of the set is constant. The curve shows that the static power on the morning of October 30 changed more than 10,000 times. The point *B* on the curve gives the effect of inducing a local signal of strength 380 microvolts—m in the loop.[8] The point *C* on the curve shows that at 8:25 A. M. the static intensity received on a 2,000-cycle wide frequency-band corresponded to the energy received from a c. w. signal of strength 3.8 microvolts—m. It would be practical always to relate static to such a c. w. signal. Experiments are now being conducted to determine whether the energy received from static is proportional to the width of the frequency-band of the receiving set and if such is found to be the case then it is proposed to have the data relate to a 1,000-cycle wide band. That static is, say, seven microvolts per meter per kilocycle (seven microvolts—mkc) would then mean that the energy of the static received on a 1,000-cycle wide frequency-band is the same as the energy received from a c. w. signal of strength seven microvolts—m.

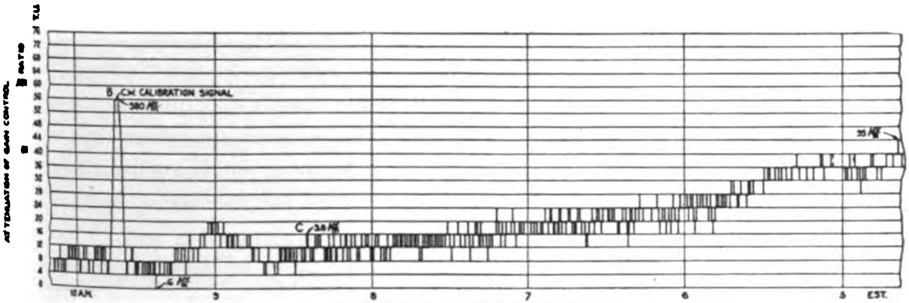

Fig. 14

Static record, morning of October 30, 1925, at Cliffwood, New Jersey.

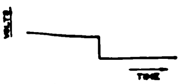

Fig. 15

Shape of impulse voltage.

Attempts have been made to calibrate the set by inducing in the loop, voltages of the shape shown in Figure 15. Relating static to such signals would have the advantage of being independent of the band width of the set. Such signals were obtained by closing and opening a mercury-switch, but one signal per second, or ten impulses per period, would overload the set (the tubes) very much. At least ten impulses per second would be required if the set should not be overloaded by each individual impulse, but this would be a difficult task to accomplish and it is therefore recommended that static be measured as explained above, by inducing a local c. w. signal into the loop. The fact that five static crashes in the course of ten seconds—one period—do not overload the set while one hundred impulses of the shape shown in Figure 15 are required to prevent overloading gives us some interesting information on static. It shows that a single static crash is not a single sudden change of the field in the ether and that it can not be represented by less than twenty consecutive impulses.

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[8] It may be worth while to have such a calibration-signal introduced automatically, for instance once every two hours.

The record of Figure 14 shows that each step on the gain control-potentiometer is four *TU* and the selection of such steps and of fifteen seconds will now be discussed. To decrease the four *TU* step to a one *TU* step would decrease the speed of the set, i. e., it would take four times longer for the recorder to register a sudden change in the static level which is particularly a disadvantage when the recorder is connected to a rotating directional antenna. On the other hand a step larger than four *TU* would not give the static level with sufficient accuracy. If the time-periods are changed from fifteen to ten seconds, then the "speed" of the set is increased, but the set is then inoperative over a large part of the period since it takes five seconds to change the gain of the set and bring the fluxmeter-needle back to zero. Besides, such a decrease in time-period would increase the probability of overloading and also it would make the energy received per period vary more irregularly especially if static consisted of separate crashes.

Table of conversion of transmission units to ratios

T. U.	Ratio
0	1
6	2
12	4
20	10
40	100
60	1,000

Dr. Friis in describing his static recorder says, "The reason for the small advance which has been made to date in the automatic recording of static is probably due largely to the lack of suitable apparatus. Certainly there has never been any doubt that automatic records would be very valuable. It is just as important to know the static level as it is to know the strength of a radio signal because it is the static to signal ratio that determines the intelligibility of the signal. A static recorder connected to a rotating directional antenna-system would tell us where static comes from and, therefore, enable the radio engineer to determine whether it is worth while to construct a directive antenna-system. Also the connection between thunderstorm-areas and static would make static recording valuable to the meteorological service. There is perhaps no reason why a suitable static recorder should not make it possible in a few years to obtain a daily static forecast just as we get our weather forecast now."

I believe another year will see possible such static forecast as Dr. Friis mentions, and with such forecasts will come a more exact knowledge of the relation of storms to static. The result of this will be the identification and tracking of many storms which are difficult especially over ocean-areas where barometer-readings are scarce, and the definite forecasting of communication-conditions.

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POSSIBLE RELATIONS BETWEEN EARTH-CURRENTS, EARTH-RESISTIVITY, AND SOME RADIO PHENOMENA

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Balfour Stewart in 1882 first suggested that electric currents may exist in the higher regions of the atmosphere and that these currents alter the magnetic field at the Earth's surface and by their changes induce currents in the Earth's crust. This view has later been developed by Schuster and Chapman[1] in turn. In the theory as it stands at present, it is assumed that the primary currents are generated by motion of a highly conductive portion of the upper atmosphere relative to the Earth's permanent magnetic field. This view has received support by the strong evidence, revealed by some radio phenomena, of the existence of a highly conductive region.

Variations in the structure, position, and constitution of this region, which doubtless arise chiefly from changes in solar activity and from the relative positions of Earth and Sun, may be expected to manifest themselves through changes in the phenomena of earth-currents, terrestrial magnetism, and radio.

Since variations in earth-currents are known to be closely related with those in terrestrial magnetism, we may expect to find about the same correlations between earth-currents and radio reception as was found by Pickard [2] between terrestrial magnetism and radio reception. However, data suitable for a statistical study of this relation are not now available. This paper must accordingly be confined to making some comparisons of a rather general nature which it is believed show the desirability of initiating earth-current observations suitable for a comparative study of these two sets of phenomena.

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[1] S. Chapman, in Dictionary of Applied Physics (2: 551-561) gives a general description of this theory and refers to the principal original papers.

During the year 1926, earth-current disturbances were of more than usual abundance both at the Ebro Observatory in Spain and at the Watheroo Observatory in Western Australia. The year was also exceptionally unfavorable for broadcast reception. The periods when the earth-current disturbances were so great as to interfere with the telegraph traffic were also reported as periods of most unsatisfactory reception.

Earth-current disturbances may be classed as either local or worldwide in character. The latter type occurs, as do many magnetic storms, nearly simultaneously at points as widely separated as the Ebro Observatory in eastern Spain and the Watheroo Observatory in Western Australia. Apparently some disturbances in radio transmission have the latter characteristic, as is indicated by the report that "a bad night for one is a bad night for all." [3]

A 27-day period of recurrence of disturbances in earth-currents, similar to that for magnetic disturbances, has been found by Peters and Ennis [4] in the data from the Ebro Observatory. Hence this may be another feature held in common with radio.

These storm conditions are believed to arise from electric pulses in the upper atmosphere, produced by corpuscular radiation from the Sun. Whether the effects manifested in radio are due to direct action of the magnetic field produced by the pulses, to a change in structure of the conducting region especially at its lower boundary, or to a change in the density and distribution of ions in regions nearer the Earth, are questions which have important bearings in terrestrial electricity, and the answer to these will probably be found in a study of parallel observations of terrestrial magnetism, atmospheric electricity, earth-currents, and radio phenomena.

The normal diurnal-changes in earth-currents have a greater range during the daylight hours than at night.[5, 6] The range is also greater in summer than in winter. Thus the variations in normal earth-current activity, daily as well as seasonal, are roughly the reverse of the variations in radio reception of waves of broadcast frequency.

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[2] G. W. PICKARD, Proc. Inst. Radio Eng., 12: 119-158 (1924).

[3] See especially BAUMLER, M., Jahrb. d. drahtlosen Teleg., 19: 325 (1922),

[4] Terr. Mag., 31: 57-70 (1926).

Another point of some interest is that the ratio of the daylight range to the night range in earth-currents is the inverse of the relation found by Taylor between day and night skip-distances. In the means for Watheroo in 1924 the daylight range was three times the night range, at Ebro (1914-1918) 2.5 times, and at Berlin (1884-1887) 3.9 times. Taylor and Hulburt[7] state that "the night skip-distances are on a rough average three or four times the corresponding day values." Since the effective height to the conducting region is proportional to the skip-distance, this may perhaps be taken to indicate that the amplitude in the normal earth-current variation depends rather directly upon the distance between conducting region and Earth.

The sunrise and sunset effects on radio reception which have been reported seem to have no counterpart in earth-currents.

Some features similar to atmospherics are observed in earth-currents. Although earth-current disturbances always accompany magnetic storms, occasionally pronounced earth-current disturbances occur at magnetically quiet times. Some of these can be associated with nearby thunderstorms, but others of equal magnitude occur when no thunderstorms are in view. These disturbances are all of periods of the order of a minute or more. There are, however, also earth-current variations of audio frequency.[8, 9, 10] These can be noted at times on telephone systems in which a ground return is used. The noises which these produce in a telephone receiver are notably of the same variety produced by atmospherics. These are reported to predominate in certain directions, according to some observers along the magnetic meridian.

An examination of a number of earth-current storms, where the period of the variations was of the order of ten minutes and occurring at widely different times and places, shows a strong predominance of change along the magnetic meridian. Schindelhauer[11] reports that "some types of

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[5] B. Weinstein, Die Erdströme im deutschen Reichstelegraphengebiet und ihr Zusammenhang mit den erdmagnetischen Erscheinungen. Braunschweig, 1900.

[6] L. A. Bauer, Terr. Mag., 27: 1-30 (1922).

[7] Phy. Rev., 27: 189-215 (1926).

[8] J. Landerer, Compt. rend., 105: 463-465 (1887).

[9] W. H. Preece, Nature, p. 554, April, 1894.

[10] R. Courant, Die Naturwissenschaften, 14: 61-64 (1926).

[11] F. Schindelhauer, Jahrb. d. drahtlosen Telegr. u. Telephonie, 22: 163 (1923).

(atmospheric) disturbances occur day after day along the direction of the magnetic meridian."

It may also be of interest in this connection to know that earth-current records at the Huancayo Magnetic Observatory, 12° south of the equator and at an altitude of 11,000 feet above sea-level, which began in October 1926, show phenomenal earth-current disturbances. During the period October 15, 1926, to January 31, 1927, the frequency of occurrence and magnitude of these far exceed any heretofore reported. The great majority of these corresponds closely with changes in the magnetic elements. Whether these disturbances are related with such atmospherics as apparently came from the tropics would seem a question worthy of investigation.

Doubtless some of these apparent correspondences will, with sufficient data, prove to be mere coincidences. Yet it is believed we have sufficient grounds to urge a closer comparative study of these terrestial phenomena. To this end, more suitable data are needed. This applies especially to earth-currents. At present good progress is being made in methods and instruments for recording the several radio phenomena. The methods and instruments for recording the earth-current features of most interest in this connection are at hand, but these are in use at present at only four places, two in Europe, one in South America, and one in Western Australia.

These earth-current installations are designed for recording variations of periods of a few minutes and upwards, and, although it is hoped that more such systems will soon be installed, especially in North America, yet they are not well suited for the individual investigator. The observation of higher frequency earth-current phenomena is, however, free from some of the obstacles attending the other type of measurement. The earthed points can be placed at much shorter distances apart, and the character of earth-plates is a matter of considerably less moment than for installations where the longer period variations are to be recorded.

Although these high-frequency earth-currents doubtless have their corresponding magnetic effect, the usual magnetic measuring instruments do not respond to them. Furthermore, according to the calculations of Chapman, and this is apparently borne out in the observations, the amplitude of the surface earth-current increases, relative to that in the Earth's magetism, as the frequency increases. It thus happens that earth-current methods are especially adapted for investigating this feature.

The resistivity of the Earth as a whole is a matter of considerable importance in theories of terrestrial magnetism. That of the core of the

Earth has been calculated by Chapman and Whitehead[12] both from magnetic diurnal-variation and from storm data; that of the oceans may be considered as known; but that of the undisturbed crust is as yet largely a matter of conjecture. A knowledge of this is needed in connection with the usual earth-current potential measurements in order to determine the current-density. The method and instruments developed by the Department of Terrestrial Magnetism and used at Watheroo Observatory, Western Australia, and at Ebro Observatory, Tortosa, Spain, for making resistivity-surveys seem to supply some of the desired information. It is believed that, if such surveys were made in conjunction with the charting of lines of equal signal-intensity about a sending station, the knowledge of earth-resistivity thus obtained would be considerably enhanced in value, not only to the geophysicist but probably also to the radio engineer.

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APPARATUS FOR RECORDING RADIO PHENOMENA[1]

T. PARKINSON

This paper is devoted to a brief description of the types of apparatus suitable for general use for the measurement of radio field-intensity, fading (fluctuation of field-intensity), and intensity and direction of atmospheric disturbances ("static"). Fundamentally the same equipment may serve all three purposes. Fading records provide relative field-intensities; it is therefore only necessary to add to a fading recorder a means of calibration in absolute units to interpret data in terms of field-intensities. In the study of atmospherics, their intensities and the location of the sources of disturbances are of prime interest; therefore the only additional apparatus required is a rotating coil-antenna, used preferably in conjunction with an open antenna as a unidirectional system.

The essentials of recording apparatus are: the receiving system, the indicating device, and the means of translating indicated changes into graphical records. To obtain measurements in absolute values a calibrating unit must also be used.

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[12] Trans. Camb. Phil. Soc., 22: (25), 463-482 (1922).

[1] Published by permission of the Director of the National Bureau of Standards of the U. S. Department of Commerce.

Receiving system.—In the receiving system we must consider both the antenna and the receiving set proper.

The antenna.—The type of antenna will vary with the purpose of the measurements. For observations of fading, any type of antenna may be used but the characteristics of the fading are likely to vary greatly according to whether we use a directional or a non-directional system. Among the directional types are the coil-antenna and also the inverted *L* when it is low and relatively long. While we are not certain just how much of fading is due to lateral and vertical direction-shifts in the received wave and how much to absorption and interference phenomena, we shall avoid the directional antenna if we wish to eliminate the effect of horizontal direction shifts from our records, and we shall choose the vertical antenna, the *T*-type, or an inverted *L* that is relatively high.

For measurement of field-intensities the coil-type of antenna, with a grounded center tap for securing balance, is most convenient because of its portability and because its constants are easily measured. The latter fact makes it possible to match with a local signal the voltage induced in the coil by the received signal and thus calculate the field. The open type of antenna is also satisfactory but necessitates a more roundabout method of measurement and calculation.

For direction measurements, whether of atmospherics or of lateral shifts in received signals, the coil-antenna of the balanced type may be used but the coil-antenna in conjunction with an open antenna is preferable. With the latter system the minimum signal or nul point, by which direction is most sharply determined, occurs in one direction only. With the coil-antenna used by itself there are found minima for two opposite positions of the coil. In the unilateral arrangement the open antenna and the closed coil-type are inductively coupled so that a composite of the signals received by the two systems is impressed on the receiving set. By means of a resistance in the tuned circuit of the open antenna, the signal is reduced until it is approximately equal to the maximum signal received in the coil-antenna alone when the latter is normal to the wave-front of the received signal. The combined signal from the two systems with the coil in this position will be double that of either alone if the voltage phases of the two are the same, but will be zero if the phases are opposite. As the coil is rotated there will always be a maximum when one side points in the direction of the received signal and a minimum when the coil is reversed. The direction in which the latter side points therefore determines the signal direction. The coil may be rotated at definite speed by any driving mechanism which will not create a field of its own in the coil. For signals at the low frequencies and for such signals at higher frequencies as do not involve considerable fading, automatic records can thus be made.

Receiving set proper.—In selecting the receiving set proper we naturally seek dependability first and then simplicity. The former requirement eliminates the crystal detector from any part of the measuring circuit, since no crystal rectifier can be counted upon to remain constant in sensitivity over long periods of time.

The simplest circuit which will give reasonable sensitivity is the autodyne, commonly known as the regenerative detector. Here the tube-circuit is so tuned that a beat-note is produced between the generated and the incoming signal. This beat must always be of the same frequency if results are to be comparable, since no receiving circuit is equally sensitive to all frequencies. For work at broadcast and higher frequencies this arrangement has disadvantages: (1) It is practically impossible to repeat adjustments, particularly so far as to secure the same amount of regeneration; (2) it distorts modulation so as to make monitoring disagreeable; (3) it practically necessitates continuous calibration by some method of matching the antenna-signal directly by a measured signal; (4) the circuit is not sufficiently selective.

Turning to radio-frequency amplifiers of the neutrodyne type we find a gain in dependability but a loss of simplicity. The advantages are: (1) Selectivity is improved by using extra tuned circuits; (2) because regeneration is avoided it is possible to duplicate tuning and amplification; (3) calibration of the amplification of the set for any given frequency is possible. The disadvantages are: (1) There are too many tuning controls; (2) any amplification-control, such as a variable resistance, will have a varying effect for signals of different frequencies, thus necessitating its complete calibration over the whole frequency-range of the receiving set.

The superheterodyne[2] or double-detection circuit-system seems most satisfactory for general use. In addition to the mentioned good points of the neutrodyne and the reduction of tuning controls from three to two, the following advantages are worth noting: (1) Beyond the first detector the frequency is always the same and therefore the amplification is uniform regardless of the frequency of the received signal; (2) for any given frequency the amplification of the signal before reaching the intermediate-frequency amplifier remains the same so that a calibration showing relative sensitivities of this portion of the circuit-system to the different frequencies can be used to correct any measurements made at the output of the receiver; (3) the fixed intermediate- or beat-frequency, of say forty kilocycles, which for practical purposes is now the carrier, can easily be isolated from modulation-effects by means of a transformer tuned to this frequency and coupled to a rectifier and indicating device; (4) one calibration of the amplification-control serves for all input frequencies, since this control affects only the amplifier and following stages, where the same beat-frequency is always involved; (5) a calibrating unit is required only from time to time for verifying and checking purposes and therefore one such unit can serve for several receiving sets, and the size of the measuring outfit can be reduced for field use.

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[2] A satisfactory superheterodyne of standard make can be purchased for $300 or less.

Figure 1 shows the slight modification of the superheterodyne which must be made to adapt it to recording purposes. The two-electrode tube-rectifier, the intermediate-frequency transformer, and the recording galvanometer shown within the broken-line rectangle constitute the additional apparatus required. This may be placed in the receiver-cabinet or may be in a separate unit outside. The wiring of the set is modified only by opening up the plate-circuit of the second detector, or of the last intermediate-frequency amplifier, to insert leads to the primary of the extra transformer. In place of the two-electrode tube-rectifier indicated it is possible to employ the three-electrode type with plate-current balanced out of the indicating device. The additional apparatus involved by the latter arrangement consists of a *B* battery, a balancing battery

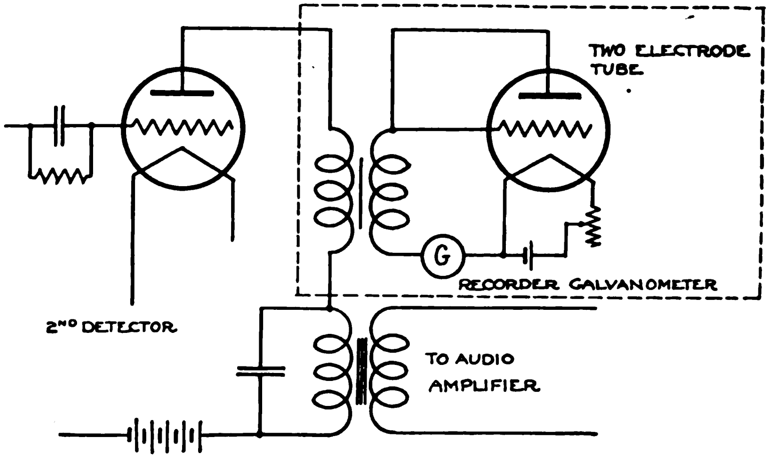

Fig. 1

Circuit connections of recorder unit to superheterodyne receiving set. (Circuits of recorder unit are enclosed within broken-line rectangle.)

and resistances. In either arrangement voltmeters or else ammeters are essential in both A and B battery-circuits to indicate whether conditions are being kept constant. In some superheterodynes it is also advisable to change the method of volume-control. Adjustment of the filament-current from a tube by means of separate rheostat does not offer a stable calibration. A very satisfactory arrangement is a variable resistance or potential-divider shunted across the primary of the first intermediate-frequency transformer. This control should be calibrated in steps which have a known ratio to the maximum amplification of the set.

Calibrating unit.—One type of calibrating unit consists of a well shielded oscillator capable of producing a current of the same frequency as the incoming signal in the receiving set. This current is introduced through a small resistance inserted in the receiving antenna, and by means of an attenuation network can be made to produce a voltage in the antenna which will match the voltage produced there by the signal to be measured. From the input current, the number of turns and the dimensions of the coil-antenna it is then possible to calculate the field-intensity. Another arrangement which requires less careful shielding is described by Van Roberts.[3] This system does away with the attenuation network and permits the measurement of the calibrating current by means of a sensitive direct-current meter.

Indicating devices.—Indicating devices may utilize sound-intensity or visual indicators of the galvanometer or the potentiometer-type. The first requires matching of pitches and intensities with those of some known values, as in Austin's comparator-method. This method is especially applicable to measurements at the low frequencies where C. W. or I. C. W. signals are interrupted by keying, sufficiently to cause variable errors in galvanometer-records. A 1000-cycle tuning fork produces the constant pitch; the autodyne receiver is adjusted to translate the incoming signal to a beat-frequency of one thousand cycles, and then the comparator-current is adjusted by means of a potentiometer until the two signals are of equal intensity in the telephone-receivers. Knowing the current in the telephones, the amplification of the receiving circuit and the antenna-constants it then becomes possible to calculate field-intensities. For measurement of the modulated and of the rapidly fading signals at broadcast and higher frequencies this method is practically impossible.

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[3] WALTER VAN B. ROBERTS. A method of generating and measuring very weak radio-frequency currents. J. Franklin Institute, 301-310, March (1926)

Of the visual indicators, the direct-current galvanometer utilizes the rectified carrier-current and gives deflections approximately proportional to the squares of the field-intensities. Its advantages are: (1) It makes continuous records possible; (2) it integrates the audio-frequency changes, thus isolating the carrier from modulation-effects, which is absolutely necessary for measurements of broadcast transmissions. The disadvantages are: (1) Extra analysis is required to translate the ordinates of a record from square-law to straight-line relationships; (2) the square-law effect reduces seriously the galvanometer-range for field-changes, thus necessitating frequent readjustment of receiving-set amplification to keep the galvanometer-pointer on the scale, and adding complications to the analysis work.

The potentiometer-method operates on what is sometimes called the slide-back principle (Figure 2). With a given signal a deflection is produced in the galvanometer. By means of an opposing current through the galvanometer the deflection is reduced to an arbitrary zero at the center of the scale. As the deflection indicates that the signal has increased, the potentiometer is readjusted so as to neutralize the increased positive charge on the detector-grid by an equal negative potential, thus bringing the galvanometer-deflection back to the central position. As the signal fades the negative potential on the grid is similarly reduced by the opposite potentiometer-adjustment and again the pointer returns to zero-position. The advantages of this method are: (1) It makes it possible to record with ordinates directly proportional to antenna-currents; (2) it can be made to control the whole amplification-system beyond the first detector, thus preventing possible saturation of amplifier-

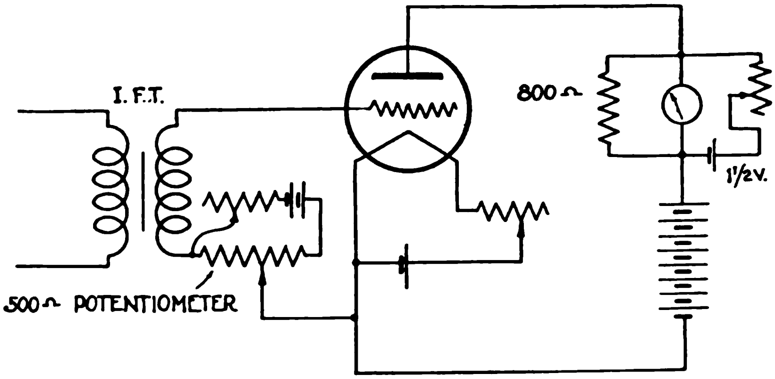

FIG. 2

Circuit diagram for potentiometer-type of recorder.

tubes. With this latter arrangement, however, there is an inverse relationship between deflections and antenna-currents. The disadvantages are: (1) It is difficult to keep the deflection even approximately constant by *manual* control of the potentiometer; (2) the inertia of the recording system in automatic apparatus restricts its usefulness for studying the character of rapid fading.

Translating into graphical records—Indicated variations of signals may be translated into records: (1) By plotting readings taken at regular intervals; (2) by following deflections with a manual control[4] linked to a recording pen which writes on a moving tape, as with the Shaw recorder (Figure 3) for example; or (3) by automatic recorders of the photographic or mechanical[5] types. Plotting eliminates the details which accompany rapid fluctuations. Manual recording is tedious and expensive in personnel if done for long periods. The mirror galvanometer is expensive and time-consuming because of photographic features. Both mirror and mechanical types of automatic recorders are

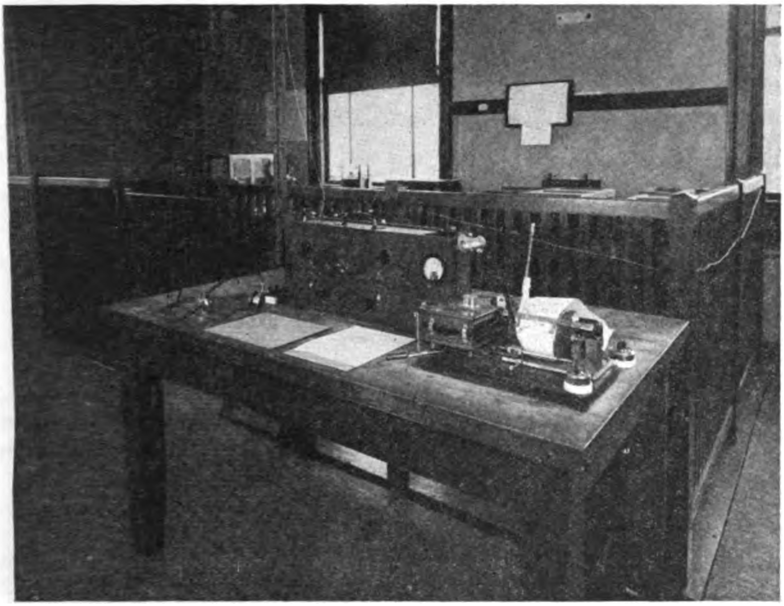

FIG. 3

Receiving set and manual recorder.

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[4] Such apparatus can be made up or can be purchased for approximately $100.

[5] A good mechanical recorder can be purchased for approximately $300.

open to the criticism of square-law complications. The automatic potentiometer[6] is open only to the criticism of slow action preventing the recording of details such as appear at broadcast and higher frequencies. Where much observing is to be done and the variations of a few seconds are not of prime interest this type of apparatus is probably the most satisfactory because it makes possible the securing of excellent records with a minimum of analysis work and of routine observing.

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[6] **A standard potentiometer-recorder which can be adapted to the purpose may be purchased for $250 to $300.**

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Washington, D. C.

SUMMARY OF SYMPOSIUM ON CORRELATIONS OF VARIOUS RADIO PHENOMENA WITH SOLAR AND TERRESTRIAL MAGNETIC AND ELECTRIC ACTIVITIES[1]

J. H. Dellinger

The papers presented on this program cover a wide scope and variety of fields of scientific work. It is significant that the general subjects of the symposium provide a sufficiently common ground so that workers in these diverse fields of scientific effort can understand and appreciate the technical results of work in the other fields. In these various fields of geophysical observation we have all for many years past been both troubled and interested by the same sorts of variations, fluctuations, and anomalies in the phenomena with which we deal. A suspicion that the similarity of character of variations in the various phenomena indicates some common underlying cause furnishes the reason for the organization of this symposium. The symposium has been most successful in the wealth of material brought forward and the clear indication offered of correlations between the several types of phenomena. It is my task to summarize the discussions and to indicate such general conclusions as may be drawn from the whole field of study.

In starting the symposium, Chairman Heck brought to our attention the recently recognized importance and value of the observation of radio wave-phenomena in their bearings upon other terrestrial phenomena.

Appropriately, the first technical paper of the session was presented by Dr. Abbot on the source and basis of the various geophysical variations—the Sun. The Sun is not only the source of the energy by which life and all of our activities are maintained, but it is also the source of the troubles we have in our various fields of scientific endeavor. Dr. Abbot gave us a clear idea of the various solar cycles with their periodicities of various types. Analyzing for certain of these cycles, he has found definite general correlations between his major subjects of study, solar radiation and sunspots. In addition, he gave us an indication of the degree of correlation between these phenomena and such things as weather, radio wave-intensity, and radio atmospheric-disturbances. There are clear general correlations in these various fields, but the specific trends of the various phenomena have either no direct correlations or else are very complicated. It is evident that the phenomena dealt with are of a complex nature and that there is a great need for methods of study and analysis to be worked out which will unscramble some of the complexities. Dr. Abbot indicated some points of departure for the unscrambling process. An example of a suggestion of this kind is the discovery that sunspottedness is more significant when relatively small; that there is, so to speak, a saturation-effect when sunspottedness is great and, consequently, correlations with other phenomena are clearer for relatively small values of the sunspot-numbers.

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[1] **Published by permission of the Director of the National Bureau of Standards of the U. S. Department of Commerce.**

In Dr. Bauer's paper, we were informed of the correlations between sunspottedness and variations of intensity of the Earth's magnetism and electricity. There are correlations in the general course of the two phenomena. Detailed records, however, do not always show a satisfactory direct correlation between sunspots and either magnetic storms or atmospheric electricity for short periods. Dr. Bauer pointed out that the magnetic phenomena and the sunspots may both be due to some undiscovered solar cause.

Dr. Nicholson's paper, presented by Dr. Adams, informed us of a most promising type of investigation of the phenomena in the Sun which are of a more basic nature than the sunspots. By means of the spectrohelioscope which is used in the investigations directed by Dr. Hale, studies are being made of the actual intensities of the Sun's magnetic field in and near the sunspots. Obviously, the sunspots are only one indication of the electric and magnetic actions on the Sun which are the actual causes of the variations which we study, and work of this kind, therefore, penetrates much nearer to basic causes. The work with the spectrohelioscope thus gives a new hope throughout the whole field of geophysical investigations, and doubtless the taking of spectroheliographs in the future will afford a new line of data with which correlations can be expected to be of a much more direct and informing nature.

Mr. Pickard presented a number of entirely remarkable correlations between radio reception and sunspot-numbers. The clear indications of direct relation which he showed do much to unscramble the complexities and vagueness of the inter-relations between the several phenomena in which we are interested. It is very interesting to note that while extraordinary agreement was shown between the shape of the curves showing the variation of sunspots, Earth's magnetism, and radio reception, in the 27-day and other solar cycles, nevertheless in all cases the phases of these curves were different (by a few days). Furthermore the shift of phase is not the same between the curves of the different phenomena or curves taken at different times or under different conditions. This lack of agreement as to phase of the various phenomena is significant in regard to the validity of sunspots as a criterion of the Sun's activity. For example, Mr. Pickard found that when data were compared with sunspot-numbers not for the whole solar disc but only for the central zone of the Sun, much better correlations were found; in other words, the electric and magnetic actions on the Sun unquestionably control the variations of the terrestrial phenomena we observe, and the sunspots are only a first approximation as an index to that solar activity. A truer index remains to be found.

Dr. Austin showed that for observations taken at relatively low radio frequencies there was little direct correlation of atmospheric disturbances and sunspots. There was definite indication of general correlation but this practically disappeared when any detailed study of the phenomena was made at specific individual times. In contrast, Dr. Austin showed the very good correlation between intensity of atmospheric disturbances and temperature. This contrast is significant and probably finds its explanation in the view that the observations of atmospheric disturbances were made at relatively low radio frequencies. The transmission of the waves producing the atmospheric disturbances at a particular point is largely determined by ground transmission-conditions and relatively little by conditions in the upper atmosphere. As the ground-conditions vary with temperature, the temperature-correlation mentioned would be expected; whereas, since the solar emanations have their effect largely in the ionized upper atmosphere, these particular radio observations would not be affected in marked degree by the solar variations.

Dr. Breit drew for us a picture of the constitution of the Earth's atmosphere which is now generally accepted as providing a mechanism for the transmission of radio waves to great distances. This involves the assumption of an effective atmosphere extending some hundreds of miles out from the Earth, the upper portions of which are highly ionized and are often referred to as the Heaviside layer. Utilizing probable values for the ionization of the upper atmospheric regions, Dr. Breit calculated the variations of radio wave-intensity received at great distances for small variations in values of the Earth's magnetic field. He concluded that as small a variation as five gammas in the Earth's field can make appreciable changes in received radio-intensity at reasonable distances. This work gives a basis for direct estimation of the rôle of terrestrial magnetic variations in the variations of radio reception. This also led to the suggestion of the value of future simultaneous measurement of received radio field-intensities and terrestrial magnetic variations.

Lieutenant Kincaid described some correlations between meteorological phenomena and radio atmospheric-disturbances which have an immediate practical application in marine navigation. By observations of the direction of the source of atmospheric disturbances and either observing the course of that direction with time or by triangulation, important correlations were worked out between the sources of these disturbances and atmospheric highs and lows. This is leading to a practical means of tracking violent storms at sea and giving a valuable direct aid both to weather forecasts and to radio forecasts. In this work valuable use is made of apparatus for the automatic recording of the intensity and direction of atmospheric disturbances.

Mr. Gish pointed out that earth-currents offer an additional type of phenomena whose variations may be of importance in determining the correlations between solar activity and terrestrial phenomena. Since there are in the world only three observatories for regular measurements of earth-currents, and these are relatively new, there does not exist a large body of data which can now be studied and the existence of correlations cannot therefore be determined at this time. A number of facts, however, indicate that results of value may be expected. It is well known that earth-current disturbances are sometimes so pronounced as to put wire telegraphs out of service. There are reasons to believe that some earth-current variations follow the same course as radio atmospheric-disturbances and radio reception-conditions. In any event this is a line of work which offers possibilities and should be kept in mind in further study of this general field.

Mr. Parkinson brought to our attention that radio observations are now of considerable importance in geophysical studies, not only because of demonstrated correlations, but also because in recent years facilities for precise and pertinent radio measurements have been highly perfected. Radio measurements have greatly advanced in recent times and accurate and useful observations can be made with relatively simple apparatus. So valuable and at the same time so easily made are radio observations that it is to be expected that observatories of many different types of organizations which have not hitherto included radio-recording will undertake observations in this field. With very simple apparatus and measurement-procedures, it is quite possible to provide for records of fading (signal-intensity fluctuations), radio field-intensity, and both the intensity and direction of atmospheric disturbances. Laboratories which care to make more detailed studies of radio phenomena can readily add observations for determination of radio wave-direction and polarization. Mr. Parkinson gave definite information as to the several components of the necessary apparatus, including antenna, receiving set, recording unit, and (where desired) automatic graphic record-maker. Observatories which are not in a position to assume any great additional expense can provide equipment for the principal radio observations at a cost of a few hundred dollars. This will be reduced in many cases as the receiving set, which is the principal part of this expenditure, would already be available at some institutions. Where the observatory desires to undertake radio measurements on a considerable scale and provide for automatic recording so as to minimize the expenditure of time on the part of personnel, the necessary equipment can be provided at a cost of a few thousands of dollars.

CONCLUSIONS

The contributions to this symposium made by the several speakers justify the following conclusions:

(1) The present time is one of emergence of a considerable amount of new knowledge on solar activity and its effects on geophysical phenomena. The numerous types of variation exhibited in these various phenomena which have been the constant object of study and oftentimes the despair of students in these several fields, are just now coming to evince a common nature and origin. The mechanism which, at least in large part, determines and explains the several types of variation is the emission of electric and magnetic radiations by the Sun and the production of effects thereby in the Earth's atmosphere. The major portions of these effects occur in the upper ionized portions of the air frequently referred to as the Heaviside layer, a region extending from approximately one hundred to four hundred miles out from the Earth's surface.

(2) Vast opportunities for the co-ordination of world-wide observations and the correlation of universe-wide phenomena is opening up. Many observatories which have been established by organizations set up to deal with one type of phenomena only (e. g., meteorological or magnetic phenomena) can most profitably and certainly reach their goal if they no longer limit their observations to the one line in which they are directly interested. The new possibility and desirability of co-ordinating research along the various geophysical lines must in the future affect the location of observatories with a view to the establishment of a co-ordinated program of study of terrestrial phenomena on a world-wide scale.

(3) Although we are far from the position where positive conclusions can be drawn as to the nature and causes of specific variations of any of the geophysical phenomena, we do have a number of most interesting and valuable correlations which have now been definitely proved by the unquestionable and repeated data. I have heard no suggestion that earthquakes show any correlations with the solar cycles but just about every other geophysical phenomenon does show some sort of correlation. We are therefore now in a position to begin to trace out the cosmic mes-

sages of which Dr. Pupin spoke last December in his Presidential address before the American Association for the Advancement of Science.

(4) Data presented from the several lines of investigation lead to the definite conclusion that sunspots are not the fundamental cause of the terrestrial variations and that direct detailed correlations with sunspots cannot be expected. They are indeed an index of the magnetic and electric activities in the Sun which are the actual cause of the terrestrial variations but, in themselves, they are far too crude an index to be considered as having any basic significance. As time goes on, no doubt they will be superseded as a solar index by actual observations of the magnetic and electric conditions in the Sun itself.

(5) There is no task more suitable for serious consideration and promotion by the September 1927 international meeting of the International Geodetic and Geophysical Union than the organization on a world-wide scale of a co-ordinated program of study of the several geophysical phenomena in the various fields covered by this morning's program. It is to be hoped that the recommendation of the American Geophysical Union that the Prague meeting in September undertake as a major topic the correlation of solar and terrestrial phenomena studies will be accepted. If this is done we may well look forward to a new world-wide organization of scientific endeavor along these lines by which the world will enter on a new phase in the study of cosmic phenomena.

Finally, the summaries presented from the various fields of geophysical endeavor show that definite correlations of far-reaching significance have been established in the following diverse fields of observation: (*a*) The various types of solar activity; (*b*) terrestrial magnetism; (*c*) atmospheric electricity; (*d*) meteorological phenomena; (*e*) earth-currents; (*f*) auroras; (*g*) radio wave-transmission.

Although these correlations are in all cases of a partial and tentative nature, they are none the less certain and significant. They indicate that study in these various fields of science in the future can have its value considerably augmented by constant regard to results in the other fields, and particularly results upon the various types of solar variation. The possible phases of solar activity with which correlations may profitably be sought include sunspots, photographic data, heat radiation, electric charge radiations, and various types of ultra-violet radiation and spectro-helioscopic data.

U. S. Bureau of Standards,
Washington, D. C.

SECTION OF OCEANOGRAPHY

(Washington, D. C., April 28, 1927)

OCEANOGRAPHIC INVESTIGATIONS ON THE NEXT CRUISE OF THE CARNEGIE

J. P. AULT

(I) TERRESTRIAL MAGNETISM

The next cruise of the *Carnegie,* to be Cruise VII, is planned to begin at Washington early in 1928, is to occupy three years' time, and is to extend around the world over all the oceans. Chief emphasis will be placed on magnetic and atmospheric-electric surveys and investigations since the vessel was designed and fitted primarily for this purpose. The practical and theoretical value of the magnetic investigations already done on the *Carnegie* is testified to by the principal hydrographic establishments of the world and by individual investigators, but the full value of the accumulated data will not be realized without additional observations to determine the secular variation (or progressive) changes in the Earth's magnetism. The chief aim of the magnetic work of the next cruise will be to increase our knowledge of these changes.

In order to make the fullest possible use of previous observations, it is planned to follow as closely as possible the tracks of previous cruises. The map in Figure 1 indicates the route tentatively selected which will be productive of the maximum of valuable data needed. Such a route and program will also give opportunity for securing additional information regarding the distribution of the magnetic elements in some large areas not already surveyed and for investigating regions of pronounced local disturbance, particularly those indicated by the previous work of the *Carnegie* over deep-sea areas, in connection with determinations of ocean depths by echo sounding devices.

(II) TERRESTRIAL ELECTRICITY

The atmospheric-electric program is to be continued and enlarged since the studies and discussions of results already accumulated indicate the value of the work done and point out the need for more data to assist in the solution of the various problems involved. In the further investigations on the origin and maintenance of the Earth's electric charge and of the relation to the Earth's magnetic condition, additional determinations of changes in the values of the atmospheric-electric elements with geographic position are needed. Recent investigations of the pentrating radiation, or of the so-called "cosmic rays," emphasize the

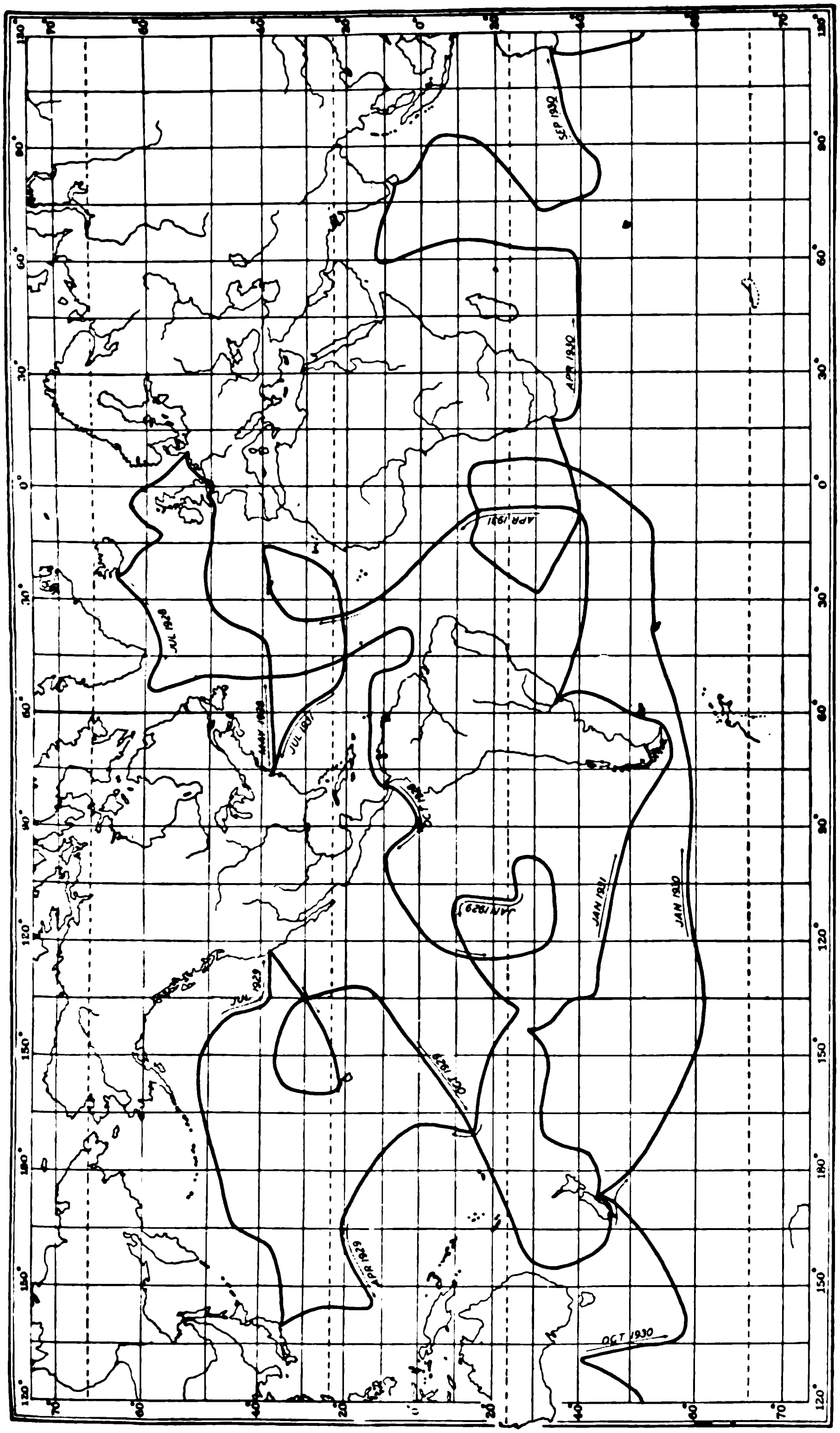

FIG. 1

Tentative route for seventh cruise of the *Carnegie*, 1928 to 1931.

needs also of additional observations of the ionization of the air over the oceans far removed from possible disturbances from land areas.

Further widely distributed and frequent determinations of the diurnal variations in atmospheric electricity will be obtained in order to settle the question whether such variations progress on the basis of universal time, a deduction first indicated by Mauchly[1] from a discussion of results obtained on the *Carnegie* and later fully confirmed by Sverdrup[2] from observations made on the *Maud* during the Arctic-Drift Expedition. To further improve and control the atmospheric-electric results, dust-count observations of the atmosphere are to be included in the program of electric work at sea.

The important contributions to the study of various geophysical problems which are being made by investigations of the Kennelly-Heaviside conducting layer and of radio transmission and variations with changing magnetic and electric conditions greatly enhance the value of the atmospheric-electric data already collected over the oceans by the *Carnegie*, and indicate cooperative investigations along similar lines for future ocean work. It is planned, therefore, to begin at sea the investigations of the conducting layer and to carry out experiments on the variations of signal-intensity, following methods already in use on land.

Important correlations between earth-current variations and changes in other geophysical and cosmical phenomena, such as solar activity, magnetic disturbances, and polar lights, have been shown from an investigation of observatory records, and the importance of these investigations in the general study of the Earth's magnetic and electric fields warrants the beginning of systematic earth-current observations at sea. Some preliminary experimental work must be undertaken to determine the best methods and instruments for such observations over the oceans.

(III) OCEANOGRAPHY

The Carnegie Institution of Washington has been interested in various problems of oceanography for a number of years through the activities of its various departments and research associates. The vast extent of ocean areas to be covered by the next cruise of the *Carnegie* offers a unique opportunity to secure much needed data in the science of oceanography in regions which have never been investigated.

With the steady improvement and simplification of methods and instrumental equipment used in making magnetic observations, gained through experience and experimentation, it is now possible to reduce very materially the time occupied in the observational work at sea as heretofore done in terrestrial magnetism. Furthermore, for the purpose of determining secular variation, observations of the magnetic elements need not be made so frequently as heretofore for the determination of their distribution, except in those regions not yet sufficiently well surveyed. Thus there will be time for the undertaking of a program of work in more general oceanography.

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[1] S. J. MAUCHLY, Phys. Rev., N S., 18: 161-162 and 477 (1921).

[2] H. U. SVERDRUP, Sc. Monthly, 22: 400-410 (1926).

The following program of oceanographic work is being planned as desirable and practicable for the *Carnegie* under the restrictions of limited space and personnel and the necessity of maintaining a non-magnetic vessel for investigations in terrestrial magnetism.

(*a*) *Physical oceanography,* for the determination of ocean circulation, conformation and topography of oceanic basins, nature of bottom sediments, and of the chemical constituents of sea-water. Temperatures and water samples are to be obtained at depths of 5, 25, 50, 75, 100, 200, 300, 400, 500, 700, 1,000, 1,500, and 2,000 meters every 150 to 200 miles, with occasional work down to the bottom, salinities are to be determined on board ship with the Wenner electric method and checked occasionally by the titration method, surface temperatures are to be recorded continuously, and ocean depths are to be determined by means of the Navy sonic depth-finder as developed by Dr. Harvey C. Hayes. The United States Navy Department is cooperating in this work by the loan of the necessary equipment. For the study of sedimentation, bottom samples will be secured whenever the wire is sent to the bottom for temperature and salinity data.

(*b*) *Microbiology,* to determine the abundance and distribution of plankton and other organisms, such observations being of interest to the biologist as well as being of economic interest in the study of fishes and of other sea food. Shallow water dredging for diatoms and foraminifera will be undertaken, and the quantitative distribution of plankton from the surface down to 100 meters will be determined occasionally by the examination of definite quantities of water brought up by means of a new water-bottle devised by Dr. W. E. Allen of the Scripps Institution of Oceanography. Some study will be made of other marine organisms by means of tow nets, and hauls both vertical and horizontal are to be made from the surface down to a depth of 150 meters. Equipment will be carried also for securing specimens of dolphins and porpoises from regions where no specimens have been secured heretofore. Limited space and time and restrictions as to power and machinery prohibit undertaking any deep-sea trawling or dredging. This work may be taken up during a later cruise when it is hoped that chief attention may be devoted to work in oceanography.

(*c*) *Meteorological oceanography,* to include observations in marine meteorology which are of great theoretical and economic importance not only as regards the sea and its life but also as regards their effects in vital relations to continental and coastal regions. One of the most important lines of investigation concerned with studies of temperatures and salinities is the study of the physical interchange between the surface of the water of the sea and the air above it. The determinations of what may be designated "meteorological sections" of the sea to moderate depths and of the air above to masthead height are certain to yield important results bearing on the significance of the seas as distributors of solar heat and on the general circulation of the air over the oceans as an aid in world-wide weather forecasting.

The kinds of meteorological observations at sea, therefore, should include, in addition to temperatures and salinities at the surface and at moderate depths: (1) Temperature lapse-rates from sea-level to masthead; (2) humidity lapse-rates from sea-level to masthead; (3) wind-direction and velocity at different heights; (4) solar and sky-radiation by day and night with cloud systems, rainfall, evaporation, and dust-content and carbonic-acid content of the atmosphere. It is hoped that methods and instruments may be devised for the increased determination of data of these kinds over the great oceanic areas to be covered by the next cruise of the *Carnegie* for the comparison of world weather with solar radiation, for the determination of the rate at which the atmosphere is being charged with water vapor so vital to life on the continents, and for the study of the dynamics of atmospheric circulation over the oceans.

To carry out the program of general oceanographic work outlined above, an additional stateroom is to be provided in the cabin of the *Carnegie* to increase the scientific personnel to seven, the additional man to be especially trained in biology and chemistry. The two lifeboats are to be moved from the quarter deck to overhead platforms amidships opposite the after observatory dome, leaving the quarter deck free for the installation and operation of the necessary equipment for handling the oceanographic work, such as a special nonmagnetic winch, special bronze davits, and racks for mounting water-bottles and deep-sea reversing thermometers. The bronze winch is to be operated electrically by means of an attached motor, the power for which is to be furnished by a generator and 25 horse-power gasoline engine mounted below decks in the engine room. Thus any additional magnetic material will be located from 50 to 75 feet distant from the position of the magnetic instruments, and no disturbance of the magnetic results is anticipated. However, the absence of deviations will be checked carefully, as heretofore, by frequent swings of vessel.

Two new laboratories are to be constructed on deck, one specially fitted for physical oceanographic, biological, and chemical work, and one fitted with radio and echo sounding equipment. Water samples and biological specimens will be stored below decks.

The difficulties attending the obtaining of deep-sea temperatures and water samples from a sailing vessel are fully appreciated. However, Helland-Hansen and Nansen have carried out similar work with great success on the *Armauer Hansen,* a sailing vessel of only one-half the size of the *Carnegie.* They state that [3] "owing to its special construction the ship is easily maneuvered in such a manner that the line along which the oceanographic instruments are suspended remains in a vertical position throughout the time of observation, even if there is a strong drift caused by wind or current." As a further aid in checking the depths at which temperatures and water samples are obtained, simultaneous use will be made of protected and unprotected thermometers, calibrated for pressure effects, a method used recently with success by the German Atlantic Expedition on the *Meteor.* To avoid rapid and excessive drift of the vessel when hove to in a strong breeze, sea-anchors will be used to check the headway.

It is planned to compute at sea, and publish promptly as the cruise progresses, the pertinent oceanographic data for use of students and investigators of oceanography, as has been done heretofore in terrestrial magnetism. The data to be published include the following results of observations and calculations at various depths: Temperature, salinity, density observed and corrected for compression, oxygen content, hydrogen-ion concentration, specific volume, and dynamic pressure and depth. The dynamic calculations will be made in accordance with the method devised by Bjerknes and as modified by Hesselberg, Sverdrup and others.

The biological and chemical work will be in charge of an observer specially trained in these fields, and the chief effort will be the collection of data and specimens. As complete a preliminary study and report as possible of the results of these investigations will be made as the cruise progresses. Interested organizations will be furnished with water samples, bottom samples, and biological specimens for further study and report.

The Department of Terrestrial Magnetism is indebted to the following institutions and organizations for cooperation in the loan of special equipment or for expert advice in planning the program of investigations: United States Navy Department, loan of latest-type sonic depth-finder complete; Scripps Institution of Oceanography; United States

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[3] HELLAND-HANSEN and FRIDTJOF NANSEN. The Eastern North Atlantic. Geofysiske Publikasjoner, 4: (2) 3-4 (1926).

National Museum; United States Weather Bureau; Museum of Comparative Zoology, Harvard University; School of Geography, Clark University; Geophysisk Institut, Bergen, Norway, loan of various oceanographic equipment; Carlsberg Laboratorium, Copenhagen, Denmark.

Department of Terrestrial Magnetism,
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Washington, D. C.

THE OCEANOGRAPHY OF THE CANADIAN ATLANTIC

A. G. HUNTSMAN[1]

The waters of the Atlantic seaboard of Canada may well be considered the most interesting in the world. Their position midway between pole and equator ensures a maximum of seasonal change. Their western position in the Atlantic Ocean gives an arctic current loaded with ice, and the jutting Newfoundland bank causes this current to debouch suddenly into the poleward-moving equatorial water. The continental shelf is broad and of diversified character, exhibiting a wealth of banks, islands, bays, basins, and deep channels. Also, there is a very heavy inflow of fresh water from land-drainage.

These waters have been well investigated from the standpoint of currents and tides by Dr. W. Bell Dawson. To Dr. Johan Hjort we are indebted for a double comprehensive survey of the Gulf of St. Lawrence and of the waters between Nova Scotia and Newfoundland and out well beyond the continental shelf in regard to temperature and salinity. The work done since his expedition of 1915 has been largely the result of the stimulus furnished by the opening of the general hydrodynamic problem as made by himself and his collaborators.

The coastal water may be considered as basically temperate and of moderate salinity, with summer temperatures of from 6° to 12° C and salinity ranging from thirty to thirty-two per thousand. However, such characteristically boreal water is considerably limited owing to the invasion of the area by other waters of quite different types. Warm coastal water over 12° C in temperature and frequently below thirty per thousand in salinity, occupies in summer many of the shallower portions of the coast, such as the Magdalen shallows of the Gulf of St. Lawrence. Arctic or cold coastal water, mostly below 0° C in temperature and about thirty-two per thousand in salinity, comes down steadily from the north in the Labrador current and is also formed locally during the winter. Such water gives an ice-cold bottom layer rather generally over the more northern banks. Tropic water, about 15° C and over thirty-five per thousand, is constantly pressing northward in the so-called "Gulf Stream" some distance outside the shelf. Moving southward below the tropic water is the bottom water of the Atlantic or slope water, about 5° C and thirty-four per thousand, which covers the slope of the continent and penetrates along the bottom into the deeper channels and basins of the shelf.

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[1] **Director of Atlantic Stations, Biological Board of Canada.**

For an understanding of the changing distributions of the various types of water a knowledge of the circulation is essential. The more pronounced movements in this circulation have become evident through their effects on vessels, through the drift of icebergs, and by direct measurement of the currents. Because much of the circulation has been undecipherable by such means, the Biological Board of Canada, with the cooperation of the United States, Newfoundland, and French government services (arranged through the North American Committee on Fishery Investigations), has put out in the last eight years over 9,000 drift-bottles in an effort to trace the circulation of the waters of the region from the Bay of Fundy on the south to the Strait of Belle Isle on the north. The results reveal a circulation of variable intensity, but showing a definite order—clockwise around isles and shoals and contra-clockwise around basins. The rate of the ultimate movement varies from about two miles per day close to land to about eight miles per day in deep water. The water of the continental shelf has neither a definitely northward trend like the tropic water, nor a definitely southward trend like the Labrador Current and bottom Atlantic water, but moving in a circular fashion shows elements of both, varying in relative proportions from time to time. No drift-bottle has yet been found to have traversed the whole region from north to south, or the reverse. Cabot Strait bottles have attained both the Strait of Belle Isle and the Gulf of Maine, but the reverse has not occurred. Extended movements have all taken place to the other side of the Atlantic, and in a general way the distribution of the bottles from north to south as found on the eastern side of the ocean has corresponded with the order from north to south, in which they were put out on the western side. Bottles put off at the Strait of Belle Isle have gone to Iceland and the north of Norway and a bottle put off from Cape Sable reached Cape Verde, Africa.

The proper elucidation of the factors concerned in the circulation is a matter for the future, but illustrations of the effects of some of them may be given. In the Gulf of St. Lawrence, the prevailing south-west winds produce a somewhat superficial current northward along the west coast of Newfoundland, thus concentrating at the inner end of the Strait of Belle Isle a body of water that has been warmed by exposure to the Sun's rays when spread over the Gulf. The return deep current passes southwestward along the other (Quebec) side of the Esquimau channel. Similarly in the Bay of Fundy the warmed surface-water is concentrated at the head, particularly in Minas Channel and basin, but the strong tidal currents mask the wind-circulation, which is nevertheless revealed by the drift-bottles.

When light and heavy water exist at the same level at different places, the Archimedean forces are set into play, and the two waters flow toward each other, the lighter above. The heavy water produced by the mixing of the tropic water with that of the Labrador Current moves south, while the tropic water flows north above it to the mixing region. The light water produced in the estuary of the St. Lawrence by the entering river water flows strongly outward past the Gaspé coast and Cape Breton.

The tidal currents, being oscillatory, are not effective in steadily progressive movements except as modified by the Earth's rotation. That they may in this way be very effective is shown by the circulation through the Strait of Belle Isle. The flooding tide is more pronounced on the north side, carrying Labrador Current water into the Gulf, but the ebbing tide is more pronounced on the south side, carrying out from the Gulf the warm water accumulated at the inner end of the Strait by wind-action. The tides pump the water in both directions through the Strait, the rotation of the Earth serving by deflection to keep the two streams somewhat distinct.

A vertical circulation is likewise effected by the tidal currents, when they flow from deep to shallow water and back again. A most pronounced example of this is to be seen in the estuary of the St. Lawrence, where the deep Laurentian Channel ends rather abruptly near the mouth of the Saguenay River. When the flooding tide comes to the end of the Channel, the deep water (cold and salt) actually comes to the surface and flows back along the south side of the Channel under the influence of the Earth's rotation. The mixing that results from this circulation produces the large volume of comparatively light water, that goes to make up the Gaspé current.

Biological Board of Canada,
Ottawa, Canada.

DYNAMIC OCEANOGRAPHY OF THE GULF OF MAINE

HENRY B. BIGELOW

Within the last few years the mathematical applications of Bjerknes' hydrodynamic theory to ocean-currents have been so simplified by his followers, especially by Ekman, Helland-Hansen, Sandstrom, Hesselberg and Sverdrup, and by Smith, that no oceanographer need longer fear to give dynamic consideration to the seas with which he is concerned.

Unfortunately, however, dynamic calculation of circulation in the sea depends on a double assumption, which is seldom—perhaps never—capable of mathematical proof in any individual case; first that somewhere in the picture there is motionless water that can be used as the basis from which to proceed (for currents deducible from the dynamic gradients are relative, not absolute); and secondly, that our choice of a base as the most nearly stationary has been correct. Every opportunity to compare the gradient currents (by whatever method of dynamic expression deduced) with the actual drift observable by direct evidence, as of the current-meter, of the drifts of floating objects, or of the distribution of temperature and salinity, is therefore most welcome. Such a test is now being carried out on the Ice Patrol, by Lieutenant-Commander Smith, in the form of a continuous comparison of dynamic gradients with the drifts of icebergs. And the Gulf of Maine has offered opportunity for attacking the problem along rather a different line, by correlating the dominant circulation with the prevailing dynamic state. This procedure, you will note, is the reverse of that which has been followed in several recent studies, proceeding backwards (if you will) from effect to cause.

As perhaps most of you know, the Museum of Comparative Zoology, and the United States Bureau of Fisheries, have been cooperating in an oceanographic exploration of the Gulf of Maine since 1912. And as our observations on temperature, on salinity, and on the distribution of the plankton have accumulated it has become increasingly clear that the circulation of the Gulf is not purely tidal, as had generally been supposed, but that, underlying the strong tidal currents, the Gulf as a whole is the site of a decidedly definite dominant drift. This, briefly, can best be described as a great anticlockwise eddy, with a movement into the eastern side of the Gulf, circling around its coastline from east and north to west and south, combined with discharges out past Cape Cod and clockwise around the eastern end of Georges Bank.

The regional distribution of salinity, as it appears on horizontal projection, is especially instructive in this partially enclosed coastwise sea, for here the only possible source for water of very high salinity is the open ocean off the mouth of the Gulf. Consequently, any tongue of water much more saline than the rest of the Gulf is the clearest evidence of an actual indraft from off-shore, shortly previous if not at the time. An indraft of just this sort, into the eastern side has appeared repeatedly on the salinity charts, for the upper stratum of water. As an example, the distribution of salinity at forty meters, for the mid-summer of 1914, is perhaps the most demonstrative, with a tongue of high salinity circling inward from the offing of Cape Sable past the mouth of the Bay of Fundy to the offing of Penobscot Bay; the successive isohalines indicating (if less clearly), a continuation of this anticlockwise drift around the western side of the basin of the Gulf.

If we descend to the one-hundred-meter level, where the Gulf is entirely enclosed except for the two narrow channels which pierce its offshore rim, the same type of circulation, with a highly saline tongue intruding by way of the eastern channel, has appeared on chart after chart, year after year, exemplified by one for February to March, 1920.

The distribution of temperature is not so good an index to the movements of water within the Gulf, because for most of the year the indraft is not sufficiently different from the basin of the Gulf in temperature to make its presence obvious in this way. But when closely analyzed, with due regard to the seasonal warming and cooling taking place *in situ*, temperature, too, gives evidence of the anticlockwise drift, especially at the season (Spring) when the icy Nova Scotian Current is flooding into the Gulf from the east.

It chances that the Gulf is an especially favorable area for mapping the drift by the distribution of the plankton, because it receives animal and plant immigrants from two dramatically opposite sources, viz., "tropicals" from the warm waters of the so-called "Gulf stream" offshore, and "Arctics" from the icy Nova Scotian Current to the east. And such of these as can only survive for a brief period in the temperatures of the Gulf—too cold for the one, too warm for the other—are the best of natural drift-bottles there. When the recorded occurrence of planktonic immigrants of this transitory category are plotted—tropics and arctics together—it develops that all, irrespective of temperature affinity, are condensed in a band extending inward from the Eastern Channel and Browns Bank, around the periphery of the basin to Massachusetts Bay and the offing of Cape Cod. And the scarcity of brief-lived "tropical" over Georges Banks, where only an odd one has been taken, and in the central part of the basin is especially interesting, because they would naturally be most frequent there, not least so, if they drifted in from the south direct.

The evidence of current-measurements also points to an anticlockwise drift around the Gulf. A considerable number of such measurements have been taken in the eastern side of the Gulf by the Tidal Survey of Canada, and in the western side by the U. S. Coast and Geodetic Survey. When plotted, the non-tidal components of these show a northward component along Nova Scotia toward the Bay of Fundy, an anticlockwise movement within the Bay, a southerly and southeasterly drift past

Massachusetts Bay and Cape Cod, and so outward past Nantucket Island and Shoals.

Still further to test whether the Gulf is as constantly dominated by an anticlockwise type of circulation as these lines of evidence suggest, we have put out about 1,500 drift-bottles in its western side, since 1922, and the Biological Board of Canada has put out many more in its eastern. In this brief talk I can show you the results of only a few of these; enough, however, I hope to make it clear that the recoveries are reconcilable only on the basis of anticlockwise drifts. Thus drifts from the Canadian series, set out in the Bay of Fundy, tend toward Massachusetts Bay and Cape Cod. Bottles set out off the southern angle of Cape Cod divide, part going out of the Gulf and to the south and west around Nantucket Shoals, but part showing a return drift across the southern side of the Gulf, some to Nova Scotia and the Bay of Fundy but others veering southward around the eastern end of Georges Bank. Bottles set out off northern Cape Cod have again shown easterly drifts, some leaving the Gulf via the route just mentioned, others circling northward to the Bay of Fundy.

Bottles from the offing of Cape Ann have for the most part gone to the Bay of Fundy with some set out close in to the Cape going southward past Cape Cod. Drifts from a line running out from Cape Elizabeth have been similarly divided, the majority crossing the Gulf to Nova Scotia and the Bay of Fundy, a few, however, coasting southward past Cape Cod. And some drifts from the offing of Mt. Desert have also coasted westward, but the majority have gone to the Bay of Fundy after time-intervals averaging about as long as those from the offings of Cape Elizabeth and Cape Ann which is reconcilable with the latter drifts only on the assumption that the Mount Desert bottles, too, circled out anticlockwise, around the basin, en route.

Thus, in summary, clearer proof of the domination of the Gulf by this type of circulation could hardly be asked than results from the cumulative evidence briefly sketched here.

Our next question is, how closely can this type of circulation be correlated with the dynamic state prevailing in the Gulf?

Our method of attack here has been projections of the dynamic contours, at the surface, and at deeper levels, by the methods developed by the Scandinavian oceanographers, and recently explained by Commander E. H. Smith, with whose very clear exposition I hope you are familiar. When it is possible to cover an entire area with a network of stations, this horizontal projection of the dynamic tendency has an obvious advantage over individual dynamic profiles, in its graphic presentation of the prevailing gradients; it also has the advantages of simplicity and of yielding a picture of a type with which the weather map has made us all familiar. I need hardly add that the dynamic contour charts take into account the regional inequalities of density for the whole column of water surface to bottom, and they are, of course, based on precisely the same hydrostatic principles as are the dynamic profiles.

In the present instance, the contour-lines are drawn for every dynamic centimeter, referred to the part of the picture where the water is heaviest, i. e., the surface lowest. And, for reasons it would take too long to explain here, how closely they stimulate the actual slope of the surface depends partly on how truly applicable, in each instance, is the arbitrary connection which Jacobsen and Jensen have shown necessary between shoal-water stations of different depths.

Such charts are to be interpreted, essentially, just as a weather map is. Thus the water tends, dynamically to drift directly inward toward the "low"; but the deflective effect of the Earth's rotation—to the right in our hemisphere—actually turns the drift nearly parallel to the contour-lines. These may then, for all practical purposes, be interpreted as stream-lines except (and this is a very important exception) as interfered with by the contour of the bottom.

From month to month the contour-charts for the Gulf show an underlying similarity, with the surface highest along shore, lowest somewhere over the basin or the Eastern Channel and outside the edge of the continent.

At the end of the winter the "low" is centered in a trough-like depression over and just within the Eastern Channel, with the surface higher over the Banks on either hand, and rising radially toward the coastline, pointing to an essentially cyclonal (i. e., anticlockwise) eddy around the northern side of the basin, directed eastward around its southern side by the submarine slopes of Georges Bank.

It is not possible here to follow the seasonal changes in detail. In April, as the margin of the Gulf is freshened and lightened by the vernal inrush of land water, the dynamic gradient is steepened along the northern and western coasts, while the "low" flattens out to include all the eastern side of the basin, pointing to a speeding up of the drift westward and southward around the coasts of Maine and of Massachusetts. By May this gradient is still steeper next the land in the western side, building up the "spring-current" to the southwest and south, with which local fishermen are familiar. In summer we have found two interrelated low (cyclonal) centers, one in the offing of the Bay of Fundy, the other over the Eastern Channel and off its mouth. Such a distribution of gradients falls, as a whole, very closely in line with the bottle-drifts recorded in summer, eddying around a point in the offing of the Bay of Fundy, and northward along the west coast of Nova Scotia; with an outdraft around the eastern end of Georges Bank. Discussion of events off the seaward slope of the latter, interesting though this would be, cannot be attempted here.

In all these cases, the Archimedean forces, as deflected by Earth-rotation, and as directed by the slopes of the bottom, tend to set the Gulf in cyclonal circulation; they must, in fact, maintain a circulation of this type unless opposed by some overpowering counter force, and there is no such, in the Gulf.

And not only do hydrostatics tend to produce essentially the same type of circulation as is actually recorded by more direct lines of evidence, but the seasonal fluctuations in dynamic contour, and the shifts in the location of the low centers in general reproduce the seasonal variations in the observed drift, especially as regards the speeding up of the longshore current in spring, and the prevailing tendency for the bottles to turn more off-shore from the western side of the Gulf, in summer. And, perhaps most significant, there is no drift clearly indicated by the dynamic contours, of which we do not already have evidence, by bottle-drifts, or by salinity, temperature, or plankton. Without entering here into the question of calculated velocities, I can add that these at least fall within the order of magnitudes demanded by the current-measurements and by the drift-bottle experiments.

Thus we can hardly escape the conclusion that the dominant circulation of the Gulf draws its motive power in large part, at least, from the Archimedean forces set in motion by the regional inequalities in the specific gravity of the water. And this has practical value, as an additional bit of proof that in sea-areas of this sort it is safe to postulate actual drift from dynamic contour, if evidence of a more direct sort is not forthcoming.

However, as Dr. Huntsman has pointed out, another motive force also tends to set the water in circulation anticlockwise around basins in the northern hemisphere, clockwise around shoals—the horizontal tidal oscillations as deflected by the Earth's rotation. In the Gulf of Maine this in general parallels the effect of the dynamic gradients. And it will need much closer analysis than has yet been attempted, to evaluate the relative importance there, of the two.

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OCEANOGRAPHIC INVESTIGATIONS OF THE INTERNATIONAL ICE PATROL

EDWARD H. SMITH [1]

A paper presented before the section of oceanography of the Geophysical Union at its meeting last year gave a brief description of the International Ice Patrol, a humanitarian service that is supported financially by the principal maritime nations of the world in order that life and property on the North Atlantic may be properly safe-guarded. Some phases of the scientific work that is being carried on by the Ice Patrol were described, especially certain investigations in physical oceanography of the ice regions south of Newfoundland where Arctic and Tropical waters meet. The Patrol, in the course of its fourteen years of ocean service, has come to recognize the great value inherent to scientific observations from which practical, useful information may be drawn.

One of the most important problems connected with the ice-work is keeping track of individual bergs that have finally drifted south of the Grand Banks and thence become governed in their movements by the particular features of the circulation prevailing at the time in that part of the North Atlantic. Bergs in such situations are distinct dangers to shipping, and extremely grave menaces, too, if, as it often happens, they are carried southward across the more populous lanes of traffic between the United States and Europe. The simplest and most obvious method of cruising over the ocean area southeast of Newfoundland, fixing the position of the ice from day to day by astronomical sights, is one not always possible. For one thing the ice often becomes widely distributed, and another difficulty is that due to fog. Almost everyone has heard of the notorious Grand Banks fogs and the danger they bring every year to fishermen. The spring and summer when fog envelops the Grand Banks and its surrounding cold waters, is the time when icebergs in greatest number invade this part of the North Atlantic.

The movement of icebergs is almost entirely determined by the ocean-currents, so that next best to actually sighting the ice from day to day, is to have a picture of the current-system into which the bergs are being carried. An accurate map of the currents as they then prevail is, moreover, the key to predicting the drift of icebergs liable to enter the trans-Atlantic shipping lanes. Such hydrographic information also, aside from its forecasting features, is bound to be of value as it throws light on the behavior of drifting ice in general in the northwestern North

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[1] **Read for Lt. Commander Smith by Capt. A. L. Gamble, U. S. Coast Guard.**

Atlantic. Realizing the practical benefits to be derived from such information, Admiral F. C. Billard, Chairman of the International Ice Patrol Board, in 1924 sent an officer of the United States Coast Guard to Europe in order that the Patrol might learn the newest methods practiced abroad in physical oceanography, particularly as they might be applied to improving the ice service.

The specialized field of physical oceanography, for which the Ice Patrol has found a practical use, has its genesis in dynamics and geophysics contributed by Professor V. Bjerknes, a Norwegian physicist, some thirty years ago. Bjerknes later became interested in applying his theories to a study of the atmosphere, and by so doing greatly improved the art of weather forecasting. It has remained chiefly for a group of Scandinavian oceanographers to apply and develop a workable method for Bjerknes' mathematics to be applied to motion in the sea. As a result of such labor, we are furnished today with a method whereby, if the temperature and the salinity of the ocean are given from known depths and places, the direction and velocity of the currents can be computed and mapped. A pamphlet of some fifty odd pages with illustrations published as Bulletin No. 14 by the U. S. Coast Guard describes this work.

The two Coast Guard cutters *Tampa* and *Modoc,* that are maintaining a continuous guard at the present time in the North Atlantic, are equipped with duplicate sets of oceanographic gear and instruments, concentrating their efforts almost wholly on observations of temperature and salinity of the water. They are each equipped with one electric hoist designed for rapid reeling, six thousand feet of one-quarter-inch wire running rope, two meter-wheels, twelve Greene-Bigelow water-bottles with messengers, twenty-four Negretti & Zambra deep-sea thermometers, six cases citrate bottles with rubber gaskets, and one electric salinity tester. A station team of six men is detailed from each watch and, as the work at all stations is identical, the men soon become very expert. With a well-trained team it is possible to complete all of the observations down to 1,200 meters within one hour, including the time necessary to stop the ship for the sounding and also to get under way. It is important that no more time than absolutely necessary be borrowed from the primary duty of the Patrol—ice scouting.

The critical area upon which the Ice Patrol's attention becomes focused is about 40,000 square miles (an area nearly equal to that of the state of Pennsylvania), off the Tail of the Grand Banks. This locality is sometimes called the gateway to the Atlantic because it is by this point that nearly all of the Arctic ice passes on its journey to temperate latitudes. The currents over this area also are subjected to devious paths and a berg may be carried slowly to the westward in one part of the current, while other ice farther off-shore is perhaps swept across the bows of approaching passenger-ships. A practical and comprehensive current-map of this 40,000 square mile rectangle requires a basis of twenty to twenty-five well-distributed stations spaced twenty to thirty miles apart. A vessel capable of maintaining a speed of ten knots is able to perform the observational work, including running time between stations, in about four days. Three times during the ice season of 1926 the Patrol ship on duty off Newfoundland conducted such a survey, and on one occasion the work was extended over an area of 100,000 square miles in the short space of four days. This is the first time in the history of oceanography that deep-sea observations have been treated immediately on board ship and the final results employed for a practical economic purpose. It demonstrates, moreover, what can be accomplished in charting the circulation in other little-known regions.

A policy of the Patrol for the past two years has been to keep an up-to-date current-map of the ice area on board at all times for reference and consultation in carrying out the work. A witness to the vessel making one of its routing current-surveys, sees the business-like side of oceanography. The ship is cruised at ten knots speed or faster over the ocean area it is desired to map, stopping at designated positions only long enough to secure a record of temperature and salinity from seven levels beneath the surface. The number of depths observed is reduced to a minimum without failing to secure an accurate record of the physical character of the water column from the surface down to homogeneous level. The following are the depths which satisfy all practical purposes and which have been adopted as standard for stations seaward of the continental shelf: 0, 25, 50, 125, 250, 450, 750, and 1,200 meters.

It may be rather surprising to some that we have today a real thoroughgoing, practical method of determining currents over thousands of square miles of ocean surface, the task being performed in the relatively short space of a few days—a great advance for, as we all know, current-compilation, especially of an ocean region, has always been a slow, laborious job. The material heretofore has been almost wholly limited to information regarding the behavior of the top few meters of an ocean. On the other hand, by computing the dynamic tendencies at a series of observational stations, and then translating these data into terms of motion, we now include consideration of the entire mass.

Current-data outside continental shelves have been amassed by previous methods from three principal sources: (*a*) ships' navigation; (*b*) floating objects; and (*c*) current-meters. The smooth, gigantic gyrals that are shown dominating each of the hemispheres in the two major oceans, have been drawn principally from the navigational records contained in thousands of log-books. The nature of such a compilation, however, admits of a general interpretation only. The success of releasing large numbers of drift-bottles depends to a great degree upon elapsed time and the percentage of recoveries which result from stranding along bordering coastlines. Obviously, an open ocean area does not lend itself to such a form of determinations. Lastly, there are great practical difficulties connected with the proper operation of current-meters in the deep ocean despite the ingenious devices that have been developed in this line of mechanical registration.

The dynamic topographical maps of the sea surface, and of the other observed isobaric surfaces, mark a real, noteworthy progress in the science of oceanography. These charts, drawn as a result of substitution in Bjerknes' formulae, show the contour in dynamic meters of the chosen decibar surface relative to that of a deep decibar surface, which is the one usually employed as a base-plane. If there is no motion on this deep abyssal surface then the contour-lines connecting equal dynamic values at the other depth will be the stream-lines of the gradient current, provided that the accelerating force of friction is negligible. Friction is of small consequence in regions exhibiting a circulatory system of even moderate character and in common gradient currents the effect to accelerate or retard is probably not greater than 0.05 to 0.07 of a knot. This figure is so small that it can be disregarded in practical determinations. All motion is included in the calculations except that due solely to friction of the winds to the surface-layers. Prevailing winds, however, such as the "trades," amassing tremendous volumes of water and initiating gradient currents, are factors included in the calculations. Winds of a temporary character on the other hand—for example, those accompanying travelling depressions—are not accounted for in Bjerknes' formulae, but the effect of this latter class, fortunately, is of a self-compensating nature and they disturb prevailing conditions for brief periods only.

Water or air unqualified by boundaries begins to flow with a tilting of the isobaric surfaces directly in proportion to the steepness of the gradient. Circulation in the sea is almost always at a maximum near the surface, and approaches zero at appreciable depths. Isobaric surfaces, therefore, undulate wider in the upper levels than they do nearer the ocean-bed and so the dynamic topography of the sea-surface can be brought in relief if it is compared with a base-plane located in the abyssal depths.

We accept as a matter of course the topographic features of landscape, but our notions of sea-level are so fixed that it may at first prove disconcerting to contemplate the surface of an ocean as having a prevailing conformation. About one hundred miles directly south of the Grand Banks, the Ice Patrol, however, on several different surveys separated by years, has found a northerly slope to the sea-surface averaging one centimeter per mile. This translated into motion is a current of one knot, which, in terms of mass means approximately two million tons of water per second is being transported eastward between these two geographical positions.

To a student of oceanography the North Atlantic presents some very interesting topographic features. Trace, for example, the outline of our eastern continental shelf, Florida to Massachusetts, and there extended along the edge in similar form we shall find a trough in the sea-surface. This is the zone where oceanic and coastal water of similar specific gravity, but otherwise greatly differing character, mix creating a mass denser than that on either side of it. Looking out across the density-wall into the basin, we find the surface of the water inclines upward, relatively boldly at first, and then less and less obliquely until off-shore a hundred, perhaps several hundreds miles, we should expect to find small changes and the topography without major definition. The North Atlantic aquain (to coin a word), according to this view, has the general shape of a broad, flat plateau, the boundary-slopes of which conform largely to conditions prevailing around the continental edges.

The direct impelling forces of an ocean-current are sought not back to a river-like source, but essentially stretched along the entire extent of flow. The difference in density between coastal and oceanic water (and we use these terms technically), is constantly producing tremendous quantities of energy; power enough to run the world is ceaselessly being converted at these places into the phenomenon of motion. Thus, finally, it is plain to see why the most striking circulation in ocean-basins takes place along the slopes, the edges, and the ridges. The great transitions in temperature and salinity, which are met when crossing from our eastern shelf into deep water, form a veritable zone of energy a thousand miles or more in length, and this is the factor which gives the Gulf Stream its often described resemblance to a river. The energy, moreover, that is being converted into moving the water-particles of the Gulf Stream at a point on the slope, say south of Marthas Vineyard, is just as vital for the identity of the Gulf Stream as the impetus that was imparted to these same water-particles months before when they were in the Carribean.

Three ocean-areas, relatively a very small portion of the North Atlantic Ocean itself, have already been surveyed in accordance with Bjerknes' formulae: The Faroe-Shetland Channel, the eastern North

Atlantic from the European coast west to the 30th meridian, and the waters off the Tail of the Grand Banks where Labrador Current and Gulf Stream meet. One of the main objects which gives stimulus to the *Meteor* Expedition, now in the South Atlantic Ocean, is this "leveling" work of hydrographic engineering. We wish in closing to emphasize the fact that delineating the dynamic topography of the oceans, at least in their main features, is a new and promising field unfolding in physical oceanography.

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A STUDY OF THE NORTHERN PART OF THE LABRADOR CURRENT

C. O. Iselin

In the discussion of the Labrador Current which follows, the observation made last summer during the cruise of the schooner *Chance* will be used as a basis. A preliminary report of this trip will shortly appear in *Science*. During the cruise two profiles were run at right-angles to the general trend of the Labrador coast, one just south of Hamilton Inlet and the other off Nachvak Bay which is about sixty miles south of Cape Chidley, the northeastern extremity of the Labrador Peninsula. Besides these physical observations on the Labrador Current, the Expedition made a collection of the flowers and plants from the Torngat Mountains and explored several of the fiords which indent the northern part of the coast. Plankton-hauls and collections of the salmon, char, and cod were also made. Since very little oceanographic work had been done off the northern part of Labrador, the region proved most profitable for a short cruise.

In general, we will limit this discussion to the Labrador Current between the Strait of Belle Isle and Davis Strait, since south of the former is the province of the Ice Patrol and therefore well known, and north of the latter there seems to be no subsurface temperature and salinity observations, which are so necessary in the modern treatment of ocean-currents. Besides the work of the *Chance* there are surprisingly few other subsurface observations from this sector. The *Scotia* in 1913 made three stations in the region of our southern profile, two of these being over the continental shelf and therefore directly comparable. From this point north to Davis Strait the information is limited to a few surface-temperatures and many observations on ice-movements. In 1924, under the direction of Dr. Hjort, a section was run across the narrowest part of Davis Strait which gives a very good picture of both the Labrador and West Greenland currents. Since this is the most northerly of the available profiles, it will be described first.

In spite of the fact that at any depth the water is about 3° colder on the western side of the Strait than it is on the eastern side, the isopicnals are caused to slant downward on nearing the Baffin Land shore by the low salinity of all the Labrador Current water. The temperature and salinity profiles show that the water to the west of that having dynamic cause for a southward movement is colder than 0° (except near the surface and below 200 meters) and much of it is less saline than 33.5 per thousand. In the center of the Strait the isopicnals in the upper stratum are forced upward by a core of North Atlantic water (with temperature of about 3.5° and a salinity of about 34.7 per thousand) lying along the outside slope of the Greenland coastal bank. This water acts as the density-wall between the Labrador and West Greenland currents and makes possible the dynamic urge which drives the one southward and the other northward. It will be noted that the gradient for both currents is only slight. Attempts at velocity-calculations are almost worthless since there are so few stations and these vary so much in depth. Yet it is evident that even at this northern point the Labrador Current is driven by the slant of the density-lines *in situ* and is not primarily an overflow of Arctic water. An interesting speculation arising from the profile is whether or not the West Greenland Current plus the land drainage into Baffin Bay does not supply enough water to account for that which is carried out of Davis Strait by the Labrador Current.

We must suppose that the current continues southward hugging the Baffin Land shore until it reaches Hudson Strait where it finds an opening to the left and urged by the rotational effect of the Earth some water undoubtely starts up the north side of the Strait. But in Hudson Strait conditions are very different from Davis Strait. There can be no central core of North Atlantic water to act as a density-wall between the lighter water along each shore since the narrow mouth of the Strait is blocked by the greater part of the Arctic Current which continues southward past it without entering. Therefore the current in Hudson Strait will not arise from gradients set up in the Strait itself, but will be dominated by the outflow of fairly fresh water from Hudson Bay. The importance of this great source of fresh water to the oceanographic conditions along the Labrador coast does not seem to have been enough emphasized. This outflow of water probably results from an actual raising of the surface of the water in Hudson Bay by the inflow of rivers and streams from the immense land-area which drains into it. Hudson Strait is also the first outlet to the right for the counter-clockwise coastal current which has been reported in the Bay itself. What Arctic water does round Resolution Island and starts into Hudson Strait meets then the outflow of water from Hudson Bay, mixes with it, and is probably mostly sooner or later carried out again around Cape Chidley, to continue south as the well-known Labrador Current.

Hudson Strait appears to be an ideal place for tidal mixing, for not only is the tidal range great (thirty feet on the Baffin Land shore) but very strong currents (four to five knots) are reported off Cape Chidley and Resolution Island. The water of Hudson Bay origin cannot differ much in character from the water of the Labrador Current with which it mixes in the mouth of Hudson Strait, for the temperature and salinity profiles made by the *Chance* just south of Cape Chidley shows water of the same character as on Dr. Hjort's profile of Davis Strait.

It is interesting from the temperature-profile of the waters of Nachvak Fiord that more than half of the water in the Fiord is half a degree colder than any of the Labrador Current. This would indicate a gradual warming up of the water over the continental shelf as the summer advances. The deeper waters of the Fiord, being motionless and protected from solar warming by a very stable surface-film of nearly fresh water, probably do not vary at all in temperature. From the mouth of the Fiord to the edge of the continental shelf is a distance of about ninety miles. This region is entirely covered by icy water. Off shore is the warmer water of North Atlantic origin and the most striking thing in this profile is the very abrupt temperature-transition which takes place between the two outer stations where in a distance of about fifteen miles the whole column of water increases about 5° in temperature.

The salinity-profile shows this same abrupt transition between the water of the Labrador Current and the homogeneous mass of the North Atlantic water just outside the edge of the continental shelf. Off the mouth of the Fiord is a region where the water has about the same salinity and temperature from top to bottom. This is probably water which has rounded Cape Chidley and which has been kept well mixed by the tidal current and waves over the many shoals and reefs just north of the profile. As in Davis Strait all the water of the Labrador Current has a salinity of less than 33.5 per thousand.

When we consider the density-profile, a puzzling fact is at once evident. The isopicnals are practically horizontal for the first sixty miles off shore, but outside of this is a narrow band where the dynamic gradient is most pronounced. Is the current confined to only this outer band as is indicated by the profile, or is all the water over the continental shelf moving south? From stations twenty-four miles apart there is a rise of 24.7 dynamic centimeters. This is enough to produce a current of about one knot along the outer edge of the icy water. Yet while making this profile we noticed particularly that, although the majority of the bergs were well off shore, a noticeable current existed in the region where the isopicnals are perfectly horizontal. My only suggestion is that possibly the in-shore water gets its movement from the water which is forced out of Hudson Strait as a result of the immense land-drainage into Hudson Bay.

Continuing south again for the next three hundred miles, the Labrador Current flows along the continental shelf constantly mixing with the warmer more saline water along its outer edge, but maintaining its low salinity by the inflow of the many streams along the coast. The characteristic low temperatures persist since as the water flows southward the solar heat cannot penetrate by convection because of the great stability of the water-layers, and the light surface-film itself cannot warm appreciably because of the melting of the many bergs which the Current carries south. Thus we find the Labrador Current off Sandwich Bay nearly the same in volume and character as when it passes through Davis Strait.

The Sandwich Bay temperature-profile shows a much greater volume of water colder than — 1.5° than did the Nachvak profile. This is another indication that the Labrador Current as a whole warms up very slightly as the summer advances because the Sandwich Bay profile was made three weeks earlier in actual time than the Nachvak profile and we must add another four weeks more for the real time interval as at the rate of twelve miles a day it would take the water twenty-eight days to move southward the three hundred and thirty miles between the two sections. Right in-shore and near the surface is a narrow band of very much warmer water. This is the coastal stream always met with by boats sailing near the coast and it consists of the very fresh water which flows out of the many bays. Since this profile is just south of Hamilton Inlet, the comparatively warm coastal stream, really a current within a current, is particularly well developed. A conspicuous gulley in the center of the profile, filled with water 2.5° warmer than that above it, is something of which the few soundings on the chart gave no indication. It is not at all characteristic of the coast in general. Off-shore we notice a narrow band of icy water extending eastward between the depths of forty and one hundred meters, bulging outward the nearly vertical isotherms of the transition region between these depths. Bottom warming is another conspicuous feature of the profile. It is present wherever the water is more than one hundred and fifty meters in depth. The Labrador Current is so very light that it tends to spread out on the surface and when it passes over places that are deeper than two hundred meters, warmer water is drawn in from off-shore and a slight mixing results.

The Sandwich Bay salinity-profile shows no important difference from the Nachvak profile except for the great freshness of the warm coastal stream the surface-waters of which have a salinity of less than 20 per thousand. But when salinities and temperatures are converted into densities a very different picture results. The slant of the isopicnals is continued across the whole profile and is not confined to a narrow band as off Nachvak. Thus all the water over the continental shelf, except in the bottom of the gulley, is being driven south by a moderate dynamic gradient and the current is distributed equally across the whole distance. Between two stations one hundred and eight miles apart there is a rise of 25.3 dynamic centimeters. This is almost exactly the same rise which was present off Nachvak in a distance of twenty-four miles.

This broadening and slackening of the gradient is well brought out by the horizontal projection of the dynamic forces. The five-dynamic-centimeter contours of only two sections make it, of course, impossible to represent with any certainty the course of the dynamic isobaths along the middle part of the coast. The possible course, however, of the strength of the Current may be based on observations on the distribution of bergs. These tended to be strung out in long rows well off-shore in the region off Nachvak and more scattered off Sandwich Bay. Along the middle part of the coast, especially near Cape Mokkovick and Cape Harrison, great numbers of bergs were aground close to the shore. On the other hand, the coast near Hamilton Inlet and also for the first one hundred and fifty miles south from Cape Chidley was comparatively free of ice.

Turning now to the velocity of the Current along the Labrador coast we find that a good average figure for the whole length of the coast is ten to fourteen miles a day. Off Nachvak, where as we have seen the gradient was steep over the edge of the continental shelf, velocity-calculations give as much as twenty-three miles a day for a narrow band, but down off Sandwich Bay velocity-calculations give at the most ten miles a day. In navigating I found that the Current was generally slightly stronger than this. The drift of the *Polaris* floe-party in March and April, 1873, averages fourteen miles a day from off Hudson Strait to a point off Hamilton Inlet where their daily average decreased to about eight miles a day until they were rescued. At this time of year the prevailing northwest winds probably accelerated their drift, while the Current itself was probably slightly slower than in summer for the main force which maintains the low density of the coastal water, that is land-drainage, had not yet come into full operation. We must expect a seasonal fluctuation in the strength and volume of the Current depending on the liberation of great quantities of fresh water each spring all along the coast.

Finally, what is the state of the warmer, more saline, North Atlantic water outside the edge of the continental shelf? Is this moving south also as most current-charts for the North Atlantic indicate? Light is thrown on this question by a profile made by the *Scotia* in 1913 just south of the Strait of Belle Isle. This consists of six stations and extends about five hundred miles off-shore in a line roughly perpendicular to the general trend of the coast. When the densities for the profile are calculated and plotted, a most interesting fact is at once evident. Just off the edge of the continental shelf the isopicnals slant upward indicating a southerly current about one hundred miles wide, but for the next three hundred miles off-shore the isopicnals at all depths are perfectly flat showing that if the warmer, more saline, off-shore water was moving south at all, it was not through dynamic forces. There is no reason to think that this same condition does not exist further north where the *Chance* sections were too short to include the flattening out of the density-lines. The Labrador Current is then primarily an icy stream over the continental shelf which flows southward inside of the motionless homogeneous mass of North Atlantic water which is present along its outer edge even to a point north of Davis Strait.

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RECENT PROGRESS IN THE DYNAMICAL OCEANOGRAPHY OF THE NORTHEASTERN PART OF THE NORTH PACIFIC OCEAN

GEORGE F. MCEWEN

INITIAL EXPLANATIONS OF COLD WATER OFF THE WEST COAST OF NORTH AMERICA

Numerous observations distributed through a period of many years have established the presence of abnormally cold surface-water contiguous to the Pacific Coast of North America. The results of several different attempts to account for this interesting phenomenon without the aid of an adequate dynamical basis have been published. Some of the conclusions thus reached are briefly reviewed here.

(1) The Japan Current in its slow and interrupted flow eastward is partly deflected to the south from Southern Alaska. It loses its heat as it flows southward and produces the fogs and rains in the region of Washington and Oregon. By the time it reaches San Francisco it is a cold current simply because it is intruding into a normally warmer area.

(2) A cold Arctic current flowing southward is powerful enough to alter materially the direction of the Japan Current as far south as Point Conception in latitude 34° north. Thus the warm Japan Current is prevented from washing the shores of North America north of Point Conception, but from here on it flows over the cold current. Both this cold in-shore current and the Japan Current flow southward and produce a narrow eddy-current flowing northward next to the shore.

(3) The warm Japan Current becomes so diffused in its passage eastward from Japan as to be ineffective before reaching North America, and the Alaskan glaciers are not extensive enough to materially cool the adjacent waters. The extensive Antarctic continental glacier cools the ocean water which sinks to the bottom and is forced northward by the supply accumulating behind it. This bottom current increases in velocity as it flows northward from the equator through a contracting section. Thus its velocity is doubled at the latitude of Vancouver's Island and it is so forced against the shore by the Earth's rotation that it rises to the surface and flows southward, having no other outlet. It is held against the coast in opposition to the westward deflecting force due to the Earth's rotation by the eastward pressure of the northward-flowing bottom current.

(4) Since observations indicate narrow belts of alternately warm and cold water, directed to the west of the coast, the hypothesis of a surface-stream flowing south from a region of low temperature is untenable. Moreover, the coldest part of the cold-water belt is at latitude 40° north near Cape Mendocino. Thus the only explanation involves an upwelling of cold water from adjacent ocean-depths. Such an upwelling results from a general drift of the bottom-water eastward in agreement with the observed direction of the surface-drift. On striking the slope of the continental shelf, this eastward drifting bottom-water is deflected upward. Local variations in the coastal water-temperatures are due to submarine valleys and other irregularities of the continental shelf.

RELATED OCEANIC AND ATMOSPHERIC CONDITIONS IN THE NORTH PACIFIC

The average or typical conditions of wind, weather, and barometric pressure, over the ocean for each month are shown on "Pilot Charts" prepared by Governmental weather bureaus and navy departments from observations furnished by navigators. The typical seasonal variation of the barometric pressure-gradient between the North Pacific "High" and the coast at latitude 35° north, and the typical seasonal variation of the ocean-winds between the center of the "High" and the coast have been found from such charts. The surface-temperature of the sea that would correspond to latitude and time of year in the absence of the disturbing effect of oceanic circulation, called the "normal" temperature, has been approximately determined from compilations of millions of temperature-observations. Typical monthly surface-temperatures have been found from a series of daily in-shore observations near the town of Hueneme at latitude 34° 09′ north begun in 1919 by the Scripps Institution. Monthly averages of the estimated normal for latitude 34° 09′ north minus actual temperatures have been found. [These values are all presented in Table 1, columns (1), (2), and (3), respectively.]

TABLE 1—*Typical seasonal variation of the barometric pressure-gradient, the ocean coastal winds, and the reduction of the surface coastal temperature below the "normal" value for the latitude 34° 09′ North*

Month	(1) Pressure-gradient in 1200 miles	(2) Ocean winds velocity per hour	(3) Temperature-reduction at the surface
	mm	*miles*	°
January	3.05	8.5	1.8
February	3.90	9.7	1.3
March	4.94	11.2	1.5
April	5.70	12.8	2.6
May	6.32	14.3	3.7
June	7.00	14.4	5.4
July	7.5	16.5	6.2
August	7.8	16.5	6.1
September	7.73	16.2	5.8
October	7.0	12.1	5.0
November	5.76	9.4	3.4
December	3.05	8.1	1.4
Averages	5.81	12.48	3.68

The table shows that all three quantities, pressure-gradient, ocean winds, and reduction of surface-temperatures, vary in the same way throughout the year. All reach a maximum value in August and a minimum value in January. The relation between the pressure gradient and accompanying winds accords with a well-known principle of meteorology based upon an equation between the pressure-gradient and the deflecting force due to the Earth's rotation and the wind-velocity at right-angles to the gradient. What is the physical basis of the indicated relation of the temperature reduction to the wind?

EKMAN'S DYNAMICAL THEORY OF WIND-DRIVEN CURRENTS

About twenty-five years ago Dr. Fridtjof Nansen's observations on winds and surface-drift in the North Polar region revealed a definite tendency of the current to deviate to the right of the wind. At his suggestion Dr. V. W. Ekman, a mathematical physicist, formulated mathematically the problem of wind-driven ocean-currents, using the general equations of the motion of a viscous fluid, in which the deflecting force due to the Earth's rotation was included. Also, he replaced the coefficient of viscosity by a coefficient of "turbulent friction," the value of this coefficient being determined by applying the solution of his formal equations to field data. His elaborate analysis of the problem, including the effects of latitude, depth of water, neighboring coasts, and topography of the bottom, has furnished oceanographers with the first mathematical equations applicable to this problem that rest upon an adequate physical basis. Certain of his conclusions are briefly summarized here:

(1) The surface current caused by wind tends to be deflected to the right of the wind through an angle of 45° in case the ocean is very deep and in the absence of coastal barriers.

(2) The velocity of the resulting current varies directly as the wind-velocity and inversely as the square root of the sine of the latitude. In general the current-velocity is roughly two per cent of the wind-velocity.

(3) The current at any level below the surface tends to deviate more to the right, the lower the level, and its speed decreases. It averages only three or four per cent of its surface-value when the direction is 180° from that at the surface.

(4) The resultant momentum of this layer of water is directed at right-angles to the wind.

(5) A difference in level at the surface resulting in a pressure-gradient will cause a current to flow at right-angles tc the gradient except near the bottom where the direction turns more in line with the gradient.

(6) The thickness of this bottom-layer within which appreciable turning of the velocity in the direction of the gradient occurs is about the same as the thickness of the upper layer within which the velocity turns through an angle of 180°. This thickness, called the "depth of frictional influence," varies directly with the wind and inversely as the square root of the sine of the latitude, and averages roughly one hundred meters.

(7) As the depth of the sea decreases from this "depth of frictional resistance" the frictional effect becomes of relatively greater importance than the effect of the deflecting force. The current tends to deviate less from the direction of the pressure-gradients or frictional drag of the wind at the surface.

COLD COASTAL WATER DUE TO UPWELLING EXPLAINED BY EKMAN'S THEORY

Observations show that, except for one or two months in the winter, the ocean-winds along the Pacific Coast of North America from about latitude 20° to 45° blow towards the southeast or southwest. This agrees with the general relation between winds and isobars. Accordingly, there is a component of the wind parallel to the coast and directed to the south. Therefore, according to Ekman's theory, surface-water is carried along the coast and to the right (to the west) of the coast by this component. This gives rise to a lowering of the water-level near the coast and an elevation to the west. Hence, along the bottom, the pressure of the water decreases toward the coast. Accordingly, near the bottom the water flows toward the coast and to the south thus replacing the water that is being removed from the layer near the surface. In between the surface-layer and bottom-layer the flow is directed to the south and parallel to the coast. The resulting replacement of warm surface-water by cold bottom-water upwelling at a rate proportional to the wind-velocity explains why the reduction of surface-temperature varies in the same way throughout the year as the intensity of the pressure-gradient out from the Pacific "High" and as the corresponding ocean-winds.

RELATION OF THE SUMMER PACIFIC "HIGH" TO IN-SHORE TEMPERATURES DURING A TEN-YEAR SERIES

There is strong evidence that the intensity of the Pacific "High," the velocity of the ocean-winds, and the reduction of surface-temperature below its normal value vary throughout the season in accordance with established physical principles. It is commonly known that this seasonal cycle obtained by combining available data pertaining to a sufficiently long period of years to obtain a characteristic or typical result does not exactly fit any individual year. Each year departs in a different way from the average, but a correlation between the departures of the related quantities should be revealed by data tabulated with reference to each of a period of years. The Imperial Marine Observatory at Kobe, Japan, has published monthly averages of sea-temperatures, barometric pressures, and cloudiness corresponding to five-degree quadrangles, based upon the meteorological logs of the Japanese merchant marine, beginning with 1916. Daily surface-temperature observations at the Scripps Institution pier, La Jolla, California, are available for the same period. A provisional index of the pressure-gradient for the four-month interval July to October was obtained by averaging the four highest pressures and subtracting from this average the average pressure at the coast for the same interval for the quadrangles between latitude 30° and 40°. In Table 2 the departure of this index from the mean for the period 1916 to 1925 is tabulated against the departure of the ocean surface-temperatures at La Jolla for the interval August 1 to October 15 from its average. Also, the departure of the seasonal rainfall for twenty stations from San Diego to Yosemite expressed in per cent of the average for the period is tabulated in the last column. As would be expected, the signs of the departures of the pressure-index are opposite to those of the temperature-departures.

The temperature-departures for the interval August 1 to October 15 indicate in advance the seasonal rainfall for the interval July 1 to July 1 of the following year. The signs of the rainfall-departures are opposite to those of the temperature-departures, except for the year 1924. In this year the pressure-departure and temperature-departure were both negative, contrary to expectations and to the general relation indicated by the table.

TABLE 2—*Departures of pressure-gradient from the Pacific "High" eastward, July to October; departures of La Jolla ocean surface-temperatures, August 1 to October 15; departures of seasonal rainfall in per cent of the average, 1916–25*

Year	Pressure-gradient per 1200 miles	Ocean surface-temperature departures at La Jolla	Seasonal rainfall-departures for Southern California
	mm	°	*per cent*
1916	.71	−1.1	14.5
1917	− .22	1.3	−14.3
1918	−1.42	1.8	−18.6
1919	4.51	−0.8	9.7
1920	−1.19	0.3	5.0
1921	....	−1.1	51.2
1922	0.14	0.3	4.6
1923	1.29	2.0	−33.8
1924	−1.92	−2.0	5.2
1925	0.66	−0.6	6.0

COMPUTATION OF UPWELLING VELOCITY FROM THE DIFFERENCE BETWEEN OBSERVED AND NORMAL SERIAL TEMPERATURES

The evidence of the presence of currents directed upward from the bottom along the California coast accounting for the low coastal temperatures suggested the following problem. Assume that the reduction of temperature at all depths is due entirely to upwelling. Compute the velocity of upwelling that would be necessary to produce the observed surface-temperature reduction below its normal value, corresponding to each month, assuming the rate of upwelling to be proportional to the component of the wind-velocity parallel to the coast found from the Pilot Charts. Below the depth of about fifty meters the Fourier equation of heat-conduction in solids or indisturbed liquids may be applied if the usual coefficient of conductivity is replaced by a larger virtual coefficient corresponding to the conditions of mixing or turbulence in the sea. Add the term

$$\left(-w\frac{\partial\theta}{\partial y}\right)$$

to the right side of the Fourier equation where w is the velocity of upwelling, θ is the temperature, and y the depth. The rate of change of temperature in terms of the virtual coefficient of conductivity, the vertical temperature-gradient, and the vertical velocity thus found are expressed by the differential equation

$$\frac{\partial\theta}{\partial t}=C\frac{\partial^2\theta}{\partial y^2}-w\frac{\partial\theta}{\partial y}$$

The observed value of the component of the ocean-wind parallel to the coast off San Diego can be well represented by a simple periodic function of the time, and the upwelling velocity w should be proportional to this periodic function. The solution of this equation was fitted to the observed monthly temperatures at a series of depths off San Diego. The average annual value of the upwelling velocity thus found was thirty-one meters per month. For a surface-layer of about ten meters thickness, the same equation may be used except for the first term in the right-hand member. This must be replaced by two terms, one denoting the rate of absorption of solar radiation and the other which is approximately a linear function of the temperature, denoting the rate of loss of heat by evaporation and back radiation. Using a deduction from the previous result that the temperature-reduction is proportional to the length of the interval and the temperature-gradient, the temperature was computed for each month at the ten-meter level. Combining these values with the temperature-reduction within the upper ten-meter level estimated from the equations that correspond to that level, the theoretical surface-temperature was found for each month. Table 3 presents the computed and observed temperatures at the forty-meter level, the predicted surface-temperatures, the observed surface-temperatures, and the normal surface-temperatures.

The agreement between the computed and observed temperatures at the forty-meter level illustrates how the formula for temperatures from that level to the bottom represents the observations, the constants being determined by fitting the formula to the observations. The agreement between the predicted and observed surface-temperatures is a severe test of the whole theory since the computations did not involve the observed surface-temperatures.

TABLE 3—*Observed and computed monthly temperatures at the forty-meter level, at the surface, and the normal surface-temperature off San Diego*

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.	Annual means
	°	°	°	°	°	°	°	°	°	°	°	°	°
Computed temperature at 40 meters...	13.4	11.9	10.6	9.9	9.9	10.6	11.9	13.4	14.7	15.5	15.5	14.7	12.7
Observed temperature at 40 meters...	13.4	13.4	13.0	11.0	10.5	10.8	11.8	12.7	14.8	16.8	15.4	14.0	13.1
Computed temperature at surface.....	15.2	14.6	14.5	15.2	16.4	17.7	18.8	19.5	19.5	18.8	17.7	16.3	17.0
Observed temperature at surface.....	15.0	14.3	14.6	15.2	16.1	17.8	19.7	20.6	20.2	18.8	16.8	15.5	17.0
Normal temperature....	16.5	15.7	15.9	17.1	18.9	20.9	22.4	23.2	23.0	21.9	20.1	18.1	19.5

An independent estimate of the average annual velocity of upwelling was made from the salinity-gradients, both horizontal and vertical, and the rate of evaporation at the surface. Since mean annual values only are considered, all conditions may be regarded as independent of the time. Therefore the total amount of water in a given volume, and the total quantity of salts in solution remain constant. Consider a rectangular volume from the surface down to a depth less than thirty meters (the depth of minimum salinity) having two of its parallel faces parallel to the coast line. Water will flow through the lower face and through the vertical face nearest the coast-line, also owing to evaporation at the surface it will flow out through the upper face. The resultant effect is a constant volume of water in the parallelopiped. The water that flows in at the base has the lowest salinity. The water that flows in through the face nearest to the coast has a different salinity from that which flows out through the face farthest from the coast. Parallel to the coast the salinity-gradient is negligible. At the upper surface fresh water leaves owing to evaporation but no salt is removed in this way. Expression of the fact of the constancy of the amount of water and of the amount of salt in this volume mathematically, results in an equation from which the vertical velocity can be computed. The value thus found agreed well with that estimated from the temperature-distribution.

HORIZONTAL SURFACE-CURRENTS COMPUTED FROM THE DIFFERENCE BETWEEN OBSERVED AND NORMAL SURFACE-TEMPERATURES

The quantity of radiant energy that passes through a unit area of the sea and is absorbed by a thin layer of water varies directly as its thickness and is assumed to be proportional to the amount of radiant energy that reaches a horizontal surface. Energy leaves this layer because of back-radiation and evaporation at a rate approximately proportional to the temperature-departure from a constant corresponding to the latitude. The time-rate of change of the energy in this volume equals the difference between the rate at which it enters and leaves. If the water flows horizontally, and there is a pressure-gradient in line with the direction of flow, the product of this gradient by the velocity is proportional to the rate at which energy enters or leaves the volume due to the current.

The average of surface-temperature corresponding to a given circle of latitude tends to approximate the "normal" for that latitude as the number of observations increases, if they are so distributed as to be representative. The "normal" temperature is defined as the temperature that would result if there were no current. From estimates of such mean annual temperatures, the seasonal range of temperatures, and solar radiation an equation has been derived between the temperatures, observed and normal, at two points along a stream-line, its angle with the meridian, and the velocity of the current. The relation between the observed and normal temperatures for only one point, the observed and normal gradients at that point, and the velocity have also been derived, and expressed in the form of an equation. The first equation is

$$e^{-R} = \frac{\theta - \theta_n - G\,(\cos\psi)\,(z - z_o)\,U}{\theta_o - \theta_{no}}$$

where

$$R = \frac{k}{H}(z - z_o), \quad U = \frac{1 - e^{-R}}{R}$$

ψ is the angle of the stream-line with the meridian, G is the normal horizontal temperature-gradient along the meridian, k is a physical constant depending upon the rate of cooling, H is the horizontal velocity, $(z - z_0)$ is the distance measured down-stream from the point where the observed temperature is θ_o and the normal is θ_{no} to the point where the observed temperature is θ and the normal is θ_n. The units are the month, the length of a degree at the equator, and Centigrade temperatures. The value of k is approximately 0.20, and G equals about 0.5. Denoting by G_o the observed gradient along the stream-line the second equation is

$$H = \frac{k\,(\theta_n - \theta_o)}{G_o}$$

The first equation is preferable, but the second may be used in case the data required in the first are not available. In general it is difficult to

TABLE 4—*Computation of surface-currents from surface-temperatures and winds based upon observations furnished by the U. S. Destroyer Fleet from San Diego southwest, March 1925*

Lat. of center of stream-line	Temperature		Angle ψ	Wind per hour	Computed current-velocity per hour		
	Obs'd θ_o θ	Normal θ_{no} θ_n			From temp.	From wind according to	
						Marmer	Ekman
°	°	°	°	*miles*	*mile*	*mile*	*mile*
25.5	17.8 18.1	20.0 20.7	241	7	.33	.10	.15
27.5	16.4 17.5	18.6 20.0	155	8	.14	.11	.15
27.5	17.4 18.4	19.4 20.7	197.5	18	.09	.25	.33
28	16.55 17.4	18.6 19.4	246	11	.15	.15	.20
29	15.5 16.2	18.2 19.0	120	13	.20	.20	.23
30	14.85 15.4	16.5 18.5	164	11	1.80	.17	.20
30	14.9 15.9	16.9 18.4	229	8	.08	.13	.14
31	14.4 15.4	15.7 18.2	172	11	.68	.18	.19

determine G_o. The result of applying these equations to data supplied by the U. S. Destroyer Fleet in March 1925, and by the U. S. Navy Transports since December 1925 are presented in Tables 4, 5, and 6. All temperatures were averaged by months and quadrangles, 30′ on a side. A uniform length of three degrees was selected for the portion $(z - z_o)$ of the stream-line, and the resultant of all winds corresponding to this part was computed. The corresponding current was assumed to be directed at an angle of 45° to the right of this resultant according to Ekman's theory. The intensive data in the region from San Diego to a position about eight hundred miles southwest, furnished by hourly observations from the Destroyer Fleet, averaged about forty to fifty ob-

Table 5—*Computation of surface-currents from surface-temperatures and winds based upon observations furnished by U. S. Navy Transports between Panama and Seattle, December 1925 to May 1926*

Date	Lat. of center of stream-line	Temperature		Angle ψ	Wind miles per hour	Computed current-velocity per hour		
		Obs'd θ_o θ	Normal θ_{no} θ_n			From temp.	From wind according to	
							Marmer	Ekman
	°	°	°	°	*miles*	*mile*	*mile*	*mile*
Dec. 1925	18.2	29.4	24.7	316	6	.38	.07	.15
		28.8	24.3					
	24.0	25.6	23.25	322	6	.18	.08	.14
		25.0	22.3					
Jan. 1926	30	16.6	17.9	182	9.5	.10	.14	.17
		17.4	19.4					
Mar. 1926	33	19.7	15.9	301	13	.62	.22	.22
		18.3	14.1					
	33.5	19.7	15.9	301	13	.21	.22	.22
		18.3	14.1					
	35	19.3	14.6	301	....	.07	....	....
		16.1	12.75					
May 1926	22.5	20.0	23.5	140	11	....	.13	.22
		24.4	24.4					
	31.5	17.5	19.4	135	13	.20	.22	.22
		17.8	20.7					
	34.7	13.3	16.55	....	....	.05	....	....
		15.6	17.7					

servations per quadrangle, while the Transport data along the coast from Panama to Seattle averaged about one tenth as much. The data and results of current computations are summarized in Tables 4, 5, and 6.

Table 6—*Computation by second method of surface-currents from surface-temperatures and winds based upon observations furnished by the U. S. Navy Transports between Panama and Seattle, January and February, 1926*

Date	Lat. of center of stream-line	Temperature			Angle ψ	Wind per hour	Computed current-velocity per hour		
		Obs'd θ_o	Normal θ_n	Gradient G_o			From temp.	From wind according to	
								Marmer	Ekman
	°	°	°	°	°	*miles*	*mile*	*mile*	*mile*
Jan. 1926	10	26.3	25.45	−.157	281	18.9	.10	.17	.62
	14.8	26.0	24.0	−.425	259	14.2	.01	.14	.35
	18.5	26.73	23.0	−.697	187	13	....	.14	.28
Feb. 1926	10.5	25.6	24.9	−.181	285	8.8	.07	.08	.27
	14	26.8	23.8	−.08	180	20	.72	.20	.49

HORIZONTAL SURFACE-CURRENTS COMPUTED FROM WINDS ACCORDING TO MARMER AND EKMAN'S THEORY

The ratio of current to wind was found empirically by Marmer from about 60,000 observations from lightships between San Francisco and the Columbia River. Winds were estimated hourly and currents were measured by a pole allowed to float from the ship through the fixed distance, one hundred feet. Contrary to Ekman's theory this ratio was found to decrease as the latitude decreased. It was found to increase slightly with a decrease of the wind-velocity. The currents computed from the ratio thus found, extrapolated to the corresponding latitudes, are entered in columns (7, 8, and 9) of Tables 4, 5, and 6. The currents computed from the winds, according to Ekman's theory are entered in columns (8, 9, and 10) of Tables 4, 5, and 6. These are seen to agree well on the average with those estimated from the temperatures, thus confirming the conclusion that the winds are the main cause of these coastal currents. In order more readily to compare estimates of the ocean-currents by each of the three methods the values are arranged according to the order of magnitude in Table 7.

TABLE 7—*Estimates of current-velocities taken from tables 4, 5, and 6, arranged in their order of magnitude*

Method	Current-velocities in mile per hour										
										Median	Mean
Temperature....	1.80	.72	.68	.62	.38	.33	.21	.20	.20	....	....
Marmer..	.25	.22	.22	.22	.20	.20	.18	.17	.17	....	....
Ekman...	.62	.49	.35	.33	.27	.23	.22	.22	.22	....	....
Temperature....	.18	.15	.14	.10	.10	.09	.08	.07	.01	0.19	0.26[a]
Marmer..	.15	.14	.14	.13	.11	.10	.08	.08	.07	0.16	0.16
Ekman...	.20	.20	.19	.17	.15	.15	.15	.14	.14	0.21	0.25

[a] Mean from the temperature-method excluding the two extremes 1.80 and .01.

HORIZONTAL SURFACE-CURRENTS COMPUTED BY MEANS OF THE BJERKNES CIRCULATION THEORY

Currents caused primarily by differences in specific gravity can be computed from serial temperatures and salinities at stations whose connecting lines are approximately perpendicular to the flow, by means of special methods based upon the hydrodynamical researches of V. Bjerknes. Also, there is reason to believe that the combined effect of winds and agencies tending to produce pressure-gradients by their effects on the distribution of specific gravity may be taken into account by applying the "Bjerknes method" in the usual way.

Off Lower California in latitudes 23° to 28°, assuming no current at the bottom, a flow to the south across each of the three sections by the Bjerknes method was found to be 0.04, 0.72, and 0.27 knot. From surface-temperatures, the corresponding velocities were found to be 0.11, 0.07, and 0.07. In latitude 43° the Bjerknes method indicated a flow of 0.03 to the south, and the temperature-method indicated 0.05.

COMPUTATION OF JAPAN CURRENT NEAR LONGITUDE 175° EAST FROM SURFACE-TEMPERATURES

A few estimates of surface-velocity in the Japan Current were made from surface-temperatures in quadrangles (41° to 45°) north, (166° to 170°) east, and (43.5° to 47.5°) north, (176° to 180°) east, the distance between being 7.46 degrees. The results of seven different computations are presented in Table 8.

The variability is rather large, but the median value of one-half mile per hour agrees well with that found by tracing temperature-departures across the Pacific from Japan.

TABLE 8—*Computation of current-velocities in the Japan stream from temperatures in quadrangles (41-45)N, (166-170)E and (43.5-47.5)N, (176-180)E (Distance between centers equals 7.46 degrees)*

Date	Lat. of middle of stream-line	Temperature		Angle ψ	Current per hour
		Observed	Normal		
	°	°	°	°	*miles*
Mar. 1919	44.2	1.4 3.9	7.7 6.3	71	.19
Oct. 1919	44.2	10.0 10.3	15.3 13.7	71	2.03
May 1920	44.2	7.4 8.0	11.3 9.8	71	.51
Nov. 1920	44.2	6.3 6.8	13.1 11.6	71	1.47
Mar. 1922	44.2	5.0 7.7	5.4 6.3	71	.41
Mar. 1923	44.2	1.5 3.5	7.7 6.3	71	.25
Apr. 1925	44.2	4.3 4.7	9.1 7.7	71	.98
Median (longitude 173° east)					0.51

In general, as illustrated by this paper, there are several different indirect methods of determining oceanic circulation from observations that can be made in any part of the ocean without special equipment. Others require intensive serial collections and observations at selected stations with the aid of special equipment. Methods appropriate to all available data should be applied in order adequately to check the results.

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OCEANOGRAPHY AND OIL DEPOSITS

PARKER D. TRASK

I think we all feel a certain amount of satisfaction whenever a practical application of a science is found. In work on a project endowed by John D. Rockefeller and carried out under the auspices of the American Petroleum Institute, the science of oceanography is being called upon to aid the oil industry. The purpose of this project is to study the conditions under which the sediments that in the future will be possible source-beds of petroleum are forming today. The work relates to the solution of the question of the origin of petroleum and should afford essential information that will aid the geologist in his increasingly more difficult task of finding new oil-pools.

Since most of the source-beds of present day oil-fields are marine in origin, it is seen that this project is closely associated with oceanography and hence is a very practical application of that science. Although attention is concentrated on the material at the bottom of the sea, nevertheless the problem is directly associated with all that goes on in the water above.

The work is being conducted under the supervision of a committee composed of Dr. W. van der Gracht (Chairman), Vice-President of the Marland Oil Co., Dr. T. Wayland Vaughan, Director of Scripps Institution of Oceanography, and Dr. Austin H. Clark, Oceanographer of the U. S. National Museum.

In attacking the problem it is necessary, first, to discover what sediments are possible future source-beds, and then, after having found them by means of a reconnaissance of certain critical areas of deposition of sediments, to choose the most likely regions for detailed studies. In the course of such a survey, information will be obtained as to the general distribution of organic matter in sediments and how the organic matter varies in different types of sediments.

In conducting such work it has been most important to find some suitable means of fairly rapidly determining the organic content of sediments. Distillations of dried sediment in the same manner that oil-shales are tested were found to be the most practical because, in addition to giving a good idea of the organic content they allow comparisons to be made with distillations of oil-shale and thus throw light on changes undergone in sediments from time of formation until the ultimate rock-product is reached.

Total ignition-tests were found to give, as a rule, a fairly good index of the organic content and hence to be a most important guide to the selection of samples to be distilled. Organic solvents, as tried out, yielded but little material. The ether-soluble portion of the alcohol-ether extract averaged only 0.06 per cent of the total sample and but slightly more than half this amount was fatty acid. Petroleum-ether, a good solvent for oils, yielded less than one-half the extract obtained by ether.

The reconnaissance of critical areas of deposition has but just begun and to date only two areas have been studied. One is the region about the Channel Islands west of Los Angeles and the other is Pamlico Sound, the large body of shallow, brackish water lying inside of Cape

Hatteras. Distillations have been made only from the latter region. The study of other areas is rapidly being carried on at the present time.

The most important result of work on this problem, which is based on the distillation of some forty-five samples, is that all types of sediment, from clay to sand, upon distillation produced oil. The yield of oil was low and, in general, varied in amount with the degree of fineness of the sediments, ranging from a maximum of 2.7 gallons per ton in a clay-silt to almost nothing in a sand. This maximum yield of 2.7 gallons per ton is but five or ten per cent the amount obtained from the better grades of oil-shale which run from thirty to fifty or even more gallons per ton. It is perhaps surprising that 2.7 gallons per ton should be the highest yield secured, but the place of a higher production apparently was taken by a black flocculent precipitate that settled in the bottom of the distillate. The amount of this precipitate varied roughly with the organic content of the sediment and reached a maximum of fourteen gallons a ton. The temperature of formation of the oil ranged from 350° to 575° C, a range comparable to the production of oil from oil-shale. The temperature of initial oil-production varied widely in the different samples but in the sands was constantly 100° higher than the average for the clays and silts, which was 390° C.

Another interesting feature brought out in work on this project is that with very few exceptions the organic content of sediments increased as the sediment became more fine-grained. This inference is based on some seventeen samples collected from the Channel Islands region of southern California, and is supported by mechanical analyses made by the U. S. Bureau of Soils. The amount of organic nitrogen, fatty acid, and ignition-loss present in each sample was found, as a rule, to increase as the sediment became more fine-grained in character. The average organic nitrogen content was 0.2 per cent, fatty acid 0.04 per cent, and ignition-loss 10 per cent. Also, the amount of water held in its natural state in the sediment varied directly with the fineness of the sediment, ranging from 29 per cent in a sand to 66 per cent in a clay. The samples brought up from the lower parts of the deep depressions in the sea-floor in the Channel Islands region had the highest organic content of all the deposits collected in that area.

Two instruments to collect the sediments in sufficient quantities for distillation have been constructed. One is a modification of the Davis[1] peat-sampler. It consists of brass plunger that fits into a brass cylinder. The sampler by means of series of pipes coupled together is shoved in the mud with the plunger forced as far as it will go in the cylinder. At the depth at which the sample is to be taken the plunger is pulled upward until a spring-catch on the inside engages top of cylinder, which, due to the friction of the mud with the wall of the cylinder, does not rise with plunger. The whole apparatus is then forced into mud and the empty cylinder becomes filled with the sample and is hauled to the surface.

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[1] C. A. Davis, Mich. Geol. Survey, Ann. Rep., 1906, Part 2, p. 317, 1907.

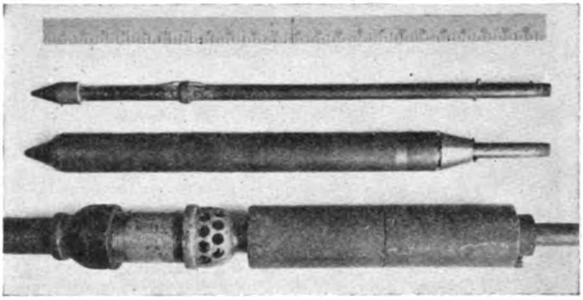

Fig. 1

Modified Davis peat-sampler (plunger with detachable top of collecting cylinder engaged in spring-catch at top and plunger inserted in cylinder preparatory to taking a sample at center) and modified Ekman sampler (with weights, perforated reducer, valve, and collecting tube at bottom).

The chief change made from the Davis sampler is that the plug that closes the top of the collecting cylinder is removable and is tapered conically upward so as to make easier the withdrawal of the instrument from the mud. This sampler will operate satisfactorily in water up to twenty-five feet in depth and will collect material, under favorable conditions, twenty-five or thirty feet deep in mud.

In deeper water the other sampler has to be used. It is constructed on the principles of the Ekman[2] and Albatross[3] samplers, but is much lighter, considerably cheaper, and much easier to transport and handle. However, it has the same disadvantage of those two samplers in that, as a rule, it will not take a sample more than four feet deep in the mud. The instrument consists of a four-foot collecting tube with beveled lower end, above which is a nipple that contains a butterfly-valve to permit water to flow through tube as the instrument sinks, but prevents the water from washing out the sample as it is raised to the surface. By means of a perforated reducer to allow the escape of water, the nipple is attached to a long stem of smaller pipe of sufficient length to insure center of gravity being below the middle of instrument. Lead-weights are fastened above perforated reducer to add momentum to sampler and to insure its falling vertically through the water. These may be re-

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[2] V. W. Ekman, Counseil Permanent International pour l'exploration de la mar. Publ. de circonstance, No. 257, 1905.

[3] F. B. Sumner, et al., Univ. Calif. Publ. Zool., 14; (5) 1914.

moved when the sampler is operated by hand. Except for the simple butterfly-valve, all parts of the instrument can be purchased in a plumbing shop, and may be wrapped in a bundle four feet long and eight inches in diameter. A paper describing these instruments is in preparation.

In view of the magnitude of the task of making a reconnaissance of the organic content of a number of critical areas of deposition, practically all our attention has to be focused on that one phase of the problem and but very little time can be given to the numerous interesting and important problems that are continually presenting themselves, a few of which, however, are mentioned briefly.

The hydrogen-ion concentration of the muds of Pamlico Sound, as determined by Brom Thymol Blue, was found to vary but slightly throughout the Sound, ranging between 6.7 and 7.0. Furthermore, no material change was noted at depth in the samples even down to a depth of twenty-three feet.

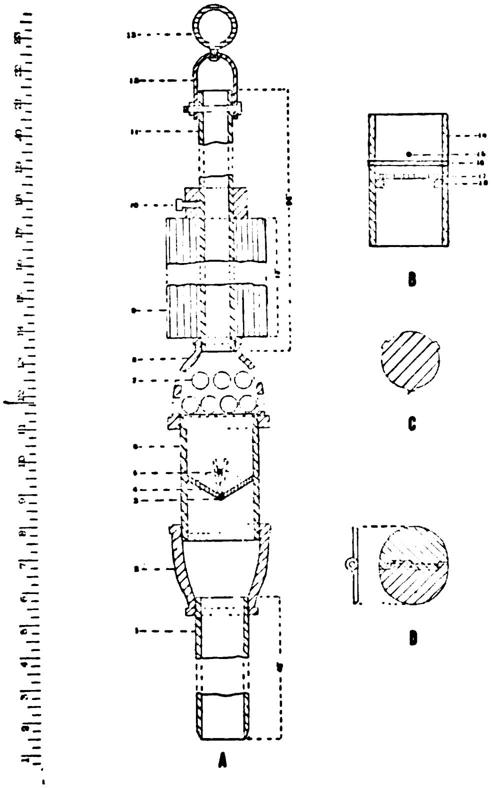

FIG. 2

Details of modified Ekman sampler: (*A*) diagrammatic cross-section, (*B*) alternative valve, (*C*) brass valve-plate, and (*D*) plan and elevation butterfly-valve. [(1) Collecting tube 1¼-inch iron pipe beveled to effect better penetration, (2) 1¼-inch by 2-inch reducer, (3) pivot-rod for butterfly-valve (4), (5) rod to prevent valve-wings passing mid-point of tube, (6) 2 by 4-inch nipple, (7) series of one-half-inch holes in 2 by ¾-inch reducer (8), (9) two 3 by 6-inch 15-pound lead-weights clamped by set-screw (10) on stem of instrument consisting of two four-foot lengths of extra-heavy ¾-inch pipe (11), (12) *U*-shaped shackle of flat iron, (13) swivel-joint attachment for rope or wire, (14) 2-inch nipple 4 inches long, (15, 16) brass rods at right-angles to each other to prevent flat circular valve-plate (17) being forced out of nipple (14), (18) supporting collar for valve-plate.]

The reducing power of the muds as determined by the reduction of dyes was found to be but moderate, notwithstanding the presence of hydrogen sulphide, being about plus 0.2 volt compared to hydrogen.

The temperature-gradient of the muds as taken in Pamlico Sound in the latter part of January was found to be much lower in sands than in clays. In the but few samples taken deep in the mud, the depth at which a constant temperature was reached was between five and ten feet beneath the surface of the mud.

Water-soluble phosphate as determined on one sample taken from the Pacific Coast in 685 meters of water was 18.4 parts per million, nearly one hundred times the concentration in the lower sea-water.

In conclusion, the writer wishes to say that the results obtained on this project in a large degree have been made possible through the enthusiastic cooperation that has been received from all sources. Time does not permit the naming of the long list of cooperators, but Dr. Kellerman of the Bureau of Plant Industry, Drs. Burgess, Brown, and Washburn of the Bureau of Standards, Dr. Vaughan of Scripps Institution of Oceanography, and Drs. Alsberg and Swain of Stanford University, should be particularly mentioned for giving laboratory facilities that have materially aided the work.

Also, Messrs. O'Malley, Radcliffe, Higgins, and Goltsoff of the Bureau of Fisheries, Captain Parker of the Coast and Geodetic Survey, and Dr. Vaughan of Scripps Institution, have kindly granted the writer permission to accompany their boats so that samples could be collected, which, of course, is the most essential feature of the entire work.

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THE CONFIGURATION OF THE BASIN OF THE PACIFIC

G. W. LITTLEHALES

The use of the sea, from early times, as the great highway of transportation and communication has made apparent to the United States the advantage of equipping its vessels with information respecting those waters through which their voyages might be directed, and for the fulfilment of this advantage a national institution, named The Hydrographic Office, has been established in the Navy Department. In relation to the Pacific Ocean, this institution produces and issues in successive editions about fifteen hundred different publications including charts ranging in continuous succession throughout its entire expanse.

The charts of fifty years ago, when deep-sea soundings began to be measured in the Pacific Ocean, were encumbered with symbols to represent the many hundreds of reports of doubtful dangers that had come down from former generations of navigators.

To furnish evidence, by measuring the depth of the ocean, that would verify or disprove these reports, often doubtless based upon misleading appearance of the surface of the sea, was, to the United States Navy, a recognized obligation which resulted in this Service becoming the pioneer in furnishing authentic measurements of the depths of this great ocean, whose expanse exceeds, by ten million square miles, the total land-surface of the globe.

It was in 1873 that the U. S. S. *Tuscarora* ran the first trans-oceanic lines of soundings in the Pacific, in the course of which the Aleutian Deep was found and the great trough bordering Japan was discovered and named Tuscarora Deep. Soon afterwards the U. S. S. *Narragansett* sounded the route from Hawaii to Australia.

The charts of the Hydrographic Office have thus become, by natural development, the repository of the accumulated information with respect to the configuration of the oceanic basins, and in them may be found the facts of observation that lend themselves to the delineation of the isobathic contours.

Although there are regions of the Pacific where the number and distribution of the soundings are sufficient for a first approximation to the mapping of the conformation of the bottom of the ocean, as in the tract between California and Hawaii and other regions where an even greater degree of completeness has been reached, as in the case of the survey, in thirty-eight days, by the U. S. destroyers *Hull* and *Cory* of a tract of the bottom of the Pacific Ocean 34,000 square miles in extent, bordering the coast of California from San Francisco to San Diego and extending oceanward to a depth of 2,000 fathoms, yet, in general, it is not possible to draw the contours of the configuration, because, on account of the sparseness of the soundings, conjecture everywhere arises in attempting to do so, and there still remain great areas in this ocean that are entirely without soundings. In the North Pacific, there is a tract twice as large as the United States which has been crossed by only a single line of soundings spaced about two hundred and fifty miles apart, and in other parts of the Pacific basin there are areas as large as the continent of Australia in which not a single authentic sounding is recorded.

Besides giving the measurement of a depth, a sounding must be accompanied by two additional coordinates determining its position, since such a measurement applies only to one place on the globe; and, to be useful in defining configuration, a sounding must be attended with other soundings at intervals apart bearing a suitable relation to the dimensions of the orographical forms whose configuration is sought to be developed.

As a formative influence upon those engaged in measuring deep-sea soundings, the Hydrographic Office undertook an investigation to determine the form and dimensions of the slenderest formation, composed of the materials of the crust of the Earth, that could be raised from the depths of the ocean to a culmination in proximity to the surface, with sufficient crushing strength to remain standing under its own weight and the weight of the superincumbent body of water, on the principle that if the intervals between soundings were regulated to disclose the presence of the slenderest formation that could stand, then any orographical formation that might exist in Nature would be likely to be found. This investigation is published in the Proceedings of the First Pan-Pacific Science Congress held in Honolulu, in 1920.

The never failing stream of deep-sea soundings from national and institutional vessels of the countries bordering or having interests in the Pacific, among which those of the United States have maintained a leading part, is year by year advancing the once distant prospect of mapping the conformation and topography of the bottom of the Pacific Ocean to a nearer stage of feasibility, as when, in 1925, the United States Fleet, from the voyage to Australia and New Zealand, brought home about 6,000 soundings—many of them from regions unfathomed before that time.

And it is with satisfaction that attention may be turned to the unnumbered thousands of soundings on the continental shelf where the diligent labors of hydrographers have made the coastline of the world the best known of the world's major geographical features.

U. S. Hydrographic Office,
Navy Department,
Washington, D. C.

COMMITTEE ON THE OCEANOGRAPHY OF THE PACIFIC

T. WAYLAND VAUGHAN

ORIGIN AND PURPOSE OF THE PACIFIC SCIENCE ASSOCIATION

Since the Committee on the Oceanography of the Pacific is one of the committees established by the Pacific Science Association, it is pertinent to state briefly the origin and purpose of the Association. Scientific men have long recognized the need for the establishment of some kind of medium through which co-operative or co-ordinated effort in the investigation of the many scientific problems presented by the Pacific region might be effected. The preparation of programs of research began years ago, and more recently there have been special discussions of the problems of the Pacific at the meeting of the British Association for the Advancement of Science in Australia in 1914, before the annual meeting of our National Academy of Sciences in 1916, and at the Semi-Centenary of the University of California in 1918. Our National Academy and National Research Council appointed a Committee on Pacific Investigations and the activity of its members resulted in the First Pan-Pacific Science Congress in Honolulu in 1920. The Second Congress was held in Australia in 1923, and at the Third Congress, in Japan in 1926, definite organization of the Pacific Science Association was accomplished. Henceforth the Pacific Science Congresses will be held by the Pacific Science Association, which is primarily an association of the National Research Councils or equivalent bodies of the countries of the Pacific region.

Although scientific papers of the usual kind are presented at the Congresses, the primary purpose is to promote investigations, and especially those that require co-operative or co-ordinated effort. The usual procedure is to prepare summaries of available information on a subject, point out deficiencies in information, suggest programs of research to supply the deficiencies, and then to try to arrange for the execution of the programs.

Resolutions, drafted by those who have studied special subjects, are presented to the appropriate divisions or sections, and, if approved, are forwarded to the Council of the Congress, which endeavors to obviate duplication and to combine resolutions dealing with the same subject. Furthermore, an attempt is made to select only those resolutions which are harmoniously supported and behind which there is a group of members determined to make them effective. The selected resolutions are submitted to a general meeting of the Congress.

The purpose of the Association is the accomplishment of work. But it needs to be emphasized that the efforts are voluntary—no compulsion is possible. Furthermore, there are no formal agreements or contracts. Those interested do what they can, and already the accomplishments in several fields are large. For instance, very nearly everything called for in the resolutions adopted in Australia in 1923 has been done. It is hoped that the record for the effective execution of the resolutions adopted in Japan will be as good.

COMMITTEE ON THE PHYSICAL AND CHEMICAL OCEANOGRAPHY OF THE PACIFIC

In Australia in 1923 a resolution was adopted authorizing the establishment of a Committee on the Physical and Chemical Oceanography of the Pacific. Nine countries were represented on the committee, which was in large measure responsible for the excellent program of oceanographic papers at the Congress in Tokyo.

OCEANOGRAPHY AT THE THIRD PAN-PACIFIC SCIENCE CONGRESS

The Third Pan-Pacific Science Congress met in Tokyo between October 30 and November 11, 1926. There were excursions prior to, during, and subsequent to the meeting. The number of papers particularly devoted to oceanography was about thirty-five, but there were also papers on marine biology, fisheries, geologic features of ocean basins, and meteorology, that might appropriately have been given dual classification. The total number of papers of importance to oceanography was about seventy. Valuable oceanographic work is being done in Canada, Japan, the Philippines, the Netherlands East Indies, and Australia, as well as in the United States; and the Russians are resuming activities. The operations of Japan and of the United States are on rather larger scale than those of the other countries, but the intensive work in the Netherlands East Indies, and the proposed expedition in 1928 to the eastern part of the East Indian Archipelago, need to be borne in mind.

The Japanese are active in all lines of oceanographic research, except the calculation of oceanic circulation from physical data, but they are ready and have competent men for that kind of work. The outstanding institutions in Japan are the Hydrographic Department of the Imperial Navy, the Imperial Marine Meteorological Observatory at Kobe, and the Imperial Fisheries Institute; other institutions of smaller scale are conducting important researches. The activities embrace studies of the depths, currents, and tides of the sea, of the chemical properties of sea-water, of marine meteorology, of marine bottom-deposits, of the quantity of plankton, of the life histories of commercial fishes, of marine mammals (especially the whales), of artificial cultures (such as culture pearls), and of fisheries technology. The intensive investigations of earthquakes by the Japanese are of value to oceanography, as well as to other branches of science.

ESTABLISHMENT OF THE COMMITTEE ON THE OCEANOGRAPHY OF THE PACIFIC

There were presented at the Congress in Tokyo several papers that dealt with international co-operation in oceanographic research in the Pacific. Among the papers was a general one by me, written in compliance with a request from the President of the Congress. Another general paper was entitled the "Pacific Ocean, Its Nature and Fauna," by P. J. Schmidt. There were several papers on more special subjects.

My paper outlined a method of international co-operation based on each nation trying to work in that part of the Pacific adjacent to its own shores and by exchanging programs of research and information on the results of research with the other countries engaged in oceanographic investigations. It was obvious that the scope of the old Committee on the Physical and Chemical Oceanography of the Pacific was too narrow. It was decided, therefore, to disband that committee and to establish in its stead a new Committee on the Oceanography of the Pacific and to give its activities sufficiently broad scope for it to take in the special functions indicated by several excellent suggestions made by different members of the Congress. In the resolution establishing the Committee on the Oceanography of the Pacific, three subcommittees for each country were authorized, viz., subcommittees on physical and chemical oceanography on fundamental marine biology, and on fisheries technology.

The text of the first two paragraphs of Recommendation No. 5 reads as follows:

"Resolved—(1) That the surface-temperature readings from ships in the Pacific should have a time-frequency of at least once per hour to be adequate for oceanographic research; but as such frequency may not be practicable, the extensive installation of thermographs is recommended.

"(2) That the Committee on Oceanography endeavor to have issued a bulletin of actual surface-temperature readings, each bulletin covering a period of not longer than three months and appearing within about two months after the close of the period, if possible."

Before I left Tokyo, Dr. Sakurai, the President of the Congress, orally requested me to serve as Chairman of the International Committee, and later he corroborated his request in a letter. I accepted Dr. Sakurai's invitation with the feeling that I was undertaking a very heavy responsibility.

ORGANIZATION OF THE INTERNATIONAL COMMITTEE

As soon as possible after my return from Japan, I sent to the appropriate organization in each country represented in the Pacific Science Association a copy of the resolutions bearing on oceanography adopted in Tokyo and preliminary suggestions regarding the organization of the different National Committees and Subcommittees. The suggested procedure for the different countries reads as follows:

"The first step would be the appointment of the chairman of the National Committee and the chairman of the three National Subcommittees in each country represented on the International Committee. The chairmen of the National Subcommittees would organize their respective subcommittees. It may be desirable to have the same person serve on more than one subcommittee.

"Each subcommittee may be subdivided into sections if desirable, with a chief for each section. . . ."

Replies are now coming from the different countries.

ORGANIZATION OF THE NATIONAL COMMITTEE AND SUBCOMMITTEES FOR THE UNITED STATES

The first step in organizing the National Committee and Subcommittees for the United States was to submit the matter to our National Research Council and to seek advice or information from institutions and individuals. Letters have been received from the Hydrographic Office, Weather Bureau, Coast and Geodetic Survey, Bureau of Lighthouses, Bureau of Fisheries, Carnegie Institution of Washington, California Fish and Game Commission, numerous water, power, and light engineers, the members of the Advisory Board to the Director of the Scripps Institution, and many individuals, including some foreigners, whose opinions I desired. It is encouraging that virtually every one from whom a reply was expected has written. Many have gone to great trouble to supply detailed information that was needed in perfecting plans. I cannot here give the names of those who have helped, but I am profoundly grateful to them.

Regarding the proposed work in physical and chemical oceanography and in fundamental marine biology, the consensus of opinion is unanimously in favor of both; and doubt regarding the desirability of the proposed work in fisheries technology has come from only one organization. Our National Research Council has appointed me the Chairman of the National Committee for the United States; Dr. Geo. F. McEwen, of the Scripps Institution of Oceanography, has been appointed Chairman of the Subcommittee on Physical and Chemical Oceanography; and Professor L. B. Becking and Professor Laurence Irving, both of the Jacques Loeb Laboratory of Physiology, Stanford University, have been appointed respectively Chairman and Vice-Chairman of the Subcommittee on Fundamental Marine Biology. The composition of the subcommittees has not yet been decided, because it is a matter that requires careful consideration from the standpoints of both co-operating institutions and individual investigators engaged in the different kinds of research.

THE ROLE OF THE UNITED STATES IN THE OCEANOGRAPHY OF THE PACIFIC

Although knowledge of what is being done in other countries is needed by each country so that its program may be properly adjusted to the programs of other countries, each country must make and execute its own program. We are now, therefore, primarily concerned with what the United States may do, and we need to consider the activities of other countries only in so far as they bear on co-ordinate action in a general program.

On the basis of a real subdivision of oceanographic activities in the Pacific, previously indicated, it would be expected that the organizations in the United States engaged in oceanographic investigations would devote their efforts principally, but not exclusively, to the northeast Pacific. Where there are countries willing and prepared to do their share of oceanographic work, the principle of a real subdivision should be adopted, but for those segments of the Pacific adjacent to whose shores no countries are prepared to conduct oceanographic investigations, other provisions are necessary. Such is the case in South America. Therefore, it is incumbent on organizations in the United States, such as our Navy, the American Geographical Society, the American Museum of Natural History, the Carnegie Institution of Washington, and the Scripps Institution, to do what they can to add to knowledge of the eastern part of the southeast Pacific.

The work which may be done in connection with the three subcommittees will be briefly outlined.

PHYSICAL AND CHEMICAL OCEANOGRAPHY

Although they may be applicable to other fields of oceanography, the accumulation, classification, and interpretation of certain kinds of data are considered within the field of physical and chemical oceanography. These data are as follows:

(1) Depth of the sea, the configuration of the sea-bottom, and explanation of its features and of those of its margins; (2) marine bottom-deposits; (3) tides and waves; (4) the temperature and density of sea-water both at and below the surface; (5) the chemical properties of sea-water, such as *pH*, the substances in solution, for example, phosphate, silica, nitrogen compounds, and gases; (6) the interrelation between the physical and chemical features of the ocean and solar radiation; (7) the interrelation between the sea and the atmosphere; (8) oceanic circulation which includes all movements of the ocean-waters, except wave-motion, both at and below the surface, the causes that produce movement, and the methods whereby the rate and direction of movement may be determined.

For certain of the subjects enumerated provisions are already adequate. Among these are the tides, by the Coast and Geodetic Survey; and seismology, by the Coast and Geodetic Survey and the Carnegie Institution of Washington. The determination of the depths of the sea belongs logically to the Navy and to the Coast and Geodetic Survey. The rôle of such an institution as the Scripps Institution in a program for physical and chemical oceanography should be to try to help others in their investigations of such subjects as tides and earthquakes; its own field would comprise the determination of the physical and chemical properties of sea-water, the study of the interrelations between the physical and chemical features of the ocean and solar radiation, the study of the interrelations between the sea and the atmosphere, and the investigation of oceanic circulation by the use of current-meters and by applying the dynamical principles discovered and elaborated by Bjerknes, Ekman, McEwen, and others. The study of marine bottom-deposits, also, appropriately belongs within the sphere of such an institution.

Although an institution of the kind indicated can do much in its laboratories and by means of small craft and fixed collecting-stations and observation-stations, it needs aid in extending its range of collections and observations. The branches of our Government that operate fleets of vessels and commercial lines of vessels can render assistance of enormous value, and it is gratifying that they are now doing this. Only the necessity of being brief deters me from saying more regarding what the Navy, the Coast and Geodetic Survey, the Bureau of Lighthouses, and certain commercial lines of steamships have done and are doing. But it is possible for them to do more in making temperature-records, especially by installing thermographs on some vessels, and by collecting water-samples for both density-determinations and plankton-counts. Some vessels might make larger numbers of vertical sections of the water from the surface to the bottom and collect additional bottom-material.

The value of expeditions, such as that contemplated for the *Carnegie,* cannot be over-estimated.

One of the needs of both oceanography and meteorology is to have made available for use the large volumes of data on sea-surface temperatures and marine meteorology deposited in the Hydrographic Office and the Weather Bureau. The method of applying such data to the solution of problems in both physical oceanography and marine meteorology are now known, and it is earnestly hoped that provisions can be made for rendering the data available for use, as is being done by the Imperial Marine Meteorological Observatory at Kobe.

The present provisions in the United States for utilizing such data in the investigation of the physical oceanography and marine meteorology of the North Pacific, except in certain Governmental bureaus, consist of Dr. Geo. F. McEwen and Mr. S. W. Chambers, at the Scripps Institution, and Prof. C. F. Brooks and a part-time assistant at Clark University. An endeavor is now being made to increase the staff of the Scripps Institution by the addition of three scientific investigators and one clerk, and to procure funds for additional thermographs. If this endeavor should be successful, and if the data on marine temperature and on meteorological observations at sea filed with the Hydrographic Office and the Weather Bureau can be made available for use, great advance in both the physical oceanography and the marine meteorology of the North Pacific may be confidently expected within the following ten years. If this proposal were made effective, it and the work now being done by the Japanese, the Canadian Meteorological Service, and the Meteorological Bureau of the Netherlands East Indies, would be mutually complementary. For, please bear in mind, that if the plan proposed for the United States can be made effective we shall be engaged in a cooperative enterprise, which at present includes at least three Pacific countries besides the United States.

FUNDAMENTAL MARINE BIOLOGY

This subject depends on the investigations in physical and chemical oceanography for data on environment. It deals primarily with the interrelation between the organisms and their environment and with the interrelations between the organisms themselves. Therefore, biophysics, biochemistry, physiology, food-chains, and life histories are all subjects fundamental and all subjects to which far too little attention has been paid. To this field should be referred investigations of photosynthesis, bacteriology, and skeleton formation, quantitative studies of plankton and the physiology and ecology of plankton organisms, the food-value of plankton and its consumption by other organisms, the food of the higher organisms, their physiological responses and the causes of migrations, the life histories of organisms, including studies of pelagic fish-eggs and larval fishes, and many other problems.

The continuing investigations must be conducted mostly at established laboratories such as the Friday Harbor Laboratory of the University of Washington, the Hopkins Marine Laboratory and the Jacques Loeb Laboratory of Stanford University, the Laboratory of the California Fish and Game Commission at San Pedro, the Scripps Institution of Oceanography at La Jolla, and the Marine Biological Laboratory at Honolulu. But more than mere laboratory work and near-shore expeditions are needed. The contributions that can be made by such expeditions as that projected for the *Carnegie* are of great and profound importance. Recurrent expeditions of magnitude are needed.

It is possible for the Governmental institutions that operate sea-going vessels to help in the investigations by collecting, while on their voyages, water-samples for plankton-counts, as well as by obtaining other data and collections already indicated. If the different possible agencies could undertake to procure additional collections of material and data they would increase their valuable aid in advancing knowledge of the biology of the sea.

FISHERIES TECHNOLOGY

The name of one of the subcommittees of each national committee on the oceanography of the Pacific is "Subcommittee on Fisheries Technology." It would have been better to have omitted the word "technology" thereby leaving the subcommittee free to consider any fisheries problem deemed desirable.

The fisheries industry should rest on a foundation of researches in physical and chemical oceanography and fundamental marine biology, for from these researches will be derived knowledge of the conditions that control the spawning and hatching of eggs, the success or failure of a hatch, and the distribution and supply of adults. Actually, fisheries problems cannot be sharply delimited from the general problems of oceanography; and it must be recognized that the needs of the fisheries industry have led to oceanographic researches of the most profound significance, for instance, many of the investigations of oceanic circulation.

The problem of the possible depletion of resources and the steps needed to preserve supplies of useful aquatic animals and plants present themselves.

The study of the technology of the fisheries industry should begin with the supply and should cover methods of fishing or harvesting, preparation and preservation of the catch or harvest, the food-value of fish, and marketing. The utilization of the parts of fisheries products not suitable for food should receive investigation for the obvious reason that any additional revenue derived from such a source will strengthen the industry.

The hope was expressed by the Congress that this subcommittee would be gradually developed so that it may function as a clearing house for information on fisheries technology for all the countries of the Pacific. In each country the work of this subcommittee should be in association with the fisheries societies.

There are at present, so far as I have ascertained, only two institutions on the Pacific Coast of the United States actually engaged in researches on fisheries technology, viz., the College of Fisheries of the University of Washington and the California Fish and Game Commission, but investigations of significance to the fisheries industry are being conducted at the Jacques Loeb Laboratory at Stanford, at the Scripps Institution, and at other places. It appears probable that there may be worked out between several institutions a program which will yield results of value, but no definite steps have yet been taken. Suggestions have also been received for the collection and distribution of information.

Although the Subcommittee on Fisheries Technology has not yet been organized, enough suggestions have been received to warrant belief that it can be organized and can be made to render valuable service.

A JOURNAL OF OCEANOGRAPHY

There is at present no regular journal of oceanography published in an English-speaking country. There is need for a journal in which both original articles and abstracts may be published. This matter might be taken up by the Geophysical Union and the other appropriate divisions of our National Research Council.

CONCLUSION

On the Pacific Coast of the United States and in the Hawaiian Islands there are at least seven institutions or laboratories engaged in oceanographic work of some kind. These institutions or laboratories are gradually getting together and before long concerted action may be expected. But only the Scripps Institution is working on physical oceanography and marine meteorology.

The valuable contributions that may be made by such institutions as the Carnegie Institution of Washington, the American Geographical Society, and the American Museum of Natural History have been indicated.

Our Navy might increase the valuable aid it is already rendering by supplying more data on sea-surface temperatures and marine meteorology and by the collection of surface water-samples for salinity-determinations and plankton-counts.

The Coast and Geodetic Survey might add to its present valuable assistance by the installation of thermographs on its vessels and by making a larger number of vertical sections of sea-water.

The Bureau of Lighthouses might add to the help it is now giving by arranging for the lighthouse-tenders to do oceanographic work while on their cruises.

The United States Shipping Board should be invited to assist in this enterprise.

A great service that can be rendered by the Hydrographic Office and the Weather Bureau is to make available for others the use of the sea-surface temperature records and the data on marine meteorology deposited in their archives.

Our National Research Council can help in the preliminary steps toward establishing a Journal of Oceanography and perhaps it can assist in procuring funds for traveling, office, and publication expenses.

This evening there are assembled with the members of the Section of Oceanography, representatives, in most instances the heads, of those governmental bureaus and of the private scientific institutions most concerned with oceanographic research in the Pacific. I trust that every one here will join with all the others and do whatever he can, as an individual or as the head of a bureau or institution, toward enabling the United States to make contributions to knowledge of the oceanography of the Pacific Ocean commensurate with its strength and standing among the nations of the world.

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THE COAST AND GEODETIC SURVEY AND OCEANOGRAPHY IN THE PACIFIC

N. H. HECK[1]

One of the resolutions passed by the Third Pan-Pacific Science Congress at its recent meeting in Tokyo emphasized the need for more knowledge of bottom-configuration in the Pacific region, especially in the vicinity of the island groups. In my opinion, there should have been added the areas adjacent to the coasts because these two classes of area include most of the interesting features of the region.

Even when the vast area of the Pacific Ocean is taken into account, it is found that the Coast and Geodetic Survey is charged with the survey of a not inconsiderable part of such area in the North Pacific. Its normal activities include surveys from the coast or from island groups to a limit usually fixed at the thousand-fathom-curve. Such surveys include the area from the Mexican boundary to the Canadian boundary and from Dixon Entrance to Attu Island, the westernmost island of the Aleutian group, a distance of 3,500 miles; along the western coast of Alaska north of the Aleutian group 1,200 miles; the entire Hawaiian group extending 1,400 miles, and the entire Philippine group, length north to south 1,200 miles and east to west 150 to 650 miles.

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[1] Representing Colonel E. Lester Jones, Director, U. S. Coast and Geodetic Survey.

In addition to this normal work, a number of vessels passing between the Atlantic and Pacific coasts of the United States have taken soundings and other observations en route from the Panama Canal to California, and the same practice has been followed between California and the Hawaiian Islands, and between the State of Washington and Alaska.

These surveys include soundings by wire and echo-methods, the obtaining of temperatures and water-samples for the determination of salinity at the surface and at various depths and of bottom-samples. Special oceanographic work has been done also for the Scripps Institution at La Jolla, California.

Some very interesting submarine features have been developed as the result of these surveys, especially a series of deep troughs extending parallel to the coasts of Mexico and Central America and the great submarine valley which approaches the California coast in Monterey Bay.

In connection with the execution of this work, the Coast and Geodetic Survey has taken an important part in the development of new methods. In connection with the echo-soundings it was the first to apply proper velocities of sound based on temperature, salinity, and pressure to soundings of all depths. Two of its officers prepared suitable tables for readily determining these velocities and later presented for the Pacific Ocean a method of applying velocity-corrections which requires knowledge only of the latitude and approximate depth of the soundings. It has applied the fathometer of the Submarine Signal Corporation of Boston, which was developed primarily for navigation, to hydrography and has developed a number of improvements which make it more suitable for such use.

The radio acoustic method of determination of position insures greater accuracy in the surveys of the critical area lying between the coast and the thousand-fathom-curve, especially in regions where frequent fogs or low-lying shores prevent objects on shore from being seen at considerable distances. The possibility of working a larger part of the time is resulting in a considerable increase of output.

The fact that the Coast and Geodetic Survey has taken up these new methods and made them effective shows its attitude in this matter and it is scarcely necessary to say that it is ready to adopt other new methods for carrying out the various operations that come directly within the field of its work, or which aid oceanography in general, provided it is not required to go outside of its field or interfere with its regular work.

It scarcely needs to be said that the Coast and Geodetic Survey is interested in the extension of oceanographic work in the Pacific Ocean. It not only appreciates the needs but the difficulties and it clearly understands that no one nation can or should undertake this great task alone. An organization under which all the nations bordering the Pacific can exchange views and discuss plans should at the very least provide for the first steps which must be taken, and it can be expected to accomplish much more.

A matter of primary oceanographic importance is the determination of accurate datum-planes. These serve as the basic reference planes from which depths in the ocean are measured and thus permit the detection of changes in the relative elevation of land to sea. The Coast and Geodetic Survey has determined such planes at a number of points on the Pacific Coast of the United States and is now maintaining in operation continuously five primary tide-stations on that coast.

The circulation of the sea near the land and the relation of this to the wind has been studied by the Coast and Geodetic Survey for several years, by means of observations on the five light-vessels stationed along the Pacific coast. The data derived establish definite results for winds of various directions and velocities.

The extension of work of this character to the entire Pacific region would not only add greatly to the value of the information that has been obtained, but would also be of international benefit.

The use of such work to oceanographers is obvious, but the Coast and Geodetic Survey is especially interested in another application. The Survey is constantly called upon to locate earthquake epicenters in the Pacific because the great earthquakes frequently occurring there are recorded on its instruments. It is important to know whether a special bottom-formation is associated with a submarine earthquake, and since the deep ocean-troughs stand first among such formations, it is at present a great handicap that we scarcely know their limits. Past work of the Coast and Geodetic Survey in the great deep north of Porto Rico shows that the ocean-deeps can be sounded to their greatest depths by the echo-method.

There is also need for the extension to the entire Pacific of the ocean gravity-surveys which have been conducted so successfully by Dr. Meinesz. This work is especially needed in the Pacific since the surface-layer of the Earth beneath this ocean appears from various evidence to differ from that elsewhere in the Earth so that the determination of all the facts by the combined efforts of oceanography, seismology, and geodesy is of the greatest importance.

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Washington, D. C.

SECTION OF VOLCANOLOGY

(Washington, D. C., April 28, 1927)

THE WORK OF THE GEOPHYSICAL LABORATORY ON HOT SPRINGS

E. T. Allen

Inception of the work. My active interest in the study of volcanic phenomena began in 1916 when I accompanied Dr. A. L. Day to the Lassen Park with the purpose of studying the hot springs in that locality. The earlier work was concerned with general observations on such matter as topography, temperature, and drainage, and with the processes of rock decomposition. It led to the discovery of a simple relation between the acid and the alkaline springs. Both are regarded as derived from the same source by the same general process of rock decomposition. The majority of these springs contain sulphuric acid together with what are plainly the products of its chemical action on the surrounding rocks, namely, the sulphates of the rock bases dissolved in the water, and silica and kaolin precipitated in it. Now, if the acid continues to react with the rock, obviously it is merely a matter of time, or of the existence of some other favorable condition, until the acid will be completely neutralized. When that point has been reached further rock decomposition by the agency of hot water and carbonic acid, which is everywhere escaping from the springs, will bring into solution carbonates, bicarbonates, or both, which impart to the water an alkaline character. The alkaline springs, therefore, are seen to be logically subsequent to the acid springs.

It was not until the Katmai region, Alaska, had been visited, and the importance of the relation of hot springs to fumaroles had been realized, that it was found possible to formulate a satisfactory theory of the cause and origin of hot springs. In its simplest terms the theory states that hot springs are fed chiefly by ground-water heated by magmatic steam. Two points in it may be stressed: the simple and rational mode of heat transfer, and the effectiveness of the agency where the heat is derived mostly from the condensation of the steam. The theory is based not only upon the close association of fumaroles and hot springs as well as upon other significant field observations, but also upon a knowledge of the composition of rocks and their behavior on heating. All types of igneous rocks, when heated to a sufficiently high temperature, give off a mixture of volcanic gases of which steam is always the preponderant constituent. The escape of such gases as are found generally in hot-spring regions is therefore a necessity, provided only that there is a body of hot rock below ground and that the gases can find access to the surface.

Cracks in the rock would of course offer the natural outlet. This is doubtless the significance of the faults which in so many cases, both in the Lassen Park and elsewhere, have already been noticed in close association with hot springs.

Phenomena at The Geysers, California. Quite different from the hot ground of the Lassen Park in its superficial features is a small area in the Coast Range of California known as The Geysers. The hot springs here are very small, the fumaroles feeble, and the exposed rocks decidedly different in nature; but in essential characteristics a marked resemblance exists between the two districts.

It was found possible at The Geysers to measure the discharge of a comparatively large group of springs, and when the measurement was repeated from time to time as the hot, dry summer advanced, the discharge was seen to fall off continuously—an effect which would have been unintelligible unless a part of the water-supply came from the surface. In the Lassen Park, seasonal changes in the temperature of the springs were observed and variations in water-level or in the volume of discharge were easily recognizable; but at The Geysers changes of this kind were not easy to follow. It was only after actual measurements had been made that all doubt in the matter was dispelled.

By far the most remarkable feature at The Geysers is the small group of steam wells developed for industrial purposes during the last few years. The thermal energy originally manifested at the surface of the ground was quite feeble, but a powerful flow of steam is developed by drilling, generally within the first hundred feet, and in the finished wells, which vary in depth from 200 to 650 feet (60 to 200 meters) pressures range from 60 to over 275 pounds per square inch (4.2 to 19.4 kg. per sq. cm). When the wells are opened enormous amounts of steam escape. When they are closed again, even after months of discharge, the pressure returns again to its former magnitude irrespective of the season. Temperatures at the top of the closed wells reach a maximum of 190° C and in one of them at a depth of 250 feet (76 meters) a temperature hot enough to melt tin (232° C) was found.

Numerous tests disclosed the presence of other volcanic gases with the steam in all the wells, and a series of temperature and pressure measurements proved that the steam was superheated. That such huge quantities of steam (a maximum of 52,000 pounds or 23.6 metric tons of steam per hour) escaping from semi-arid ground with a poor capacity for the conservation of the water-supply, could be derived directly from surface water, seemed highly questionable from the first; and when its superheated character and its constant association with other volcanic gases were discovered the view was rejected as untenable. On the other hand, steam escaping from a magmatic solution would necessarily show a drop in pressure below that of steam escaping from a body of water at the same temperature; in other words it would be superheated. Again, the apparently unlimited store of the steam as well as all its other characteristics is in harmony with the view that it originates in a great magmatic reservoir.[1]

Investigations in the Yellowstone Park. The importance of magmatic steam as facilitating the explanation of geyser action is so obvious that when observations were extended to the Yellowstone Park one of our first concerns was the search for superheated steam vents. Notwithstanding the relatively large supply of ground-water here, superheated steam was found escaping from vents in five different localities in the Park. The maximum temperature found was 138° C, where water boils at a temperature of 92° to 93° C.

Superheated steam was found also in the Lassen Park in one locality. It occurs in many of the Katmai vents and finally in the Tuscan steam wells, as measurements have proved. Superheated steam in volcanic areas is thus by no means a sporadic occurrence. Drilling and casing would probably reveal its presence not far below ground in many other places but we already have observations enough, it seems to me, to afford a thoroughly sound basis for the theory of heat transfer which has been expressed here.

When we compare the springs of the three localities under discussion, we find all to be conditioned as to size and concentration by climate and topography. In the Coast Range of California where the climate is semi-arid and the water-supply poorly conserved, the smallest and most concentrated springs occur; and in the Yellowstone Park where the drainage basins are the largest and the water-supply best conserved, the springs are much the largest and deepest and the waters quite dilute. This statement is an obvious acknowledgment of the influence of surface water on all the springs.

In all three localities the chemical nature of the rocks is clearly reflected in the mineral waters which issue from them. The springs of the Yellowstone Park which issue from rhyolite are highest in the alkalies, and calcium and magnesium drop to a very low figure. In the springs of the Lassen Park which issue in dacite, the alkalies and alkaline earths approach the same order of magnitude. The aluminum derived from the rocks of both localities is generally found precipitated in the sediments in the form of kaolin or alunite. The springs of the

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[1] The primary mass of steam is regarded as magmatic; a subsidiary portion may very well be taken up on its way to the surface.

Coast Range which are closely associated with serpentine are highest in magnesium and lowest in the alkalies, which almost vanish in most of them. It may be noted that these facts are in harmony with the view that the spring-waters derive their mineral content from slight depths, although, of course, they do not prove it since the surface rocks may be of very considerable thickness.

A detailed chemical survey of the Norris Basin in the Yellowstone Park indicates that the acid and alkaline springs there are similar in derivation and in mutual relationship to those of the other two localities. In all three of them the condition generally most favorable to the formation of an alkaline water seems to be a concentration of acid so low that it quickly becomes neutralized in the water that feeds the spring, for in the majority of cases the concentration of the SO_4 radical in the alkaline waters is of a lower order of magnitude than it is in the acid springs.

Among the new problems that present themselves in the Yellowstone Park is a more complete and satisfactory explanation of geyser mechanism. Bunsen discovered the condition of first importance when instead of thermal equilibrium in the geyser tubes he found hotter temperatures at the bottom; but much more work will probably have to be done before the great diversity of behavior in different geysers can be reasonably explained. Fortunately we now have a number of facts which have an obvious bearing on the problem but which have not yet been applied to it.

The waters of the Norris Basin contain several elements in amounts which can be explained only by assuming some kind of concentration intermediate between the decomposition of the rock and the solution of the products. Thus arsenic occurs in concentrations far greater than it does in igneous rocks, so great that its deposition in the form of sulphides is in progress in many places. Unlike the waters of the Lassen Park and The Geysers, those of the Norris Basin contain chlorine (as chlorides) often in relatively large amounts, but even here the distribution is very unequal; some springs contain practically none. This recalls the old problem of the high concentration of chlorine in sea-water. Whether the two cases are comparable is open to question. Problems of the same kind occur in other places, but so far as I know they have never been satisfactorily handled.

The place of hot springs in the system of volcanism. Hot springs are commonly regarded as the last *temperature*-phase of volcanism. This view is perfectly rational. At the same time another factor is equally vital—the *activity of the volcanic gases.* All phases of volcanism are related to one another through their connection with the volcanic gases. It is the expansive force of the hot gases which conditions the rise of lava in craters and which supplies the energy for the major volcanic eruptions. It is the combustion of inflammable gases which prolongs the life of the lava flows. In a less intensive stage of volcanism the gases escaping through vents in the ground constitute the fumaroles and when the fall of temperature permits the percolation of ground-water it is still the volcanic gases which supply the heat of hot springs. Finally, it is steam and its chemically active associates, hydrogen sulphide and carbon dioxide, which cause the decomposition of rocks, the separation of silica, the formation of salts, mineral waters, and other secondary products which form such striking features of the volcanic landscape. The volcanic gases are thus not only the constant associates of every phases of volcanism, but together with the heat they are the active agency in them all. Because the gases are not so easily handled and tested as rocks, and because they form an insignificant portion of the rocks by weight, geologists generally have not yet realized their importance and what the volcanic processes would be without them.

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THE PRESENT STATUS OF THE VOLCANO-GAS PROBLEM

E. S. SHEPHERD

Prior to 1911 my notions of volcanoes were determined by textbook diagrams, fire insurance calendars, and the conventional Neapolitan postcards. In that year I visited Kilauea under the guidance of F. A. Perret, and found some revision of my ideas necessary. I am still revising them. At that time, and during my stay, Kilauea was entirely normal; that is, there was a circular pool of lava about a hundred meters in diameter some thirty meters below the rim of Halemaumau. Nothing happened except an occasional rise of a few meters with overflow, or drop of a few meters with some falling in of the lake wall. Otherwise the lake bubbled away with quite regular fountaining, flow, and hissing cracks. Since it was generally accepted that beneath the lake there was an unimpeded conduit leading down to a convenient magma chamber, and also that the lake had no outlet, a chemist would instinctively deduce that the energy supply must come from the gases, since these were the only products evolved. In 1912 I accompanied Dr. A. L. Day on another expedition, with the object of collecting some of these gases and determining their composition. The results of that expedition are familiar to you.

Since then, collections have been made both by ourselves and by Professor Jaggar, to whom we owe the best collection of volcano-gases ever secured. As a result, we have analyses of some twenty-four vacuum-tube specimens taken at various times and from various promising sources over a period of seven years. These data have been published with little or no comment in the Bulletin of the Hawaiian Volcano Observatory.[1]

Meanwhile we turned our attention to the gases which still remained in solution in the cooled lava, thinking that, special solubilities aside, we might find further clues as to the possible energy supply. It develops that the volatiles which can be pumped out of the lava when it is heated in vacuo are practically identical with those collected from the lake itself. The agreement is much better than we had any reason to expect.

We then extended our experiments to volcanoes of different type, such as Lassen Peak and Mt. Pelée. Here, unfortunately, we cannot compare the lava gases with those erupted; but our Hawaiian results as well as general theory incline us to believe that the differences will not be great.

Regardless of origin, the volatiles which can be obtained from lavas seem to agree upon one thing, and that is that the water-content shall be about 80 per cent of the total. There are exceptions, of course; but for active lava the figure seems quite general. Whether this water is magmatic or whether it is to be derived from deep-seated combustion of primitive hydrogen, or perhaps has been absorbed by the magma on its way to the Earth's surface, must remain for the present a matter of personal opinion. It is difficult for a chemist to imagine a gas leaving the depths and finding its way to the surface without being contaminated by the volatiles resident in the various rocks and strata through which it must pass. In fact such a passage resembles an absorption-tower of greater or less efficiency so that any discussion as to whether these gases are really magmatic or not has very little objective foundation. Of course, if one believes in unimpeded straight conduits leading directly down into the magma chamber, with a rapid rise of magma through such a conduit to the surface, the probability of the gas being magmatic is considerably increased. Such an assumption seems hardly warranted. At Kilauea it ought to be fairly clear by this time that while there is a more or less well-defined central vent, the "unimpeded conduit" turns out to be, at least in its upper portion, a breccia through which the lava must froth its way to the surface. At the andesitic volcanoes the conduit seems equally ill-defined, though for a different reason. Probably the mechanism of eruption is entirely different, being in these cases the result of a crystallizing mass, as at Mt. Pelée and Lassen Peak.

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[1] Bull. Haw. Volc. Obs., 7: 94-97 (1919); 8: 65-67 (1920); 9: 83-88 (1921).

Of the other gases present we note that hydrogen and carbon monoxide are present in very small amounts. In other words the chemical energy of these gases seems to have pretty well exhausted itself before they reached the surface.

Sulfur and the halogens are usually considered as of more interest in connection with fumaroles and chemical alteration than as sources of heat, while the rare gases are present in discouragingly small amounts.

In his paper on the chemistry of volcanism,[2] Allen examined the reactions usually invoked to supply volcanic heat and dismissed them as insufficient. They can furnish heat but the quantities of gas required become rather incredible. The oxidation and reduction reactions can probably be dismissed as of only local import and that a minor one. It may be noted in passing that the classic equation between ferrous oxide and water may be permanently retired, since we are dealing not with ferrous oxide but with ferrous silicate, in which case the sign of the heat change is just the opposite. The chief failure of these combustions lies rather in the means of getting an adequate supply of oxygen.

On a former occasion I called your attention to the chemical results of such engulfments as Professor Jaggar had demonstrated in the recent great subsidence at Kilauea, and pointed out that in this way large quantities of oxygen could be fixed by the iron silicates and buried in the heart of the mountain. The actual amount of heat which can be obtained in this way is, I find, too small to be significant in the general reactions of the volcano, though it adds appreciably to the heating up of the volcano edifice. As a general mechanism we may say that the fixation of oxygen by iron silicates will not suffice, since with the Kilauea basalt we find that with all of the available iron oxidized and then reduced again to ferrous silicate the iron oxides in a unit of lava would generate only enough heat to warm that unit some eight degrees Centigrade, and that with the lesser iron-content of more acid lavas it becomes negligible. Furthermore, to be of any service in volcanism proper it would be necessary to devise some mechanism for repeatedly oxidizing the iron, and this involves the same fatal defect of all the oxidation reactions—the introduction of large quantities of oxygen into the lava. As I see it, Professor Jaggar's surface oxidation hypothesis, although based upon good observation, is negatived by the fact that the ratio of nitrogen to water actually found in the gases is about sixty-eight parts of water to eight of nitrogen, whereas there should be seventy parts of nitrogen to thirty parts of water if the oxygen was derived from the air. It is difficult to imagine oxygen diffusing into the lava column and leaving the nitrogen behind.

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[2] **Journ. Franklin Inst., 193: 29-80 (1922).**

Nearly all of the suggested reactions involve moving large quantities of gas through the lava at a considerable rate of speed if the four hundred million calories per second are to be supplied—this figure being that estimated by Daly from his measurement of the heat-radiation of the lake. Diffusion seems much too slow to move the amounts of gas required, and a little simple arithmetic readily shows vesiculation at any great depth, or the movement of vesiculated material, to be improbable.

It was probably some such considerations as these which caused Day to suggest that the shifting of chemical equilibrium with changing temperature and pressure in the lava might furnish some portion of the heat necessary for the observed phenomena. This hypothesis has one great advantage in that the heat is generated, so to speak, *in situ* and requires no gross movement of the gases within the magma. Of the suggested reactions, that between sulfur and carbon dioxide could furnish the most heat, and if the total heat of this reaction could be transported and transformed in this way the reaction could supply the necessary heat for a hundred-meter lake without an incredible rate of gas evolution at the lake surface. In this connection it is interesting to find that the final state of this reaction calls for a ratio of sulfur to carbon dioxide which is sensibly identical with this ratio as determined from the average composition of all of our vacuum-tube samples. It does not follow that this reaction furnished all of the heat at Kilauea, but it does imply that this reaction is the dominant one. It is important to note that this ratio does not hold for the andesitic volcanoes, Lassen Peak and Mt. Pelée, if we may judge from the gases still retained in solution in the chilled glassy lavas.

At present I believe that the necessary heat for maintaining the lava lake at Kilauea could be obtained by gas heating without too incredible amounts of gas, but feel that such an hypothesis is unnecessary. The same amount of heat could be easily supplied by the rise of a few cubic meters of lava a second with only a two-hundred-degree drop in temperature. It will be recalled that the gas-heating and gas-fluxing hypotheses were invented primarily to account for the lava lake on the explicit assumption that it was an undrained lake, a closed system.

If, on the other hand, we make the entirely probable assumption that Kilauea is merely a lava spring with not one but many irregular subterranean outlets, these hypotheses are less necessary. However improbable such a conception may appear to those familiar only with the continental mountains, a short examination of the porous structure of the Hawaiian range and the history of the various flows and eruptions on the active island will show that no absurdity is involved, rather the contrary. A more detailed explanation of this revived hypothesis will be presented in a forthcoming paper on the Hawaiian volcanoes. Meanwhile I would suggest that a plausible case can be presented for believing that volcanoes are developed, not from some one magma chamber, but from numerous roots, which, in the case of the basaltic volcanoes, go to considerable depths; that they are the fundamental volcanoes, and that among the more superficial roots of the basaltic volcanoes the acid volcanoes take their origin. If this prove to be feasible, then we can withdraw the gas-heating and gas-fluxing hypotheses from their present sadly strained position and apply them readily to those superficial, though highly spectacular and even terrifying, phenomena which a chemist more or less unconsciously comes to regard as the quintessence of volcanism.

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THE CONCENTRATION OF METALLIC CONSTITUENTS BY VOLCANIC EMANATIONS

E. G. Zies

This paper discusses the concentration of metallic constituents by means of the vapor-phase activity of an igneous body. Many geologists and geochemists have considered the possibility of such concentration; and its probability became apparent after studying the incrustations found in the volcanic area of Alaska known as the Valley of Ten Thousand Smokes. In this Valley, approximately a cubic mile of a rhyolitic pumice was deposited after having been blown through the old floor of the Valley. Many fumaroles are located in the pumiceous area. The persistence of relatively high temperatures (97° to 650°) and great volumes of steam since the eruption in 1912 indicate that the emanations derive their heat from a fairly deep-seated igneous body.

The emanations contain over 99 per cent of steam, together with approximately 0.12 per cent HCl, 0.03 per cent HF, and 0.03 per cent H_2S. These acid gases have greatly altered the pumice in the vicinity of the vents of the fumaroles. The metallic constituents *Ag, Pb, Cu, Sn, Mo,* and *Zn* were found in nearly all of the incrustations so far examined. Especially noteworthy is the concentration of molybdenum as "molybdenum blue" in several large fumarolic areas.

Analysis of the pumice of the Valley and analysis of the rhyolitic glass obtained from a new volcano in the Valley revealed the presence of the metallic constituents mentioned above. This fact together with the extensive alteration of the pumice by the acid gases indicates strongly that the *Mo, Cu,* etc., in the "molybdenum blue" areas were obtained from the pumice itself.

Concentration has taken place, however, by still another method. Magnetite (Fe_3O_4) was found in 1919 in a number of fissures. It was deposited in large masses as loosely coherent octahedra against the roofs of the fissures. A simple calculation based on the iron-content of the pumice and the approximate extent of the magnetite-deposit showed that the entire cubic mile of pumice would have been required to furnish the iron for magnetite in one fumarole area. Obviously this is out of the question. It is believed that the magnetite was derived from some combination of iron and halogen that had been brought up from below in a vapor-phase. All of the metallic constituents mentioned above were also found in the magnetite.

Thallium, gallium, arsenic, antimony, bismuth, and cadmium were likewise found in many of the incrustations. The amounts were quite small (0.002 per cent and less) but could readily be determined by a carefully controlled chemical and spectrographic technique.

All of the metallic constituents mentioned have the following property in common—either their halides or oxides have appreciable vapor-pressures at the higher temperatures found in the Valley. The amount of material that can be brought up from below by such a mechanism will depend on the partial pressure of the acid gases, on the vapor-pressure of the metallic halide or oxide, on the ease with which hydrolysis will take place, and finally on the velocity with which the steam is moving. It is believed that the steam at the more elevated temperatures plays an important rôle in sweeping out the reaction products formed below.

As the temperature of escaping steam falls, important changes are likely to take place both in the metallic constituents formed from the decomposition of the pumice by the acid vapors and in those brought up by the acid vapors. One illustration will suffice. On a subsequent expedition in 1923, Dr. C. N. Fenner had no difficulty in determining the location of the fumarole area from which the magnetite was obtained. He did not, however, find any magnetite but found instead a siliceous material impregnated with crystals of sphalerite and galena. He also noted that the temperature of the steam had fallen to 97°. As the temperature falls water will condense near the orifice and provide an acid environment in which the magnetite is no longer stable, but the lead and zinc in the presence of H_2S will be stable as sulfides, provided the acid concentration is not too high.

Differential concentration can be brought about by virtue of the fact that some of the halides are more readily hydrolyzed than others. Thus steam and hydrochloric acid passing over a body containing Fe_2O_3 and TiO_2 will remove both iron and titanium; but since the titanium chloride is more readily hydrolyzed than the iron chloride, the titanium will not travel as far as the iron.

It is quite probable, therefore, that in a volcanic area such as the Valley of Ten Thousand Smokes, or even in the more deep-seated hearth of a volcano which is gradually becoming quiescent, concentration of metallic constituents will take place provided, of course, that the igneous body or adjacent material contains such constituents. The prime requisites are differences in temperature and pressure.

Such concentration is to be looked upon as the preliminary concentration, since lowering of temperature will eventually permit water to be present and this together with the acids (HCl, HF, H_2SO_4) will, as we have seen in the case of magnetite, produce a new set of conditions under which a new set of minerals will be formed. In this new set of minerals the metallic constituents can be concentrated to a still greater degree.

No extensive ore body is likely to be formed in the Valley of Ten Thousand Smokes on account of the great porosity of the pumiceous deposit. This great porosity permits free access of surface water which can carry the acid waters together with the metallic constituents into the drainage system of the area. A volcanic hearth, on the other hand, would prove a much more favorable locality.

Even if the igneous mass is deep-seated, the activity associated with the vapor-phase may play an important rôle in the preliminary concentration of the metallics. If a difference in temperature and a difference in pressure can be established then the volatile constituents of the magmatic materials can escape; they will very likely be acid even though the igneous body is alkaline or neutral. These emanations can carry upward large amounts of metallic constituents and can profoundly alter the rock materials which they penetrate. This type of concentration will at first be independent of the concentration of the metallic sulfides or oxides from the alkaline or neutral magmatic fluids; subsequently, however, the two types of concentration are likely to merge gradually into one another.

In a paper about to be published, the concentration of the metallic constituents in the Valley of Ten Thousand Smokes will be described in detail.

Geophysical Laboratory,
Carnegie Institution of Washington,
Washington, D. C.

THE PRESENT STATUS OF SANTORINI

H. S. Washington

Contrary to the expectations that naturally followed from the history of the eruptions of 1707 and 1866, the present eruption of Santorini appears to be subsiding within two years of its beginning and to be reaching a fumarolic stage. It began beneath the sea in August 1925, and the center of eruption has subsequently shifted. Detailed data on the present stage of subsiding activity are not yet available.

RESEARCH WORK IN VOLCANOLOGY IN JAPAN

Arthur L. Day

(This was an informal report, of which no abstract was prepared)

FACTORS BEARING ON THE FORMATION OF CRATERS ON THE MOON

F. E. Wright

The craters and other surface features of the Moon have long attracted the attention of astronomers and geologists and many hypotheses have been suggested to account for their origin. Although a host of detailed observations have been made, we do not yet know definitely how any one of these features was formed, nor do we know definitely what the materials are that we see exposed on the Moon. The chief difficulty that confronts the geologist in his study of lunar physiography is the visualization of the conditions that obtain there; namely, lack of running water and of an atmosphere, and gravity only one-sixth that at the Earth's surface. Terrestrial features are so profoundly modified and altered by the action of water and of air that our whole geological experience is built up around these factors. The first task of the geologist who wishes to study the physiography of the Moon is to ascertain the physical environment of its surface materials and then to gather experience in the effects produced by such conditions. Erosion, weathering, and cementation in the ordinary terrestrial sense are inactive forces on the Moon's surface. Sedimentary rocks are apparently absent; they do not, as on the Earth, constitute the history-book of the Moon. In the absence of an atmosphere and of circulating water, geological history of the kind we know on the Earth cannot be recorded.

The lunar day is 29½ times as long as our day; the temperature range from midday to midnight is about 200°C. If water were on the Moon's surface, this temperature range would, in itself, mean disintegration of the rocks at a rapid rate. Because of the lack of an atmosphere, both fine dust and heavy blocks ejected into space above the Moon settle down to the Moon's surface with equal rapidity; there is no differential sorting of volcanic ash according to size. A blast of dust and ash-laden volcanic gas encounters no great resistance as in the case of a terrestrial explosion; once started, it will therefore tend to maintain its straight course better than on the Earth. This is no doubt one of the factors responsible for the long streaks on the Moon that at full Moon appear to radiate from several different craters. Winds and wind-erosion are inactive agents on the Moon; volcanic blasts may accomplish something, but only now and then.

The geologist faced with these conditions has not much of a background of experience on which to build. The obvious course for him to take is to commence with the simple facts at hand and to ascertain by test and by computation the behavior of materials under lunar conditions of experiment. As an example of this method of approach let us consider the flight of materials hurled out from a volcano on the Moon as compared with similar materials ejected by a terrestrial volcano.

A long series of computations on the flight of projectiles on the Moon ejected under different angles and with different muzzle-velocities shows that the ranges attained on the Moon are from twenty-five to fifty-fold those attained on the Earth for the same muzzle-velocities. The shape

TABLE 1—*Ranges of projectiles (angle of ejection = 50° or angle of departure = 40°) for different muzzle-velocities from 300 to 1000 meters per second on the Moon and on the Earth with and without an atmosphere and for the ballistic coefficients $C = 1$, 2, and 3*

Initial velocity meters per second	Meters	300	400	500	600	700	800	900	1000
	Feet	984.3	1312.3	1640.4	1968.5	2296.6	2624.7	2952.8	3280.8
Moon..........	Km	55.8	100.6	160.2	236.2	330.8	446.5	586.8	755.8
	Miles	34.7	62.5	99.5	146.8	205.5	277.4	364.6	469.6
Earth without atmosphere...	Km	9.2	16.6	26.5	39.0	54.6	73.7	96.9	124.8
	Miles	5.7	10.3	16.5	24.2	33.9	45.8	60.2	77.5
Earth with atmosphere:									
$C=1$.......	Km	4.8	5.8	6.6	7.3	8.0	8.5	9.2	9.9
	Miles	3.0	3.6	4.1	4.5	5.0	5.3	5.7	6.2
$C=2$.......	Km	6.3	8.2	9.4	10.5	11.7	12.7	13.7	14.8
	Miles	3.9	5.1	5.8	6.5	7.3	7.9	8.5	9.2
$C=3$.......	Km	6.9	9.4	11.2	12.8	14.4	16.1	17.7	19.4
	Miles	4.3	5.8	7.0	8.0	8.9	10.0	11.0	12.1

and density of the projectile on the Moon are not factors to be taken into account as is the case in exterior ballastics on the Earth. In Table 1 are listed the ranges of projectiles on the Moon for an angle of ejection $\beta = 50°$ for different muzzle-velocities from 300 meters to 1,000 meters per second; for comparison the corresponding ranges on the Earth without an atmosphere, and for the Earth with its atmosphere, and for projectiles with ballistic coefficients, $C = 1$, 2, and 3, are given. This table shows clearly the ranges that may be obtained on the Moon in comparison with those actually attained on the Earth. For example, an artillerist on the Earth firing a 75-millimeter gun may attain a range of five to eight miles; on the Moon his ranges for the same muzzle-velocities would be 230 to 280 miles. Were he to fire a Big Bertha, such as the Germans used against Paris during the war, with a muzzle-velocity of a mile a second, his range, instead of 72 miles on the Earth, would be 2,250 miles, or over one-fourth the distance around the Moon.

The significance of this extreme dispersion is realized when we consider a volcano in action on the Moon. Its materials are hurled into space with velocities like those on the Earth, but instead of falling back into the crater as on terrestrial craters they are flung far and wide. The crater-floor is left cleaned out down to the molten lava, if lava be there. It is characteristic of lunar craters that their floors are in general relatively flat, often with a central spire or peak, and that the level of the floor is below that of the surrounding plains. This factor of great dispersion must be adequately considered in any discussion of the Moon's surface-features. Under these conditions we should expect to find large craters greatly exceeding terrestrial craters in size, and this we do find. Lunar craters range in size from 800 miles maximum diameter to a minimum of less than one mile; in fact down to the limit of resolving power of the telescope. Lunar crater-rims rise above the surrounding plains to a height of 4,000 feet or more; the depth from the rim to the crater-floor may be 5,000 to 10,000 feet or more; the height of the central hill may be 1,000 to 5,000 feet. The largest terrestrial crater is less than 30 miles in diameter. According to Gilbert, the average diameter of the ten largest lunar craters is 275 miles (442 km); that of the ten largest terrestrial craters is eleven miles (18 km). Large lunar craters range from 8,000 to 15,000 feet (2,400 to 4,600 meters) in depth; terrestrial from 2,000 to 4,000 feet (600 to 1,200 meters). The floor of the lunar crater is commonly below the level of the surrounding plain; that of terrestrial craters is commonly well above the level of the adjacent plains.

With these relations in mind, let us consider the two theories of the formation of lunar craters that have aroused the most discussion; namely, the volcanic and the meteor-impact theories. Objection to the volcanic theory has been made because lunar craters do not resemble terrestrial craters in detail. I submit that under the conditions that obtain on the Moon, one cannot expect much similarity between lunar and terrestrial craters. The criteria for testing the theory of volcanic origin of lunar craters have not yet been established.

On the other hand, it has been asserted as a strong argument against the meteor-impact hypothesis that the craters are almost uniformly circular in outline, whereas one should expect elliptical shapes to be developed by meteors striking the surface at low angles. In the first place, it is not certain that this is mechanically necessary; in the second, when we realize that meteors strike the Moon at full speed without retardation by an atmosphere as on the Earth, a situation arises that needs investigation. The meteor projectile traveling, say, ten to twenty kilometers per second on impact penetrates some distance into the Moon's crust, sets up an aureole of intense compression which rebounds elastically and ejects some material. Moreover, by the abrupt stopping of the meteor its kinetic energy of translation is available both for mechanical deformation and for conversion into heat sufficient to melt the meteor itself and the adjacent rocks, to set free occluded gases and even to volatilize some of the material, thus producing an explosion analogous to a volcano in its action. In case the meteor were to penetrate deeply to a molten layer of rock a real volcano would result. In either case the original shape of the crater would be much changed by the action of the forces set up by the impact. The circular shape of the craters on the Moon, therefore, may not be construed as an argument against the meteor-impact hypothesis. More data are required before a definite decision can be arrived at regarding the mode of origin of these craters, to say nothing of other perplexing features on the Moon.

Geophysical Laboratory,
Carnegie Institution of Washington,
Washington, D. C.

VOLCANO STATIONS OF THE U. S. GEOLOGICAL SURVEY

W. C. Mendenhall

On July 1, 1924, Congress transferred the Hawaiian Volcano Observatory, with its staff and equipment, from the Weather Bureau in the Department of Agriculture, to the Geological Survey in the Department of the Interior. This transfer brought to the Survey a going scientific concern. Since 1911, Dr. T. A. Jaggar, Jr., in charge of this observatory, at first under the auspices of the Massachusetts Institute of Technology and later under the Weather Bureau, with the aid of local support in the Hawaiian Islands, has been engaged in a study of the volcanological problems exhibited at Kilauea. Throughout this period systematic records have been made of all the phenomena of volcanism that have been observed. Seismographs have been installed and records of tremors accumulated for systematic study; fluctuations of lava-levels in the pit of Halemaumau, the overflows from this pit and from the rifts in adjacent regions, including the outbursts along the slopes of Mauna Loa, have been studied. The composition of lavas and their gases have been made the particular subject of a joint research between the Observatory and the Geophysical Laboratory of the Carnegie Institution of Washington; variations in land-level, including tilt of the land toward and from the volcanic centers, have been recorded. Systematic reports on all of the observed phenomena have been published locally[1] and summaries of them are in course of publication in more permanent form.

On July 1, 1926, the Federal appropriations available for this type of work were increased and corresponding extensions of the work are now under way. Precise level-lines have been run by the U. S. Coast and Geodetic Survey, under a cooperative arrangement, from sea-level at Hilo to the summit of Mauna Loa. The local lines about Kilauea are tied to this base-line, so that both relative and absolute changes of level can now be detected; and by repeating the line to Mauna Loa in the future any changes in its elevation will be revealed. A large number of shallow borings are being made over the floor of the pit of Kilauea. These will be used to measure temperature-variations and temperature-distributions beneath the surface of the recent flows which cover this floor.

In the fall of 1926 an observatory was installed at Mineral, California, near the base of Mt. Lassen. Records are now being secured at this station in consonance with the general plan worked out at Kilauea. In the spring of 1927, Doctor Jaggar went to Alaska, where he will first install a station at Kodiak, convenient to Katmai Volcano and Augustine Island. It is expected that another station will be established later on Akutan Island between Shishaldin and Bogoslof. Seismographs, built at the Hawaiian Observatory, will be installed at these stations and local observers will be secured to keep them in continuous operation.

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[1] See especially the Monthly Bulletin of the Hawaiian Volcano Observatory, and The Volcano Letter, a weekly news-leaflet of the Hawaiian Volcano Research Association.

The developments from the Alaskan stations are not yet determined. The field is one exhibiting more active volcanological phenomena than any field on the mainland of the United States and, with continued support, an expansion of the observations from the seismologic records, with which the work will be started, to the broader field of volcanology in general will doubtless be undertaken.

U. S. Geological Survey,
Washington, D. C.

SPECIAL REPORTS

NOTE ON ACTIVITIES IN ATMOSPHERIC-ELECTRIC INVESTIGATIONS[1]

W. F. G. Swann

The only matters needing report in connection with my activities in atmospheric electricity are briefly noted in the following paragraphs.

(1) I have developed a theory which is explained in its title as follows: A Modification of the Electromagnetic Equations consistent with restricted relativity and providing a possible explanation of the origin of the Earth's electric charge, of the Earth's magnetic field, and of gravitation. This paper was published in abstract form towards the end of my paper on "The Earth's Electric and Magnetic Phenomena," in the Journal of the Franklin Institute for February, 1926. It is now in process of publication in extended form in the Philosophical Magazine. It will also appear in the publication of the papers presented at the International Mathematical Congress in Toronto a couple of years ago.

(2) I have completed in Norway an investigation concerning a search for a possible corpuscular radiation of cosmic origin and the result is contained in the following sentence: Experiments indicate that the absolute magnitude of a current absorbed by a copper cylinder 20.6 cm in diameter and 19.3 cm high is no more than 1.5 per cent of that which would have been obtained by the complete absorption of a vertical corpuscular-current of densities sufficient to account for the replenishment of the Earth's charge.

(3) I have completed an investigation on the measurement of residual ionization in air with pressure up to 1,000 pounds per square inch, measurements having been carried out at the summit of Pike's Peak, Colorado Springs, and at New Haven. The results of this investigation will shortly be published.

(4) Extending number (1) to the magnetic field of sunspots it appears that peripheral velocities of the order of 80 kilometers per second are capable of giving rise to magnetic fields of the order of 2,000 gauss. Moreover, the indications are that Jupiter should have a very large magnetic field, a field of the order of 60,000 gauss.

Sloane Laboratory,
Yale University,
New Haven, Conn.

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[1] Abstract-report prepared at the request of the Union for the Prague Assembly of the International Geodetic and Geophysical Union.

REPORT UPON THE ORGANIZATION OF THE DEPARTMENT OF GEOPHYSICS AT THE COLORADO SCHOOL OF MINES, GOLDEN, COLORADO[1]

C. A. HEILAND

In 1922, geophysical methods were used for the first time in the United States as an aid to detect mineral deposits, especially oil-bearing structures, which could not be found by geological investigations alone. The first instrument employed for that purpose was the Eoetvoes torsion balance; shortly after, seismic investigations were started, and recently magnetic and electric methods have been taken up. Thus far, the mining industry has used geophysical methods comparatively little. Magnetometers have been applied sometimes in search for iron ores in our northern states; for other ore deposits the electrical methods have been employed; extensively, however, only in Canada.

Geophysical measurements for such practical purposes as outlined above have been made in this country by private corporations exclusively. Consulting engineers, as well as companies, have done contracting work with one or more geophysical methods, or the large operating oil concerns have organized their own geophysical staffs. An idea of the amount of geophysical work as done at present may be obtained perhaps by conceiving that at present about five consulting engineers and about twelve companies are making geophysical measurements on the contract basis and that, on the other hand, about one hundred torsion balances, twenty seismographs, and fifty magnetometers are in use by the geophysical staffs of oil companies.

The development of applied geophysics in this country as characterized above has given rise to two serious problems. The first is the need of men who are trained equally well in the theory and the operation of geophysical instruments. The second is the need of a central and impartial place where researches on problems of applied geophysics are done and made public. In view of the fact that the essential features of most geophysical methods are kept secret yet, and that every company which starts this kind of work has, usually, to go back to the very beginning, the need for scientific public research for the benefit of the industry is obvious in order that time shall not be wasted in useless experimenting.

In consideration of these needs, the President of the Colorado School of Mines, Dr. M. F. Coolbaugh, decided in 1926 to establish a Department of Geophysics at the school. The organization of this Department, which began in January, 1927, was entrusted to the writer. The establishment of such a department, the first of its character in this country, at a mining school also aims to encourage the mining industry to apply the methods in question more than it has done before.

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[1] Report prepared at the request of the Union for the Prague Assembly of the International Geodetic and Geophysical Union.

The program, which has been outlined for this work, consists of two parts, (1) the educational program, and (2) the research program.

We feel that a sufficient ability to handle any applied geophysical method intelligently cannot be attained by any person without a good foundation in general geophysics. Therefore a course in general geophysics is given at the school which is open to seniors and postgraduate students, and required for members of the special courses which concern theory and operation of instruments. From the next academic year on, the general course will be given in two semesters and will cover the following subjects: (1) The history of the Earth (cosmogonical hypotheses, paleogeography, paleoclimatology); (2) constitution and dynamics of the atmosphere (meteorology, atmospheric electricity, propagation of sound, etc.); (3) constitution and dynamics of the hydrosphere (hydrology, oceanography, tides); (4) the constitution and dynamics of the lithosphere (density, gravity, isotasy, earth-magnetism, chemical and physical constitution of the interior of the earth, radio-activity, earth-electricity, movement of the earth in space, bodily tides, volcanism, earthquakes, etc.). Because of its importance in applied geophysics, more emphasis is placed upon item (4) than upon items (1) to (3).

The operation of any geophysical instrument is confined to the special courses which are open for postgraduates only. In these courses, first in general the theory of the respective instruments is taught in detail, then some work is done in the laboratory, and finally the instruments are taken out to obtain practical field work. In the first semester, after the organization of the department, a special course in the theory and operation of the torsion balance was given. In the next semester, the theory and operation of magnetic variometers will be taken up. After that, a seismological course probably will follow; a course in electrical methods is also planned for the future. At present the school has a photographic recording torsion-balance and two magnetic field-balances for the horizontal and vertical intensity. In order to present the magnetometer course as well as is desirable, the recording of magnetic variations is indispensable; therefore, the very next thing planned is the erection of a magnetic observatory where three components—probably D, H, and Z—are to be recorded continuously. By the erection of a continuously recording magnetic observatory we also hope to contribute to the advancement of theoretical investigations, which are made in regard to the relationship of terrestrial magnetism and earth-electricity to atmospheric electricity and activity of the Sun, especially in this country by the well-known Department of Terrestrial Magnetism of the Carnegie Institution of Washington and the Division of Terrestrial Magnetism and Seismology of the U. S. Coast and Geodetic Survey.

The purchase, or construction, of a seismograph will also be a necessity sooner or later. We are also contemplating having our own precision-mechanics workshop on the campus in the very near future, in order to make all necessary instrument work as inexpensive as possible.

As far as the activities in the new department, aside from the educational program, are concerned, we felt that the first requirement for all studies in applied geophysics would be a thorough knowledge of the respective literature; however, at present, this is comparatively difficult to obtain because, first of all, a textbook on applied or general geophysics is not available yet in this country; secondly, because the literature on applied geophysics is scattered in journals of very different character. We have decided, therefore, to compile all articles—on applied geophysics as well as on general geophysics, the study of which is important for applied geophysics—which have been published, especially recently, in all countries of the world. This "Bibliography on Applied Geophysics" will be published as soon as possible.

The research work planned for the future in the department is, naturally, stipulated by the equipment which we have at present. As far as the torsion balance is concerned, it is our aim to make this method, which so far has been applied to the greatest extent to oil propositions, also practical for use in mining by practical and theoretical investigations. Our research work pertaining to magnetic methods is to start with investigations about the physical, especially thermic, properties of their fundamental constituents—the magnets. The generosity of one of the leading manufacturers of geophysical instruments has enabled us to start these and the ensuing investigations. The school has been presented with three apparatuses especially designed for these purposes. Extensive researches are also planned on the magnetization of rocks, in view of the fact that that is one of the most complex phenomena, and it seems to be worth while, especially from the physico-chemical standpoint, to investigate the conditions under which an induced and a permanent magnetization of rocks is originated.

Colorado School of Mines,
Golden, Colorado.

GEOPHYSICAL EXHIBIT

GEOPHYSICAL EXHIBIT

(Washington, D. C., April 22 to 29, 1927)

GENERAL DESCRIPTION

An exhibit of geophysical instruments, researches, and applications was displayed from 9 A. M. to 5 P. M. during Friday, April 22, through Friday, April 29, 1927 (except Sunday, April 24), in the southwest and northeast exhibit-rooms of the building of the National Academy of Sciences and National Research Council, Washington, D. C.

The exhibits, arranged for largely by the executive committees of the different sections, were furnished by the following institutions and individuals: American Petroleum Institute; Roald Amundsen and H. U. Sverdrup, *Maud* Expeditions of 1918-1925; Astrophysical Observatory of the Smithsonian Institution; Blue Hill Meteorological Observatory; Bureau of Standards; Carnegie Institution of Washington, through its Department of Terrestrial Magnetism, Geophyscial Laboratory, and Mount Wilson Observatory; Coast and Geodetic Survey; Georgetown University; L. Gorczynski; T. A. Jaggar, Jr., for the Volcano Observatory, Hawaii; National Geographic Society; Naval Observatory; G. W. Pickard; Scripps Institution of Oceanography; Smithsonian Institution; University of Michigan Expedition to Greenland, 1926; Weather Bureau; Weston Electrical Instrument Corporation. The appreciative thanks of the Union are due to these organizations and individuals for their cooperation and interest which contributed so largely to the success of the exhibit and of the meetings.

The exhibits were classified under subject-headings corresponding to the fields of activity of the different sections and are briefly described below.

GEODESY

(*a*) Invar-pendulum apparatus for measuring the intensity of gravity.

(*b*) Geodetic level and level-rod used in extending precise level-lines.

(*c*) Tape stretcher and invar tape used in accurate measurements of distances.

(*d*) Type of theodolite designed and constructed by the Coast and Geodetic Survey used on most of the precise triangulations in the United States since 1897. (This theodolite has twelve-inch circle, five-minute graduations, and is read by each of three micrometer-microscopes to one second of arc.)

(*e*) Personal-equation apparatus for measuring personal equation of observers in astronomical observations.

SEISMOLOGY

(*a*) Earliest type of Milne seismograph for photographic recording of distant earthquakes, originally installed by Milne in 1899 at Oahu College, Honolulu.

(*b*) The Wood-Anderson torsion-seismograph, devised recently by H. O. Wood and J. A. Anderson under the auspices of the Carnegie Institution of Washington for recording nearby or distant earthquakes.

(*c*) Typical earthquake-records (seismograms) obtained by Milne and by Wood-Anderson seismographs.

(*d*) Photographs of various types of seismographs and earthquake-effects, and diagrams showing transmission of earthquake-waves.

(*e*) Photographs, paintings, instruments, and models showing development of the seismograph:

(1) First seismograph after Choko, A. D. 136.
(2) A painting of the ancient dragon-frog seismoscope.
(3) De Haute Feuille seismoscope, A. D. 1703 (first improvement on Choko's machine).
(4) A simple mantle-type tromometer to show slight earth-tremors.
(5) One of the early seismographs constructed on modern principles.
(6) A model of the famous "Sekiya's wires" showing the actual path of an earth-particle during an earthquake.
(7) Vertical seismograph of Galitzin operated at Georgetown.

(*f*) Seismograms recorded by great earthquakes:

(1) Japan, September 1, 1923, record at Tokyo.
(2) Sumatra, June 28, 1926, record of vertical seismograph at Georgetown.
(3) Saguenay Basin, Canada, February 28, 1925, record at Georgetown and showing possible cause of misinterpretation.
(4) Heavy microseismic disturbance recorded at Georgetown.

METEOROLOGY

(*a*) The Marvin copper-cone pyrheliometer for measuring solar radiation. (The cone with small angular opening has interior surface specularly reflecting, thus insuring nearly complete absorption.)

(*b*) Smithsonian silver-disk pyrheliometer. (The bent tube mercurial thermometer, with bulb imbedded in the silver disk, measures temperature-changes in the latter as it is alternately exposed to and shielded from radiation.)

(*c*) Gorczynski's type of solarimeter. (A Moll thermopile with a Weston microammeter for measuring the current generated by the radiation absorbed at the thermal junctions.)

(*d*) McAdie's multiple-scale thermometer. (A mercurial thermometer provided with six adjustable scales for direct readings of (A) temperature by (1) Fahrenheit scale, (2) Centigrade scale, (3) absolute scale, (4) kilograd scale, and of (B) water-vapor showing (1) saturation density and (2) pressure.)

(*e*) Fergusson balloon-meteorograph. (Distinctive features are (1) a mechanism providing a two-part scale for more accurate reading of pressure, (2) a single time arc for all elements simplifying reading of records, and (3) light weight, permitting use of small, inexpensive balloons.)

(*f*) Photographs taken on 1926 Greenland Expedition of University of Michigan. (Aerological data and synoptic charts for Greenland and vicinity, August, 1926.)

(*g*) Series of seven graphs showing Sun and weather variations.

(*h*) Slide-rule extrapolator. (A special arrangement of six slide-rules on one mounting to facilitate the computation of solar radiation-observations; the time of reduction is reduced nearly five-fold by this extrapolator.)

TERRESTRIAL MAGNETISM AND ELECTRICITY

(*a*) Shaw fading recorder for taking graphic records of radio-signal fluctuations.

(*b*) Temperature-compensated variometer with diagrams showing increased efficiency through such compensation in magnetic-observatory work and compilations.

(*c*) Scaling device for reading photographic records directly in absolute units.

(*d*) Transparencies from photographs in the Arctic Sea off the North Coast of Siberia, 1918 to 1925, showing types of aurora.

(*e*) Transparencies of correlations between solar activity and (1) the magnetic and electric activities of the Earth, (2) radio reception.

(*f*) Earth inductor and galvanometer for detailed study of anomalies and geological formations.

(*g*) Apparatus for observing radio signal-intensities at magnetic and electric observatories.

OCEANOGRAPHY

(*a*) Portable automatic tide-gauge to record tides for reduction of soundings to datum (reference) plane in hydrographic surveys.

(*b*) Pressure sounding-tube to determine depths down to 100 fathoms.

(*c*) Allen-type water-bottle, five-liter capacity, to secure water-samples for quantitative study of plankton and other microscopic forms of sea life.

(*d*) Improved bottom-samplers of the Davis, and Eckman and Albatross types to secure samples of mud and ooze from the bottom of the ocean.

(*e*) Tentative route planned for Cruise VII of the *Carnegie* from 1928 to 1931, during which oceanographic, as well as magnetic and electric, investigations will be undertaken.

(*f*) Graphs for solving the equation for calculation of surface-currents from temperature-data and for determining density or salinity.

(*g*) Graphs showing sub-surface distribution of diatoms.

VOLCANOLOGY

Photographs of volcanoes, hot springs, and lunar craters, as follows:

(*a*) Aso Volcano, Kyushu Island, Japan, 1927. (The original caldera-type crater is 60 miles in circumference, and is said to be the largest in the world; the present mountain mass arises from the center of this caldera to a height of about 5,000 feet and contains five active craters.)

(*b*) The Kilauea (Halemaumau crater, Hawaii, in 1917 when nearly filled with lava.

(*c*) "Valley of Ten Thousand Smokes" near Mt. Katmai, Alaska, National Geographic Society's expedition of 1917.

(*d*) Lava lake at Kilauea showing the islands frequently formed by the separation of great sections of the surrounding basin.

(*e*) Hot springs of the "Porcelain Basin," Yellowstone National Park.

(*f*) Craters of the Moon photographed at Mount Wilson Observatory, September 15, 1919:

(1) Northern portion of Moon at last quarter.
(2) Southern portion of Moon at last quarter.
(3) Region about Crater Copernicus at last quarter.

GENERAL

STATUTES AND BY-LAWS OF THE AMERICAN GEOPHYSICAL UNION

(As amended to July 1, 1927)

PREAMBLE

The American Geophysical Union is hereby established by the National Research Council. It supersedes the "American Section" of the proposed International Geophysical Union." The American Geophysical Union is the American National Committee of the International Geodetic and Geophysical Union and the Executive Committee of the American Geophysical Union is the Committee on Geophysics of the National Research Council. The American Geophysical Union is constituted as follows:

ARTICLE 1

Objects

(a) To assist in carrying out the objects of the International Geodetic and Geophysical Union, which are: To promote the study of problems concerned with the figure and physics of the Earth; to initiate and co-ordinate researches which depend upon international co-operation, and to provide for their scientific discussion and publication; and to facilitate special researches, such as the comparison of instruments used in different countries.

(b) To promote and co-ordinate the study of the various branches of geodesy and geophysics in the United States of America and its outlying territories.

ARTICLE 2

Membership

The membership of the American Geophysical Union shall be as follows:

(a) Those who composed the membership of the "American Section of the proposed International Geophysical Union" on July 1, 1919.

(b) Those who may be appointed by the Executive Board of the National Research Council upon nominations submitted by the Executive Committee of the American Geophysical Union.

(c) The Chairman of the Division of Foreign Relations, the Chairman of the Division of Physical Sciences, the Chairman of the Division of Chemistry and Chemical Technology, the Chairman of the Division of Geology and Geography, and the Chairman of the Division of Biology and Agriculture, of the National Research Council, members *ex officio*.

(d) The American officers of the International Geodetic and Geophysical Union and of its Sections, members *ex officio*.

ARTICLE 3

Sections

The American Geophysical Union shall be divided into Sections, in accordance with the organization of the International Geodetic and Geophysical Union, as follows: (a) Geodesy; (b) Seismology; (c) Meteorology; (d) Terrestrial Magnetism and Electricity; (e) Oceanography; and (f) Volcanology.

The members shall designate the Section, or Sections, to which they will belong. The *functions of the Sections*, in their respective fields, shall be the promotion of the objects of the American Geophysical Union.

ARTICLE 4

Officers of the American Geophysical Union

The *Officers of the American Geophysical Union* shall consist of a chairman and a vice-chairman, elected to serve for a period of three years, and not to be eligible for immediate re-election to their respective offices; and a general secretary, to serve for a period of three years.

ARTICLE 5

Officers of Sections

The *Officers of the Sections* of the American Geophysical Union shall consist for each Section of a chairman, a vice-chairman, and a secretary, whose eligibility and periods of office shall correspond to those of the Union. The Chairman of each Section shall have the power to appoint, with the right to vote, a representative of the Section to such meetings of the Executive Committee of the Union as he may be unable to attend.

ARTICLE 6

Executive Committee

The *Executive Committee* shall consist of the chairman, the vice-chairman, and the general secretary of the American Geophysical Union, and of one representative from each of the sections of the Union, who shall be the chairman thereof or a member appointed by him; and, in addition, *ex officio*, the American officers of the International Geodetic and Geophysical Union, and of its Sections, the chairman of each of the Standing Committees of the American Geophysical Union, and the chairman of each of the following Divisions of the National Research Council: Foreign Relations, Physical Sciences, Chemistry and Chemical Technology, Geology and Geography, and Biology and Agriculture. This Committee shall have charge of all administrative matters of the American Geophysical Union, and shall be the representative of the Union in its relations with the National Research Council and with the International Geodetic and Geophysical Union. The Executive Committee, in consultation with the Sections, shall nominate to the Executive Board of the National Research Council delegates, or representatives, to the meetings of the International Geodetic and Geophysical Union and of its Sections, in accordance with the regulations fixed by the International Research Council for the voting power of the United States of America.

ARTICLE 7

Annual Meetings

The American Geophysical Union and its Sections shall meet annually at such time and place as may be chosen by the Executive Committee of the Union. The officers of the Union and of the Sections shall be elected at this annual meeting.

ARTICLE 8

Special Meetings

Special Meetings of the American Geophysical Union, or of its Sections, may be called when necessary by the respective chairmen.

ARTICLE 9

By-Laws

The American Geophysical Union is authorized to enact such by-laws as may be necessary for the performance of its functions and duties.

ARTICLE 10

Amendments

Amendments to the Statutes of the American Geophysical Union shall be subject to the approval of the National Research Council.

BY-LAWS

The fiscal year of the American Geophysical Union shall conform to the fiscal year of the National Research Council.

Standing committees on Statutes and By-laws, on Membership, on Meetings, and on Budget, shall be appointed for the ensuing fiscal year by the Chairman-elect of the Union prior to the beginning of each year.

All committees of the American Geophysical Union shall terminate at the end of the fiscal year for which they may be appointed.

Meetings of the Union shall be called at least thirty days in advance, and those present at the meeting shall constitute a quorum.

Meetings of the Executive Committee shall be called by the Chairman, and those present shall constitute a quorum.

Final action shall not be taken by the Executive Committee on matters affecting any Section in the absence of the Chairman of such Section, or his representative.

No mail ballot shall be closed until at least thirty days after the date of issue.

After thirty days' notice has been given, amendments to the Statutes (all of which are subject to the approval of the National Research Council) may be voted on either by mail or at a regular meeting of the Union.

Amendments to the By-Laws may be proposed either by mail or at a regular meeting; when voted on by mail they require for adoption a majority vote of the members; when voted on at a regular meeting they require for adoption a three-fourths vote of the members present.

Any amendment to the Statutes or By-Laws proposed by at least five members shall be submitted for vote to the members of the Union by the Executive Committee.

A member of the American Geophysical Union may be dropped from the Union by a three-fourths vote of the members of the Executive Committee.

Statutes and By-Laws adopted by a Section shall not conflict with the provisions of the Statutes and By-Laws of the Union.

Matters pertaining to a Section or Sections, and not pertaining to the Union as a whole, shall be acted upon by the Section or Sections concerned.

ALPHABETICAL ADDRESS LIST OF MEMBERS OF THE AMERICAN GEOPHYSICAL UNION[1]

Dr. C. G. Abbot, (*c, d*)
Smithsonian Institution
Washington, D. C.
Mr. L. H. Adams (*b, f*)
Geophysical Laboratory
2801 Upton St. N. W.
Washington, D. C.
Dr. John A. Anderson (*b*)
Mount Wilson Observatory
Pasadena, California
Capt. J. P. Ault (*c, d, e*)
Department of Terrestrial Magnetism
36th St. and Broad Branch Road
Washington, D. C.
Dr. L. W. Austin (*d*)
Radio Physical Laboratory
Bureau of Standards
Washington, D. C.
Professor W. D. Bancroft (*c*)
Cornell University
7 East Avenue
Ithaca, New York
Dr. S. J. Barnett (*d*)
University of California
Southern Branch
Los Angeles, California
Dr. Louis A. Bauer (*a, b, c, d*)
Department of Terrestrial Magnetism
36th St. and Broad Branch Road
Washington, D. C.
Dr. H. B. Bigelow (*e*)
Harvard University
Cambridge, Massachusetts
Major W. R. Blair (*c, d*)
Care Office Chief Signal Officer
Washington, D. C.
Dr. N. L. Bowen (*b, f*)
Geophysical Laboratory
2801 Upton St. N. W.
Washington, D. C.
Major E. H. Bowie (*c*)
U. S. Weather Bureau
San Francisco, California
Dr. William Bowie (*a, b, e, f*)
Coast and Geodetic Survey
Washington, D. C.
Dr. L. J. Briggs (*a, e, f*)
Bureau of Standards
Washington, D. C.
Professor E. W. Brown (*a, b, f*)
Yale University
New Haven, Connecticut
Dr. Austin H. Clark (*e*)
Smithsonian Institution
Washington, D. C.
Dr. F. W. Clarke (*e, f*)
Geological Survey
Washington, D. C.
Dr. H. H. Clayton (*c*)
1410 Washington Street
Canton, Massachusetts
Dr. Whitman Cross (*f*)
101 East Kirke Street
Chevy Chase, Maryland
Professor R. A. Daly (*b, e, f*)
University Museum
Harvard University
Cambridge 38, Massachusetts
Professor W. M. Davis (*e*)
Professor Emeritus
Harvard University
23 Hawthorn Street
Cambridge 38, Massachusetts
Dr. A. L. Day (*b, f*)
Geophysical Laboratory
2801 Upton St. N. W.
Washington, D. C.
Dr. J. H. Dellinger (*d*)
Bureau of Standards
Washington, D. C.

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[1] Total number of members on July 1, 1927, is 72. The letters following each name indicate the sections with which the member is affiliated, these being (*a*) Geodesy, (*b*) Seismology, (*c*) Meteorology, (*d*) Terrestrial Magnetism and Electricity, (*e*) Oceanography, and (*f*) Volcanology.

Capt. R. L. Faris (*a, b, d, e*)
Coast and Geodetic Survey
Washington, D. C.

Mr. J. A. Fleming (*b, c, d, e*)
Department of Terrestrial Magnetism
36th St. and Broad Branch Road
Washington, D. C.

Mr. O. H. Gish (*c, d*)
Department of Terrestrial Magnetism
36th St. and Broad Branch Road
Washington, D. C.

Mr. W. R. Gregg (*c, d*)
Weather Bureau
Washington, D. C.

Dr. George E. Hale (*a, c, d*)
Mount Wilson Observatory
Pasadena, California

Dr. Harvey C. Hayes (*c*)
Naval Research Laboratory
Bellevue, Anacostia, D. C.

Mr. D. L. Hazard (*a, b, d*)
Coast and Geodetic Survey
Washington, D. C.

Commander N. H. Heck (*b, d, e*)
Coast and Geodetic Survey
Washington, D. C.

Mr A. J. Henry (*c, d, e*)
Weather Bureau
Washington, D. C.

Professor L. M. Hoskins (*a, b*)
Stanford University
1240 Waverley Street
Palo Alto, California

Dr. W. J. Humphreys (*b, c, d, f*)
Weather Bureau
Washington, D. C.

Dr. T. A. Jaggar, Jr. (*b, f*)
Hawaiian Volcano Observatory
Volcano Post Office, T. H.

Dr. Lewis V. Judson (*a, b*)
Bureau of Standards
Washington, D. C.

Dr. H. H. Kimball (*c*)
Weather Bureau
Washington, D. C.

Mr. W. D. Lambert (*a, b, e*)
Coast and Geodetic Survey
Washington, D. C.

Dr. A. C. Lawson (*b*)
University of California
Berkeley, California

Professor A. O. Leuschner (*a, b*)
University of California
Berkeley, California

Professor G. W. Littlehales (*a, c, d, e*)
Hydrographic Office
Washington, D. C.

Professor A. G. McAdie (*b, c, d, e*)
Blue Hill Meteorological Observatory
Harvard University
Readville, Massachusetts

Rev. Professor J. B. Macelwane, S. J., (*a, b, c, d, f*)
St. Louis University
St. Louis, Missouri

Dr. George F. McEwen (*e*)
Scripps Institution of Oceanography
La Jolla, California

Mr. H. A. Marmer (*e*)
Coast and Geodetic Survey
Washington, D. C.

Mr. C. F. Marvin (*b, c, d, f*)
Weather Bureau
Washington, D. C.

Dr. S. J. Mauchly (*c, d, e, f*)
Department of Terrestrial Magnetism
36th St. and Broad Branch Road
Washington, D. C.

Professor A. A. Michelson (*a, b, e*)
University of Chicago
Chicago, Illinois

Professor R. A. Millikan (*c, d*)
California Institute of Technology
Pasadena, California

Dr. H. F. Moore (*e*)
Linville Falls, North Carolina

Professor F. R. Moulton (*a, c, e*)
University of Chicago
Chicago, Illinois

Dr. Robert C. Murphy (*e*)
American Museum of Natural History
New York City

Dr. G. P. Paine (*c*)
P. O. Box 131
Newark, Delaware

Commander W. E. Parker (*c, d, e*)
Coast and Geodetic Survey
Washington, D. C.

Mr. F. A. Perret (*f*)
Geophysical Laboratory
2801 Upton St. N. W.
Washington, D. C.

Mr. W. J. Peters (*d, e*)
Department of Terrestrial Magnetism
36th St. and Broad Branch Road
Washington, D. C.

Professor H. F. Reid (*a, b, c, f*)
Johns Hopkins University
Baltimore, Maryland

Commander G. T. Rude (*b, c, e*)
Coast and Geodetic Survey
Washington, D. C.

Professor Frank Schlesinger (*a*)
Yale University Observatory
New Haven, Connecticut

Dr. R. B. Sosman (*f*)
Geophysical Laboratory
2810 Upton St. N. W.
Washington, D. C.

Mr. George Steiger (*f*)
Geological Survey
Washington, D. C.

Professor W. F. G. Swann (*a, c, d*)
Yale University
New Haven, Connecticut

Rev. Dr. Francis A. Tondorf, S. J. (*b*)
Georgetown University
Washington, D. C.

Professor Sidney D. Townley (*a, b*)
Stanford University
Stanford University
California

Mr. C. E. Van Orstrand (*a, b, e, f*)
Geological Survey
Washington, D. C.

Dr. T. Wayland Vaughan (*b, c*)
Scripps Institution of Oceanography
La Jolla, California

Professor R. DeC. Ward (*c*)
Harvard University
Cambridge, Massachusetts

Dr. H. S. Washington (*a, b, f*)
Geophysical Laboratory
2801 Upton St. N. W.
Washington, D. C.

Dr. Roger C. Wells (*e, f*)
Geological Survey
Washington, D. C.

Professor Bailey Willis (*b, c, e, f*)
Stanford University
Stanford University
California

Mr. H. O. Wood (*a, b, f*)
Care Seismological Laboratory
220 N. San Rafael Ave.
Pasadena, California

Dr. Fred E. Wright (*a, b, f*)
Geophysical Laboratory
2801 Upton St. N. W.
Washington, D. C.

EX OFFICIO MEMBERS

The Chairman, Division of Foreign Relations, National Research Council.

The Chairman, Division of Physical Sciences, National Research Council.

The Chairman, Division of Chemistry and Chemical Technology, National Research Council.

The Chairman, Division of Geology and Geography, National Research Council.

The Chairman, Division of Biology and Agriculture, National Research Council.

OFFICERS OF THE UNION

Chairman, H. S. Washington (1926-1929)
Vice-Chairman, G. W. Littlehales (1926-1929)
General Secretary, J. A. Fleming (1925-1928)

OFFICERS OF SECTIONS

SECTION	CHAIRMAN	VICE-CHAIRMAN	SECRETARY
Geodesy	William Bowie (1926-1929)	F. E. Wright (1926-1929)	W. D. Lambert (1925-1928)
Seismology	L. H. Adams (1926-1929)	N. H. Heck (1926-1929)	D. L. Hazard (1925-1928)
Meteorology	H. H. Kimball (1926-1929)	G. W. Littlehales (1926-1929)	A. J. Henry (1926-1929)
Terrestrial Magnetism and Electricity	N. H. Heck (1926-1929)	J. H. Dellinger (1926-1929)	J. A. Fleming (1926-1929)
Oceanography	T. W. Vaughan (1926-1929)	G. T. Rude (1926-1929)	Austin H. Clark (1926-1929)
Volcanology	T. A. Jaggar, Jr. (1926-1929)	F. E. Wright (1926-1929)	R. B. Sosman (1925-1928)

MEMBERSHIP OF SECTIONS[1]

SECTION (a) GEODESY (number of members, 23)

Louis A. Bauer
William Bowie, Chairman (1926-1929)
L. J. Briggs
E. W. Brown
R. L. Faris
George E. Hale
D. L. Hazard
L. M. Hoskins
Lewis V. Judson
W. D. Lambert, Secretary (1925-1928)
A. O. Leuschner
G. W. Littlehales
J. B. Macelwane
A. A. Michelson
F. R. Moulton
H. F. Reid
Frank Schlesinger
W. F. G. Swann
S. D. Townley
C. E. Van Orstrand
H. S. Washington
H. O. Wood
F. E. Wright, Vice-Chairman (1926-1929)

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[1] The affiliations with sections of the ex officio members are not shown in these lists, owing to inevitable changes from year to year.

SECTION (b) SEISMOLOGY (number of members, 33)

L. H. Adams, Chairman (1926-1929)
J. A. Anderson
Louis A. Bauer
N. L. Bowen
William Bowie
E. W. Brown
R. A. Daly
A. L. Day
R. L. Faris
J. A. Fleming
D. L. Hazard, Secretary (1925-1928)
N. H. Heck, Vice-Chairman (1926-1929)
L. M. Hoskins
W. J. Humphreys
T. A. Jaggar, Jr.
Lewis V. Judson
W. D. Lambert
A. C. Lawson
A. O. Leuschner
A. G. McAdie
J. B. Macelwane
C. F. Marvin
A. A. Michelson
H. F. Reid
G. T. Rude
Francis A. Tondorf
S. D. Townley
C. E. Van Orstrand
T. Wayland Vaughan
H. S. Washington
Bailey Willis
H. O. Wood
F. E. Wright

SECTION (c) METEOROLOGY (number of members, 27)

C. G. Abbot
J. P. Ault
W. D. Bancroft
Louis A. Bauer
W. R. Blair
E. H. Bowie
H. H. Clayton
J. A. Fleming
O. H. Gish
W. R. Gregg
George E. Hale
A. J. Henry, Secretary (1926-1929)
W. J. Humphreys
H. H. Kimball, Chairman (1926-1929)
G. W. Littlehales, Vice-Chairman (1926-1929)
A. G. McAdie
J. B. Macelwane
C. F. Marvin
S. J. Mauchly
R. A. Millikan
F. R. Moulton
G. P. Paine
W. E. Parker
G. T. Rude
W. F. G. Swann
R. DeC. Ward
Bailey Willis

SECTION (d) TERRESTRIAL MAGNETISM AND ELECTRICITY (number of members, 25)

C. G. Abbot
J. P. Ault
L. W. Austin
S. J. Barnett
Louis A. Bauer
W. R. Blair
J. H. Dellinger, Vice-Chairman (1926-1929)
R. L. Faris
J. A. Fleming, Secretary (1926-1929)
O. H. Gish
W. R. Gregg
George E. Hale
D. L. Hazard
N. H. Heck, Chairman (1926-1929)
A. J. Henry

W. J. Humphreys
G. W. Littlehales
A. G. McAdie
J. B. Macelwane
C. F. Marvin
S. J. Mauchly
R. A. Millikan
W. E. Parker
W. J. Peters
W. F. G. Swann

Section (e) Oceanography (number of members, 31)

J. P. Ault
H. B. Bigelow
William Bowie
L. J. Briggs
A. H. Clark, Secretary (1926-1929)
F. W. Clarke
R. A. Daly
W. M. Davis
R. L. Faris
J. A. Fleming
Harvey C. Hayes
N. H. Heck
A. J. Henry
W. D. Lambert
G. W. Littlehales
A. G. McAdie
G. F. McEwen
H. A. Marmer
S. J. Mauchly
A. A. Michelson
H. F. Moore
F. R. Moulton
Robert C. Murphy
W. E. Parker
W. J. Peters
H. F. Reid
G. T. Rude, Vice-Chairman (1926-1929)
C. E. Van Orstrand
T. Wayland Vaughan, Chairman (1926-1929)
Roger C. Wells
Bailey Willis

Section (f) Volcanology (number of members, 24)

L. H. Adams
N. L. Bowen
William Bowie
L. J. Briggs
E. W. Brown
F. W. Clarke
Whitman Cross
R. A. Daly
A. L. Day
W. J. Humphreys
T. A. Jaggar, Jr., Chairman (1926-1929)
J. B. Macelwane
C. F. Marvin
S. J. Mauchly
F. A. Perret
H. F. Reid
R. B. Sosman, Secretary (1925-1928)
George Steiger
C. E. Van Orstrand
H. S. Washington
Roger C. Wells
Bailey Willis
H. O. Wood
F. E. Wright, Vice-Chairman (1926-1929)

OFFICERS SINCE ORGANIZATION

OFFICERS OF THE AMERICAN GEOPHYSICAL UNION[1]

CHAIRMAN	VICE-CHAIRMAN	GENERAL SECRETARY
William Bowie, 1920-22	Louis A. Bauer, 1920-22	H. O. Wood, 1919-21
Louis A. Bauer, 1922-24	Arthur L. Day, 1922-24	W. J. Humphreys, 1921-22
H. F. Reid, 1924-26	H. S. Washington, 1924-26	William Bowie, 1922-25
H. S. Washington, 1926-29	*G. W. Littlehales*, 1926-29	*J. A. Fleming*, 1925-28

OFFICERS OF THE SECTIONS[1]

SECTION	CHAIRMAN	VICE-CHAIRMAN	SECRETARY
(*a*) Geodesy	William Bowie, 1920-22	J. F. Hayford, 1920-22	H. O. Wood, 1920-22
	J. F. Hayford, 1922-24	R. L. Faris, 1922-24	William Bowie, 1922-25
	R. L. Faris, 1924-26	J. F. Hayford, 1924-25	*W. D. Lambert*, 1925-28
	William Bowie, 1926-29	William Bowie, 1925-26	
		F. E. Wright, 1926-29	
(*b*) Seismology	H. F. Reid, 1920-22	J. C. Branner, 1920-22	H. O. Wood, 1920-22
	W. J. Humphreys, 1922-24	J. B. Woodworth, 1922-24	*D. L. Hazard*, 1922-28
	J. B. Woodworth, 1924-26	L. H. Adams, 1924-26	
	L. H. Adams, 1926-29	*N. H. Heck*, 1926-29	
(*c*) Meteorology	C. F. Marvin, 1920-22	W. J. Humphreys, 1920-22	*A. J. Henry*, 1920-29
	E. H. Bowie, 1922-24	R. DeC. Ward, 1922-24	
	C. G. Abbot, 1924-26	H. H. Kimball, 1924-26	
	H. H. Kimball, 1926-29	*G. W. Littlehales*, 1926-29	
(*d*) Terrestrial Magnetism and Electricity	Louis A. Bauer, 1920-22	W. F. G. Swann, 1920-22	*J. A. Fleming*, 1920-29
	W. F. G. Swann, 1922-24	Louis A. Bauer, 1922-24	
	Louis A. Bauer, 1924-26	N. H. Heck, 1924-26	
	N. H. Heck, 1926-29	*J. H. Dellinger*, 1926-29	
(*e*) Oceanography	G. W. Littlehales, 1920-22	A. G. Mayor, 1920-22	J. T. Watkins, 1920
	J. P. Ault, 1922-24	G. W. Littlehales, 1922-24	W. E. Parker, 1920-23
	N. H. Heck, 1924-26	W. J. Peters, 1924-26	G. T. Rude, 1923-26
	T. W. Vaughan, 1926-29	*G. T. Rude*, 1926-29	*A. H. Clark*, 1926-29
(*f*) Volcanology	H. S. Washington, 1920-22	R. A. Daly, 1920-22	H. O. Wood, 1920-21
	L. H. Adams, 1922-24	T. A. Jaggar, Jr., 1922-24	*R. B. Sosman*, 1921-29
	T. A. Jaggar, Jr., 1924-29	N. L. Bowen, 1924-26	
		F. E. Wright, 1926-29	
(*g*) Geophysical Chemistry[2]	Arthur L. Day, 1920-22	F. W. Clarke, 1920-22	R. B. Sosman, 1920-24
	H. S. Washington, 1922-24	Whitman Cross, 1922-24	

DATES OF ANNUAL MEETINGS OF UNION AND OF ITS SECTIONS

First, April 23, 1920
Second, April 18-20, 1921
Third, March 6, 8, 1922
Fourth, April 17-19, 1923
Fifth, May 1-3, 1924
Sixth, April 30, May 1, 1925
Seventh, April 29, 30, 1926
Eighth, April 28, 29, 1927

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[1] The names of the officers for the eighth annual meeting are indicated by italics.

[2] Discontinued May 31, 1924, as International Union failed to provide for such a Section.

DECEASED MEMBERS

J. C. Branner, died March 1, 1922
A. G. Mayor, died June 24, 1922
R. S. Woodward, died June 29, 1924
J. F. Hayford, died March 10, 1925
J. B. Woodworth, died August 5, 1925

Publications of the National Research Council

Bulletin Series

Volume 1

Number 1. The national importance of scientific and industrial research. By George Ellery Hale and others. October, 1919. Pages 43. Price $0.50.

Number 2. Research laboratories in industrial establishments of the United States of America. Compiled by Alfred D. Flinn. March, 1920. Pages 85. Price $1.00. [Out of print. See Number 16.]

Number 3. Periodical bibliographies and abstracts for the scientific and technological journals of the world. Compiled by Ruth Cobb. June, 1920. Pages 24. Price $0.40.

Number 4. North American forest research. Compiled by the Committee on American Forest Research, Society of American Foresters. August, 1920. Pages 146. Price $2.00.

Number 5. The quantum theory. By Edwin Plimpton Adams. October, 1920. Pages 81. Price $1.00. [Out of print. See number 39.]

Number 6. Data relating to X-ray spectra. By William Duane. November, 1920. Pages 26. Price $0.50.

Number 7. Intensity of emission of X-rays and their reflection from crystals. By Bergen Davis. Problems of X-ray emission. By David L. Webster. December, 1920. Pages 47. Price $0.60.

Number 8. Intellectual and educational status of the medical profession as represented in the United States Army. By Margaret V. Cobb and Robert M. Yerkes. February, 1921. Pages 76. Price $1.00.

Volume 2

Number 9. Funds available in 1920 in the United States of America for the encouragement of scientific research. Compiled by Callie Hull. March, 1921. Pages 81. Price $1.00.

Number 10. Report on photo-electricity, including ionizing and radiating potentials and related effects. By Arthur Llewelyn Hughes. April, 1921. Pages 87. Price $1.00.

Number 11. The scale of the universe. Part I by Harlow Shapley. Part II by Heber D. Curtis. May, 1921. Pages 47. Price $0.60.

Number 12. Cooperative experiments upon the protein requirements for the growth of cattle. First report of the Subcommittee on Protein Metabolism in Animal Feeding. By Henry Prentiss Armsby, Chairman. June, 1921. Pages 70. Price $1.00.

Number 13. The research activities of departments of the State government of California in relation to the movement for reorganization. By James R. Douglas. June, 1921. Pages 46. Price $0.60.

Number 14. A general survey of the present status of the atomic structure problem. Report of the Committee on Atomic Structure of the National Research Council. By David L. Webster and Leigh Page. July, 1921. Pages 61. Price $0.75.

Number 15. A list of seismologic stations of the world. Compiled by Harry O. Wood. July, 1921. Pages 142. Price $2.00.

Volume 3

Number 16. Research laboratories in industrial establishments of the United States, including consulting research laboratories. Originally compiled by Alfred D. Flinn; revised and enlarged by Ruth Cobb. December, 1921. Pages 135. Price $2.00.

Number 17. Scientific papers presented before the American Geophysical Union at its second annual meeting. March, 1922. Pages 108. Price $1.50.

Number 18. Theories of magnetism. By members of the Committee on Theories of Magnetism of the National Research Council. A. P. Wills, S. J. Barnett, L. R. Ingersoll, J. Kunz, S. L. Quimby, E. M. Terry, S. R. Williams. August, 1922. Pages 261. Price $3.00.

Volume 4

Number 19. Celestial mechanics. Report of the Committee on Celestial Mechanics of the National Research Council. E. W. Brown, G. D. Birkoff, A. O. Leuschner, H. N. Russell. September, 1922. Pages 22. Price $0.40.

Number 20. Secondary radiations produced by X-rays, and some of their applications to physical problems. Arthur H. Compton. October, 1922. Pages 56. Price $1.00.

Number 21. Highway research in the United States. Results of census by Advisory Board on Highway Research, Division of Engineering, National Research Council, in cooperation with the Bureau of Public Roads, United States Department of Agriculture. William Kendrick Hatt. October, 1922. Pages 102. Price $1.50.

Number 22. Mechanical aids for the classification of American investigators, with illustrations in the field of psychology. Harold C. Bingham. November, 1922. Pages 50. Price $0.75.

Number 23. Certain problems in acoustics. Compiled by the Committee on Acoustics, National Research Council. November, 1922. Pages 31. Price $0.50.

Number 24. Electrodynamics of moving media. Report of the National Research Council Committee on Electrodynamics of Moving Media. W. F. G. Swann, John T. Tate, H. Bateman, and E. H. Kennard. December, 1922. Pages 172. Price $1.50.

Number 25. Celestial mechanics. A survey of the status of the determination of the general perturbations of the minor planets. Appendix to the report of the Committee on Celestial Mechanics, National Research Council. A. O. Leuschner. December, 1922. Pages 73. Price $1.00.

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BULLETIN
OF THE
NATIONAL RESEARCH COUNCIL

Number 62

ALGEBRAIC NUMBERS—II

Report of the Committee on Algebraic Numbers

„National research council,

H. S. VANDIVER, *Chairman;* G. E. WAHLIN

PUBLISHED BY THE NATIONAL RESEARCH COUNCIL
OF
THE NATIONAL ACADEMY OF SCIENCES
WASHINGTON, D. C.
1928

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BULLETIN

OF THE

NATIONAL RESEARCH COUNCIL

February, 1928 Number 62

ALGEBRAIC NUMBERS—II

Report of the Committee on Algebraic Numbers[1]

H. S. VANDIVER, *Chairman;* G. E. WAHLIN

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[1] The members of this committee of the Division of Physical Sciences are: H. S. Vandiver, *Chairman,* University of Texas, Austin, Texas; R. Fueter, University of Zürich, Zürich, Switzerland; G. E. Wahlin, University of Missouri, Columbia, Missouri.

PREFACE

The National Research Council Committee on Algebraic Numbers published in February, 1923 a report[1] on algebraic numbers which aimed to cover all the literature on the theory of algebraic numbers after 1895. The committee now plans to publish from time to time reports on various topics in this subject designed to cover all the literature on those topics which has appeared since the beginning of their history. The present volume is the first of these reports. It is hoped that, ultimately, all the published chapters will be combined, after revision, into two or three volumes.

The present work is not intended as an introduction to the subjects treated. It was planned primarily for investigators who have had considerable familiarity with the theory of algebraic numbers. Naturally, much use has been made of the previously mentioned bulletin on Algebraic Numbers as well as of Dickson's History of the Theory of Numbers.

The notations of the authors quoted herein are employed in every case except when they are confusing or very cumbrous.

The committee is indebted to Professor L. J. Mordell, who has read critically all the proof sheets and through whose suggestions many obscurities have been removed.

H. S. VANDIVER.

October, 1927

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[1] Algebraic Numbers, National Research Council Bulletin No. 28.

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CHAPTER I

THE CLASS NUMBER IN AN ALGEBRAIC NUMBER FIELD

INTRODUCTION

In attempting to discover a proof of Fermat's last theorem Kummer enlarged the domain of rationality by the adjunction of roots of unity to the domain of rational numbers. In the field formed by the adjunction of a primitive cube root of unity to the rational numbers, an arithmetic similar to that of the rational numbers is possible and, due to this fact, by this method it is possible to show that the equation

$$x^3+y^3=z^3$$

has no solution in rational integers x, y, z. It therefore seems reasonable to expect that, by the adjunction of $e^{\frac{2\pi i}{l}}$, it should be possible to prove that

$$x^l+y^l=z^l$$

has no solution in rational integers. The problem is, however, much more complex in the general case and has not been settled except under certain restrictions on the exponent l.

Most of the results found by Kummer are considered in the report by HILBERT[1] and the rest will be considered later in this report in connection with the more recent discoveries.

A brief exposition of the main points in the connection between the theory of algebraic numbers and Fermat's theorem is given by MORDELL.[2]

The fundamental theorem in the theory of rational numbers states that any rational integer can be resolved into the product of prime factors in one and only one way. Kummer discovered that this theorem is, in general, not true in the cyclotomic number fields. (Cf. Hensel[40] of Chapter II.) This led him to the introduction of ideal numbers and by means of prime ideal numbers he was able to establish unique factorization in the enlarged number fields.

These ideas of Kummer were later extended by Kronecker and by Dedekind who defined ideals and by means of them established unique factorization in any algebraic number field.

---

[1] Jahresber. Deut. Math. Ver., 4: 175-546 (1897). French translation by Levy and Got.

[2] Three Lectures on FERMAT'S Last Theorem. Cambridge Univ. Press, 1921.

After having defined an algebraic integer as a root of an algebraic equation with rational integral coefficients and that of the highest power of the unknown unity, let us consider the integers of a number field $K(\alpha)$. It may happen that the greatest common divisor of two integers β and γ of the field is an integer of the field or it may be an algebraic integer which does not belong to the field. In either case the totality of all integers of the field which are divisible by this greatest common divisor is an ideal of the field according to the Dedekind definition of an ideal. In the case when the greatest common divisor is an integer of the field under consideration, the ideal is called a principal ideal of the field.

When all the ideals of a field are principal ideals it is possible to develop the arithmetic of the field in a manner similar to that of the rational numbers. An example of such a field is the field of the cube roots of unity mentioned above.

In the general case many problems reduce to the consideration of the question whether or not certain ideals are principal ideals.

1. THE EQUIVALENCE AND CLASSIFICATION OF IDEALS

For any given ideal $\mathfrak{i}$ in an algebraic number field $K(\alpha)$ there exist in the field ideals $\mathfrak{j}$ such that each product $\mathfrak{i} \cdot \mathfrak{j}$ is a principal ideal. Two ideals $\mathfrak{i}_1$ and $\mathfrak{i}_2$ are equivalent when for the same ideal $\mathfrak{j}$ the products $\mathfrak{i}_1 \cdot \mathfrak{j}$ and $\mathfrak{i}_2 \cdot \mathfrak{j}$ are principal ideals. This equivalence shall be expressed by

$$\mathfrak{i}_1 \sim \mathfrak{i}_2.$$

The equivalence of ideals has the following properties:

(1) $\mathfrak{i} \sim \mathfrak{i}$,

(2) From $\mathfrak{i}_1 \sim \mathfrak{i}_2$ follows $\mathfrak{i}_2 \sim \mathfrak{i}_1$,

(3) From $\mathfrak{i}_1 \sim \mathfrak{i}_2$ and $\mathfrak{i}_2 \sim \mathfrak{i}_3$ follows $\mathfrak{i}_1 \sim \mathfrak{i}_3$,

(4) From $\mathfrak{i}_1 \sim \mathfrak{i}_2$ and $\mathfrak{i}_3 \sim \mathfrak{i}_4$ follows $\mathfrak{i}_1\mathfrak{i}_3 \sim \mathfrak{i}_2\mathfrak{i}_4$.

The ideals are then separated into classes as follows. All ideals equivalent to a given ideal $\mathfrak{i}$ are put in the same class and constitute the class. If $\mathfrak{i}_1$ belongs to a class C_1 and $\mathfrak{i}_2$ belongs to a class C_2 then $\mathfrak{i}_1 \cdot \mathfrak{i}_2$ belongs to a class C_3 which contains all the products of an ideal of C_1 and an ideal of C_2. We then define the multiplication of classes by saying that the class C_3 is the product of the classes C_1 and C_2.

With multiplication so defined, the classes of ideals constitute an abelian group with the class containing the principal ideals, known as the principal class, as the unit element of the group.

The totality of all ideals of a field do not constitute a group. If, however, we consider fractional ideals[3] the ideals do constitute an abelian group with the principal ideals as a sub-group. In this case the group of classes is the quotient group, and the class number, which in all cases is the number of classes, is the index of the group of principal ideals under the group of all ideals.

In each class of ideals there exists an integral ideal whose norm does not exceed $|\sqrt{d}|$, where d is the discriminant of the field. From this fact it follows that the class number is finite in any algebraic number field. This class number will be denoted by h.

The above theorem together with other theorems due to Minkowski furnish a method for the computation of the number of classes in any given field.

The determination of the general expression for the class number of an algebraic number field is one of the most interesting and also one of the most difficult problems in the theory of algebraic numbers. The following pages contain a report of the work which has been done on this problem.

Sometimes variations of the problem are considered. One such variation is that in which restricted equivalence is considered. Two ideals i_1 and i_2 are equivalent in the restricted sense when and only when an ideal j exists such that $i_1 j = (\beta)$ and $i_2 j = (\gamma)$ are principal ideals and $N(\beta)$ and $N(\gamma)$ have the same sign.

Other variations of the problem are the considerations of the classification of the ideals of a ring in an algebraic number field and such problems of equivalence as are considered by Weber and by Landau in articles mentioned in a later part of this report.

2. ON THE DENSITY OF IDEALS IN A CLASS AND IN A FIELD

The investigations of DIRICHLET,[4] which were extended to a general algebraic number field by Dedekind, show that we can speak of the density of the ideals of a certain class and that this density is the same for all classes of a given field. This fact is expressed by the following well-known theorem:

Let $K(\alpha)$ be any algebraic number field and C any class of ideals in $K(\alpha)$. Furthermore, denote by T the number of ideals in C whose norms do not exceed t, then

$$\lim_{t \to \infty} \frac{T}{t} = \kappa$$

---

[3] HECKE; Vorlesungen über die Theorie der algebraischen Zahlen. Leipzig, Akad. Verlagsges., 1923.

[4] J. reine angew. Math., 19: 324-369 (1839); Werke, I, 411-461.

where κ is independent of the class C and only dependent on the field $K(\alpha)$. The actual expression for κ is

$$\kappa = \frac{2^{r_1+r_2}\pi^{r_2}R}{w|\sqrt{d}|}$$

where r_1 is the number of real, and r_2 the number of pairs of conjugate imaginary fields among $K(\alpha)$ and its conjugates, R is the regulator, d the discriminant and w the number of roots of unity in $K(\alpha)$. From the form of κ it is easy to see that it is independent of the class C and only dependent on the field under consideration.

If we next let T denote the number of ideals of the field whose norms do not exceed t, we see that the last stated theorem yields the following:

$$\operatorname*{Lim}_{t\to\infty}\frac{T}{t} = h\cdot\kappa.$$

The general problem of the determination of the class number for any field has thus been reduced to the evaluation of

$$\operatorname*{Lim}_{t\to\infty}\frac{T}{t}$$

for all ideals of the field.

The evaluation of this limit leads to the well-known Dedekind zeta function which is defined by the following equation

$$\zeta_K(s) = \sum_{\mathfrak{a}}\frac{1}{N(\mathfrak{a})^s}$$

where the summation extends over all ideals $\mathfrak{a} \neq 0$ of the field. The function $\zeta_K(s)$ is a continuous function of s for all values of the real part of s greater than 1 and its relation to the class number problem is expressed by the equation

$$\operatorname*{Lim}_{s\to 1}(s-1)\zeta_K(s) = h\cdot\kappa.$$

By a ring in an algebraic number field we understand a system of integers of the field which includes the unit element and has the further property that the sum, difference, and product of any two integers in the system is an integer in the system. The classification of ideals in a ring is effected in a manner similar to that used for the field, and similar expressions for the class number relations may be developed. A treatment of this problem and the development of the relations existing between the class number of a field and that of a ring in the field is given by BACHMANN.[1]

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[1] Allgemeine Arithmetik der Zahlenkörper. Leipzig, Teubner, 1905. Chap. 9.

The details of the development of the expression for the class number in a general algebraic number field in terms of the zeta function are given by DEDEKIND,[6] WEBER,[7] Hilbert,[1] Bachmann,[5] LANDAU,[8] Hecke,[3] and FUETER.[9]

In this connection may also be mentioned the theorems on the finiteness of the class number due to Minkowski.

MINKOWSKI[10] shows that in a general algebraic number field having the discriminant d, each class of ideals contains at least one ideal whose norm is less than $|\sqrt{d}|$. The proof of this theorem is included in several of the treatises on the theory of algebraic numbers.

MINKOWSKI[11] has proved the theorem that each class of ideals in a general algebraic number field contains at least one ideal whose norm is less than

$$\frac{\Gamma\left(1+\frac{n}{2}\right)}{\Gamma(\frac{1}{2})^n} \cdot \frac{2^n}{n^n} |\sqrt{d}|,$$

where n is the degree and d the discriminant of the field.

Minkowski[11] has also proved that if in an algebraic number field of degree n and discriminant d, the irreducible equation which one of its generating numbers satisfies has s pairs of imaginary roots, then in each class of ideals there is at least one ideal whose norm is less than

$$\left(\frac{4}{\pi}\right)^s \frac{n!}{n^n} |\sqrt{d}|.$$

3. THE GENERAL ZETA FUNCTION

The reduction of the expression for the class number to a finite form is dependent on the specific laws governing the factorization of the rational primes in the field considered and hence has been accomplished only in certain special cases. LANDAU[12] has studied the zeta function for a general algebraic number field and obtained a reduction in the expression for the class number, which, though it is not in finite form, is of interest in that it appears as a generalization of the corresponding expressions in the cases of the more special fields in which the problem has been studied.

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[6] DIRICHLET; Vorlesungen über Zahlentheorie, Braunschweig, 1874.

[7] Algebra, 2nd Ed. Bd. 2. 1899.

[8] Algebraische Zahlen und der Ideale; Leipzig, Teubner, 1918.

[9] Synthetische Zahlentheorie; Leipzig, Teubner, 1925.

[10] J. reine angew. Math., 107: 295 (1891); Werke.

[11] Diophantische Approximationen, Leipzig, 1907.

[12] J. reine angew. Math., 127: 162-174 (1904).

Let $\zeta(s)$ be the Riemann zeta function. Then from the fact that

$$\lim_{s\to 1}(s-1)\zeta(s)=1$$

and the relation

$$\lim_{s\to 1}(s-1)\zeta_k(s)=h\cdot\kappa$$

it follows that

$$\lim_{s\to 1}\frac{\zeta_k(s)}{\zeta(s)}=h\cdot\kappa.$$

The quotient $\zeta_k(s)/\zeta(s)$ for $s>1$ can be developed in a Dirichlet series

$$\sum_{n=1}^{\infty}\frac{c_n}{n^s}$$

corresponding to the series

$$\sum_{n=1}^{\infty}\frac{\left(\frac{d}{n}\right)}{n^s}$$

in the theory of the quadratic number-fields. This series $\sum\frac{c_n}{n^s}$ is unique.

The possibility of such a representation follows from the relation

$$\zeta_k(s)=\sum_{n=1}^{\infty}\frac{F(n)}{n^s}$$

where $F(n)$ is the number of distinct ideals in $K(a)$ whose norms are n; and from

$$\frac{1}{\zeta(s)}=\sum_{v=1}^{\infty}\frac{\mu(v)}{v^s}$$

where $\mu(v)$ is the Möbius function. By multiplication this gives

$$\frac{\zeta_k(s)}{\zeta(s)}=\sum_{n=1}^{\infty}\sum_{v=1}^{\infty}\frac{F(n)\mu(v)}{(nv)^s}$$

$$=\sum_{n=1}^{\infty}\frac{\sum_{v/n}\mu(v)F\left(\frac{n}{v}\right)}{n^s}$$

where $\sum_{v/n}$ denotes a summation extending over all divisors v of n. Hence we have a Dirichlet series $\sum\frac{c_n}{n^s}$ in which

$$c_n=\sum_{v/n}\mu(v)F\left(\frac{n}{v}\right).$$

Since a Dirichlet series converging for $s=s_0$ is continuous for $s>s_0$ it follows that if the series

$$\sum_{n=1}^{\infty} \frac{\sum_{v/n} \mu(v) F\left(\frac{n}{v}\right)}{n^s}$$

converges for $s \rightarrow 1$ then

$$h \cdot \kappa = \sum_{n=1}^{\infty} \frac{\sum_{v/n} \mu(v) F\left(\frac{n}{v}\right)}{n}. \tag{1}$$

Landau[13] shows that this convergence condition is fulfilled and hence the equation (1) gives an expression for the class number in terms of an infinite series.

4. QUADRATIC FIELDS

Dedekind[6] uses the quadratic number field as an example for the illustration of his general theory and develops an expression for the class number of such fields. A finite expression for the class number for quadratic fields is given by Hilbert.[1]

SOMMER[13] gives a proof of the theorem: If the discriminant of a quadratic field contains only one prime factor, the class number is odd.

If we consider an imaginary quadratic field $k(\sqrt{-m})$ having the discriminant d and the class number $h(d)$, LANDAU[14] has shown that if the inequality

$$h(d) < d^{\frac{1}{2}-\delta}$$

has an infinite number of solutions for some positive δ, then, for each positive w, after a certain number of fields, the discriminants d_1 and d_2 of any two successive fields satisfy the inequality

$$|d_2| > |d_1|^w.$$

In the proof of this fact he uses, besides the series

$$L_1(s) = \sum_{n=1}^{\infty} \frac{X_1(n)}{n^s}; \qquad X_1(n) = \left(\frac{d_1}{n}\right);$$

and

$$L_2(s) = \sum_{n=1}^{\infty} \frac{X_2(n)}{n^s}; \qquad X_2(n) = \left(\frac{d_2}{n}\right);$$

also

$$L_3(s) = \sum_{n=1}^{\infty} \frac{X_1(n) X_2(n)}{n^s}.$$

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[13] Vorlesungen über Zahlentheorie, Leipzig u. Berlin, 1907.

[14] Göttingen Nachr., Math. Phys. Kl., 1918: 277-284.

LANDAU[15] has also shown that when for an imaginary quadratic field $k(\sqrt{-m})$ the function $\zeta_k(s)$ has no zeros in the region $\frac{1}{2}<s<1$ then the class number for the field is at least $|\sqrt{d}|/\log|d|$, and from this he deduces the fact that for each positive w there exists a c such that, from a certain point on, any two successive solutions d_1 and d_2 of

$$h(d)<c\frac{|\sqrt{d}|}{\log|d|}$$

will also satisfy the inequality

$$|d_2|>|d_1|^w.$$

LANDAU[16] has also developed expressions for the upper and lower bounds for the class number of an imaginary quadratic field. Let $s=\sigma+\tau i$ be the variable in $\zeta(s)$.

I. If $\zeta(s)\neq 0$ for $\sigma>\frac{1}{2}$ or also only for $|s-1|\leqq\theta$ where θ is free from m, then for $m>e^e$ there exist positive constants b_3 and b_4, dependent only on θ, such that

$$b_3\frac{\sqrt{m}}{\log\log m\cdot\log\log\log m}<h<b_4\sqrt{m}\log\log m\cdot\log\log\log m.$$

II. If $L(s)\neq 0$ for $\frac{1}{2}\leqq\sigma<1$, $|\tau|<3$ then

$$h<e^{a_2\log^{a_3}m}$$

where a_2 and a_3 are positive absolute constants and $a_3<1$.

For a method of determining the class number of a quadratic field without the use of the zeta function, see Hecke.[8]

5. CUBIC FIELDS

DEDEKIND[17] has made an exhaustive study of the cubic fields defined by a binomial equation. He determines the laws of factorization of the rational primes in such a field and applies them in making the necessary reductions in the zeta function.

Any cubic field defined by a binomial equation can be considered as defined by an equation of the form

$$x^3-ab^2=0$$

where a and b are relatively prime and neither is divisible by the square of a rational prime. Two types of fields present themselves for consideration. I, in which $a^2-b^2\not\equiv 0 \mod 9$; and II, in which $a^2-b^2\equiv 0 \mod 9$.

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[15] Göttingen Nachr., Math.-Physik. Kl., 1918: 285-295.

[16] Math. Annalen, 79: 388-401 (1918).

[17] J. reine angew. Math., 121: 40-121 (1900).

If k be used to denote $3ab$ for fields of type I and ab for fields of type II, then the field discriminant is $3k^2$ and the class invariant

$$\kappa = \frac{2\pi \log \epsilon}{k\sqrt{3}}$$

where ϵ is the fundamental unit. For the study of the zeta function for these fields Dedekind uses the expression of the function as an infinite product

$$\zeta(s) = \prod_{\mathfrak{p}} \frac{1}{1 - \frac{1}{N(\mathfrak{p})^s}}$$

the product extending over all prime ideals of the field.

For the determination of the laws of factorization and their application, a consideration of the quadratic fields defined by a primitive cube root of unity becomes necessary. This leads to the definition of the character of the prime ideals $\psi(\mathfrak{p})$ as follows:

(a) $\psi(\mathfrak{p}) = 0$ if $\mathfrak{p}$ is a factor of k,
(b) $\psi(\mathfrak{p}) = 1$ if $\mathfrak{p}$ is a factor of 3 but not of k,

and for all other prime ideals

(c) $$\psi(\mathfrak{p}) = \left(\frac{ab^2}{\mathfrak{p}}\right)_3$$

where $\left(\frac{ab^2}{\mathfrak{p}}\right)_3$ is a cube root of unity and is the cubic residue character of ab^2 with respect to the modulus $\mathfrak{p}$.

Let $F(p)$ denote that part of the infinite product for which the prime ideals are factors of the rational prime p. We may then write

$$\zeta(s) = \prod_p F(p),$$

the product extending over all rational primes. Moreover, using the character $\psi(\mathfrak{p})$ defined, it may be shown that

$$F(p) = \frac{1}{1 - \frac{1}{p^s}} \cdot \prod_{\mathfrak{p}} \frac{1}{1 - \frac{\psi(\mathfrak{p})}{N(\mathfrak{p})^s}}$$

the product now extending over all ideal prime factors of p.

If v is an algebraic integer of the field of the cube roots of unity and x any rational integer, then all algebraic integers of this field which for some x are congruent to xv with respect to the modulus k constitute a modul k_v. These moduls are of three types according as in $\psi(v) = \rho^i$, the exponent i, is 0, 1, or 2. We can speak of these as types 0, 1, and 2. For each modul select a base (a_j, β_j), the subscript being used to denote the type, and put $w_j = \frac{a_j}{\beta_j}$ and denote its conjugate by w'_j.

Further

$$\eta(w_j) = e^{\frac{\pi i w_j}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i w_j n})$$

and

$$H(w_j) = \eta(w_j)\eta(-w'_j)\sqrt{i(w'_j - w_j)};$$

and then

$$\epsilon^h = \frac{\Pi H(w_1)}{\Pi H(w_0)}$$

where the two products extend over all moduls of the types indicated by the subscripts.

By using Minkowski's[17] theorem that in every class of ideals there exists an ideal j satisfying the condition

$$N(j) < \left(\frac{4}{\pi}\right)^r \frac{n!}{n^n} |\sqrt{d}| \tag{2}$$

where n is the degree of the field and r the number of imaginaries among the n conjugate fields, Reid[18] discusses the computation of the class number for a general cubic field. He determines the least exponent t such that j^t is a principal ideal. If C is the class containing j, he forms the classes C, C^2, C^3, . . . C^t and determines whether or not all ideals satisfying the condition (2) belong to the classes thus found. If such is not the case, he forms new classes in the same way until all have been determined. The article concludes with two tables. Table I lists 172 cubic fields defined by equations of the form $x^3 + ax + b = 0$ and Table II, all fields defined by equations of the form $A_0x^3 + A_1x^2 + A_2x + A_3 = 0$, all coefficients having absolute values less than 3. For each field listed he gives the discriminant of the equation, the discriminant of the field, the class number, a base, units and the factors of some rational primes.

Westlund[19] considers the cubic sub-fields of the cyclotomic fields $K(e^{\frac{2\pi i}{p}})$ defined by the $\frac{p-1}{3}$ term periods, where p is a prime congruent to 1 mod. 3. The equation for the three periods is of the form[20]

$$x^3 + x^2 - \frac{p-1}{3}x - \frac{3p-1+pA}{27} = 0$$

$$A^2 + 27B^2 = 4p.$$

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[18] Am. J. Math., 23: 68-83 (1909). Göttingen Dissertation.

[19] Jahresber. Deut. Math. Ver., 22: 135-140 (1913).

[20] Weber's Algebra, 2nd Ed., Bd. 1: 629.

A is defined by

$$\sum_{t=1}^{p-2} \rho^{\operatorname{ind}(t^2+t)} = \frac{A+b\sqrt{-3}}{2}, \qquad \rho = \frac{-1+\sqrt{-3}}{2}.$$

He gives the laws of factorization for these fields and at the end gives a table of all primes which enter into the determination of the class number for all p from 7 to 97.

Reid[18] and Westlund[19] do not make use of the general formula or the zeta function.

6. Cyclotomic Fields

A large part of the work connected with the class number has been devoted to the cyclotomic fields. Hilbert's report[1] contains an account of the work up to 1895. Attention may here be called to an error noted by Mitchell[28] in the finite expression for the class number as given by Hilbert.[1]

The class number problem arose in the early investigations of the arithmetic theory of the cyclotomic fields. Kummer,[21, 22] following suggestions taken from the methods of Dirichlet in his investigation of quadratic forms, develops the general expression for the class number of the cyclotomic fields generated by the lth root of unity, l being a prime. The same problem is also solved in Hilbert,[1] Dedekind,[6] Weber,[7] Fueter,[8] and Got.[23]

The general expression for the class number of the field of the primitive nth roots of unity (n composite) was found by Kummer.[24]

If ξ is a primitive nth root of unity, and $f(\xi)$ a polynomial such that $f(\xi^\kappa) = f(\xi)$ for some κ prime to n, then $f(\xi)$ generates a subfield of $K(\xi)$. Fuchs[25] has made a study of these subfields and computed an expression for their class number.

The class number of subfields of the fields of the roots of unity has also been studied by Weber.[7] Weber[7, 26] contains the investigation of the 2^mth roots of unity and shows that in this case the class number is odd.

The works of Kummer on the class number are reported in H. J. S. Smith.[27]

[21] J. reine angew. Math., 40: 93-116 (1850).

[22] J. de Math., 16: 377-488 (1850).

[23] Annales de la Faculté de Toulouse, Sec. 3, 3: 17-20 (1911).

[24] Monatsberichte, Berlin, 1863: 21-28.

[25] J. reine angew. Math., 65: 74-111 (1866).

[26] Acta Math., 8: 193-263 (1886).

[27] Report on the Theory of Numbers, part II, § 50. Collected Mathematical Papers. 1: 112-114. Report of the British Ass'n, Part II, 1860: 134-136.

[28] Bull. Nat. Research Council, No. 28: 17 (1923).

The following theorem is due to Kummer.[21] If K_1 and K_2 are two subfields of the field of the primitive lth roots of unity, l a prime, and K_1 is a subfield of K_2 then the class number of K_1 is a factor of the class number of K_2. Hilbert[1] has called attention to the fact that there is an error in Kummer's proof of this theorem.

FURTWÄNGLER[29] gives a proof of the following theorem which is easily seen to include that of Kummer[21] as a special case.

If K_1 and K_2 are two subfields of the field of the n^{th} roots of unity, where n is a power of a single prime, and if K_1 is a subfield of K_2 then the class number of K_1 is a divisor of the class number of K_2.

In the proof two cases are considered. First, let p be a rational prime which is not a factor of the relative degree of K_2 with respect to K_1. It is then seen that p must occur in the class number of K_2 with an exponent at least as great as its exponent in the class number of K_1.

Next, if p is a divisor of the relative degree it is seen that a reduction in the power of p in the class number in passing from K_1 to K_2 is possible only when between K_1 and K_2 there are two fields K'_1 and K'_2 such that K'_2 is relative cyclic with respect to K'_1 and has the relative discriminant ± 1. This is seen to be impossible since, by a known theorem regarding the factorization of prime ideals in cyclotomic fields, the relative discriminant contains at least one prime.

By an example it is shown that the theorem is not in general true when m is not the power of a single prime.

WESTLUND[30] has investigated the relation between the class numbers of $K(r)$, $(r=e^{\frac{2\pi i}{p^n}})$, and $K(r')$, $(r'=e^{\frac{2\pi i}{p^{n-1}}})$, p being an odd rational prime. The method used is similar to that of Weber[7] for $p=2$. Denote by h and h' the class numbers of $K(r)$ and $K(r')$ respectively and let h_2 be the class number of $K(r+r^{-1})$ and h'_2 that of $K(r'+r'^{-1})$. We may then write $h=h_1h_2$ and $h'=h'_1h'_2$, $h_1=h'_1A$, $h_2=h'_2B$ and $h=h'H$. The numbers h_1 and h_2 are known as the first and second factors of the class number of $K(r)$ and h'_1 and h'_2 as the first and second factors of the class number of $K(r')$.

Westlund (*l. c.*) develops expressions for A and B following the method used by Weber and shows that A is an integer and that B has the form $p^{-\sigma}I$ where I is an integer dependent on the regulator of a fundamental system of normal units. A unit in $K(r+r^{-1})$ which is not ± 1 and whose norm relative to $K(r'+r'^{-1})$ is ± 1 is a normal unit.

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[29] J. reine angew. Math., 134: 91-94 (1908).

[30] Trans. Am. Math. Soc., 4: 201-212 (1903).

Whether or not A and I are divisible by p and what is the value of σ remains unsettled. Cf. Vandiver.[41]

Weber[7, 30] as already noted, has proved that the class number of the field of the 2^mth roots of unity is odd. This has been generalized by FURTWÄNGLER[31] who proves that if l is an arbitrary prime number, then the class number of the field of the l^nth roots of unity is divisible by l when and only when l is a factor of the class number of the field of the l^{th} roots of unity. Cf. Furtwängler[40] of Chapter II.

If the class number of the field of the lth roots of unity is divisible by l then the first factor h_1 of the class number is divisible by l, but the second factor h_2 is not necessarily divisible by l. The only cases of $l<100$ for which h is divisible by l are $l=37$, 59, and 67 and in none of these cases is the second factor divisible by l. Cf. KUMMER.[32]

Kummer[31, 32] shows that in the fields of the lth roots of unity, l being a prime, the first and second factors of the class number are integers. Kummer[32] contains a table of values of the first factor for all primes less than 100. Without proof Kummer[32] also gives the following asymptotic value for the first factor of the class number

$$\frac{l^{\frac{l+3}{4}}}{2^{\frac{l-3}{2}}\pi^{\frac{l-1}{2}}}.$$

The latter part of Kummer[32] is devoted to a consideration of the class number for $l=37$, 59, 67. These are the only primes less than 100 whose class number is divisible by l.

Kummer[31] also has shown that the first factor of the class number is divisible by l when l is a factor of the numerator of one of the first $\frac{l-3}{2}$ Bernoulli numbers.

KUMMER[33] gives a table of values of the first factor of the class number for all n from 2 to 100. In this it appears that the first factor is not necessarily integral when n is composite. KRONECKER[34] shows that 2 is the only number which can occur as a denominator of the first factor.

Let g be a primitive number with respect to the modulus l and denote by g_σ the least positive residue of g^σ. The cyclotomic units $e_\sigma(r)$, $\sigma=0, 1, 2\ldots, \mu-1$, $\mu=\frac{l-1}{2}$, $r=e^{\frac{2\pi i}{l}}$, are defined by the equation

$$e_\sigma(r)=\frac{r^{g_{\sigma+1}}-r^{-g_{\sigma+1}}}{r^{g_\sigma}-r^{-g_\sigma}}.$$

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[31] J. reine angew. Math., 140: 29-32 (1911).

[32] J. reine angew. Math., 40: 117 (1850).

[33] Monatsberichte, Berlin, 1861: 1051-1053.

[34] Werke, Bd. 1: 125.

Put $c_\sigma = 0$ when e_σ is positive, and $c_\sigma = 1$ when e_σ is negative, and let w be a primitive μth root of unity. Put

$$\psi(w) = c_0 + c_1 w + c_2 w^2 + \ldots + c_{\mu-1} w^{\mu-1}$$

and

$$\Psi(w) = w^\rho \sum_{i=1}^{\frac{q-1}{2}} \frac{1}{i} \left(w^{\operatorname{ind}(v_i-1)} + w^{\operatorname{ind}(v_i-8)} + \ldots \right).$$

Here q is a prime and $\rho = \operatorname{ind} q$.

$$v_i = \left| \frac{q-2i}{q} \right|$$

where the symbol $\left| \frac{a}{b} \right|$ is used to denote the least positive solution of the congruence

$$bx \equiv a \bmod l.$$

KUMMER[35] proves the following theorems:

I. For all those primes l for which the first factor of the class number of $k(e^{\frac{2\pi i}{l}})$ is not divisible by 2 all real units which, with their conjugates, are positive are the squares of units.

II. The second factor of the class number is not divisible by 2 when the first factor is prime to 2.

III. If the second factor of the class number is divisible by 2 then $\psi(w)$ and $\psi(w^{-1})$ are divisible by the same factors of 2.

IV. The odd prime q is not a factor of the class number if $\Psi(w)$ is relatively prime to q.

Using the method of KRONECKER,[36] MIRIMANOFF[37] develops a new criterion for the divisibility of the second factor of the class number of $K(e^{\frac{2\pi i}{l}})$ by l.

If we suppose that $h_1 = q_1 l^{t_1}$ and $h_2 = q_2 l^{t_2}$ and that C is any class of ideals in the field $K(r+r^{-1})$, $(r = e^{\frac{2\pi i}{l}})$ and consider the q_2th power of C expressed by means of the base of the abelian group of all q_2th powers of the classes in $K(r+r^{-1})$ then

$$C^{q_2} \sim C_1^{x_1} C_2^{x_2} \ldots C_e^{x_e} \qquad 0 \leqq x_i < l^{b_i}.$$

The classes of $K(r)$ whose ideals have relative norms with respect to $K(r+r^{-1})$ which belong to the principal class of this field constitute a group of order h_1 and if $\overline{C}$ is any class of this group

$$\overline{C}^{q_2} \backsim \overline{C}_1^{y_1} \overline{C}_2^{y_2} \cdot \ldots \cdot \overline{C}_{e_1}^{y_{e_1}} \qquad 0 \leqq y_i < l^{\overline{b}_i}.$$

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[35] Monatsbericht Kgl. Preuss. Akad., 1870: 855-880.

[36] J. reine angew. Math., 93: 1-52 (1882).

[37] J. reine angew. Math., 109: 82-88 (1892).

Hecke[38] has studied these class groups. By means of **Furtwängler's**[39] general existence proof for class fields, he shows that there are at least e independent classes $A_1, A_2, \ldots, A_e$ such that their l^{th} powers and their relative norms are in the principal class of $K(r+r^{-1})$ and no equivalence

$$A_1^{m_1}A_2^{m_2}\ldots A_e^{m_e} \sim C$$

exists where C is a class in $K(r+r^{-1})$ if $m_1, m_2, \ldots, m_e$ are not divisible by l. The group with the base $\overline{C}_1, \overline{C}_2 \ldots, \overline{C}_e$ is the largest such group and hence $e_1 \geqq e$. If $e_1=0$, $e=0$ and from this is obtained another proof of Kummer's theorem that the second factor of the class number is divisible by l only when the first factor is divisible by l.

Vandiver[40] has determined the residue of the first factor of the class number of $K(e^{\frac{2\pi i}{l}})$ with respect to the modulus l^b.

$$h_1 \equiv \frac{\prod\limits_{s=1}^{l-2} l(-1)^{(sl^b-1)/2} B_{(sl^b+1)/2}}{2^{\frac{1}{2}(l-3)}} \bmod l^b$$

where B_i is the i^{th} Bernoulli number.

If h_1 is the first factor of the class number of $K(e^{\frac{2\pi i}{l^m}})$ and h'_1 that of $K(e^{\frac{2\pi i}{l^{m-1}}})$ and if $a=\frac{h}{h'_1}$ **Vandiver**[41] shows that

$$a \equiv \Big[\prod_{i=1}^{\frac{l-3}{2}} B_i\Big]^{l^{m-2}(l-1)} \bmod l$$

and makes application to Fermat's last theorem.

Wolfskehl[42] shows that if ξ is a primitive l^{th} root of unity and $\eta_0, \eta_1, \ldots, \eta_{e-1}$ the e periods of f terms and if i_1 is any ideal of $k(\eta)$ then there exists an ideal i_2 such that $i_1 \cdot i_2$ is a principal ideal and $N(i_2) \leqq l^{\frac{l}{2}}$. From this it is then shown that in the fields of the 11th and 13th roots of unity the first factor of the class number is 1.

Minkowski[10] has shown that in the fields of the 17th and 19th roots of unity the second factor of the class number is equal to one.

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[38] Göttingen Nachr., Math. Phys. Kl., 1910: 420-424.

[39] Math. Ann., 63: 1-37 (1906).

[40] Bull. Am. Math. Soc., 25: 458-461 (1919).

[41] Proc. Natl. Acad. Sci., 6: 416-421 (1920).

[42] J. reine angew. Math., 99: 173-178 (1886).

Let p and l be two primes such that $p-1$ is divisible by l. Then the field $K(e^{\frac{2\pi i}{p}})$ contains a cyclic subfield of degree l, and its discriminant contains only one prime p. The class invariant κ for this cyclic field is

$$\frac{2^{l-1}}{p^{\frac{l-1}{2}}}\cdot R,$$

where R is the regulator.

This field is the only cyclic field of degree l having p as the only divisor of its discriminant. FUETER[43] has determined the laws of factorization of the rational primes in this field and the finite expression for the class number.

Since the field is cyclic and p is the only prime in its discriminant, p is the l^{th} power of a prime ideal and every other prime is either prime or the product of l distinct primes in the field.

Let ξ be a primitive l^{th} root of unity and g a primitive root of p. Then every integer n which is not divisible by p is congruent to $g^{\operatorname{ind} n}$. If we now set

$$\left(\frac{n}{p}\right)_l=\xi^{\operatorname{ind} n} \qquad \text{if } n\not\equiv 0 \bmod p,$$

$$\left(\frac{n}{p}\right)_l=0 \qquad \text{if } n\equiv 0 \bmod p,$$

then every rational prime q is, in the cyclic subfield, the product of l distinct primes or is itself a prime according as $\left(\frac{q}{p}\right)_l$ is or is not equal to 1.

The following theorem is a generalization of one due to Kummer.

If $f(x)=\sum\limits_{m=1}^{\infty} a_m \cos mx \qquad 0<x<2\pi$

is an absolutely convergent series, then for $n=1, 2, 3, \ldots, l-1$

$$\sum_{\sigma=1}^{p-1}\left(\frac{\sigma}{p}\right)_l^n f\left(\frac{2\pi\sigma}{p}\right)=\Lambda_n\sum_{m=1}^{\infty}a_m\left(\frac{m}{p}\right)_l^{l-n}$$

where

$$\Lambda_n=\sum_{\nu=0}^{p-2}\xi^{\nu n}Z^{g^\nu}$$

$Z=e^{\frac{2\pi i}{p}}$.

By this theorem it is found possible to evaluate

$$\lim_{s\to 1}\,(s-1)\zeta(s)$$

[43] J. reine angew. Math., 147: 174-183 (1917).

and hence determine the class number for this field. The final form of the result is

$$h = \frac{\Delta}{R}$$

where R is the regulator and

$$\Delta = (-1)^{\frac{l-1}{2}} \begin{vmatrix} lg\, e_1, & lg\, e_2, \ldots & lg\, e_{l-1} \\ lg\, e_2, & lg\, e_3, \ldots & lg\, e_l \\ \cdots & \cdots & \cdots \\ lg\, e_{l-1}, & \ldots\ldots\ldots & lg\, e_{l-3} \end{vmatrix}$$

e_i being the relative norm in the cyclic subfield of the unit

$$\frac{(Z^{g^i}-1)(Z^{-g^i}-1)}{(Z^{g^{i-1}}-1)(Z^{-g^{i-1}}-1)},$$

and $lg\, e_i = \log |e_i|$.

Hilbert[1] calls attention to the fact that KRONECKER[44] made an error in his attempt to establish some of Kummer's results by arithmetic means.

7. RELATIVE ABELIAN FIELDS

Every relative abelian field which forms an absolute Galois field with respect to an imaginary quadratic field $K(\sqrt{m})$ is obtained by the adjunction of $e^{2\pi i z}$ and $j(w)$ where z is a rational number, w a number of the imaginary quadratic field and $j(w)$ the complete invariant of the modular function.

FUETER[45, 46] has made a study of the class number for such fields.

$K(\sqrt{m})$ is an imaginary quadratic field. Let f be a rational integer whose separation into rational prime factors is given by the equation

$$f = l_1^{r_1} l_2^{r_2} \ldots l_n^{r_n}.$$

All numbers of $K(\sqrt{m})$ which satisfy the condition

$$\alpha \equiv 1 \bmod f$$

constitute a number ray whose conductor is f. A ray ideal consists of all those numbers of a field ideal, prime to f, which belong to the ray. Two ray ideals i_1 and i_2 are said to be equivalent in the restricted sense when $i_1 = \alpha i_2$ where α is a number of the ray. Two ray ideals i_1 and i_2 are said to be equivalent in the wider sense if $i_1 = \alpha i_2$ and for each power l^r of a prime factor of f there exists in $K(\sqrt{m})$ a unit e such that $e\alpha \equiv 1 \bmod l^r$. Ray ideals which are equivalent belong to the same class.

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[44] Werke, Bd. 1: 276.

[45] Göttingen Nachr. Math. Phys. Kl., 1907: 288-298.

[46] Rendiconti Circolo Matematico di Palermo, 29: 380-395 (1910).

Equivalence as above defined is spoken of as equivalence with respect to the modulus f.

All numbers of $K(\sqrt{m})$ which, with respect to the modulus f, are congruent to rational numbers constitute a ring with the conductor f.

As above, by a ring ideal is meant all the numbers of a field ideal, relatively prime to f, which belong to the ring. Two ring ideals i_1 and i_2 are equivalent or belong to the same class when $i_1 = \alpha i_2$ and α is a number of the ring.

The norms of all field ideals belonging to the same class can be represented by a quadratic form

$$F = AX^2 + BXY + CY^2$$

where X and Y vary over all rational integers.

Fueter's first problem is that of the determination of the restrictions which must be imposed on X and Y in order that the numbers represented by the quadratic form shall be the norms of the ideals of a ray class or a ring class. This problem is solved and is expressed by the theorem: Let d be the negative discriminant of $K(\sqrt{m})$ and let us consider all ideals i_1 such that $i_1 = \alpha i$,

$$\alpha \equiv \frac{a + b\sqrt{d}}{2} \bmod f, \qquad 0 < a < f, \qquad 0 \leqq b < f$$

then F gives the norms of all ideals i_1, when

$$X = \frac{a + bB}{2} + f \cdot x, \qquad Y = -Ab - f \cdot y$$

and x and y vary over all rational integers.

Each norm is represented once if $f \neq 2$ and twice if $f = 2$.

The evaluation of $\sum \frac{1}{F^s}$ where the summation extends over all values of x and y leads to a generalization of a formula due to Kronecker.

By setting $b = 0$ we obtain an expression for $\sum_i \frac{1}{N(i)}$ where the summation extends over all ideals of a ring class or a ray class.

As has already been noted above, the abelian field relative to an imaginary quadratic field consists of cyclotomic fields and fields of complex multiplication. To each such field there corresponds in $K(\sqrt{m})$ a ring or a ray according as this field is the complex multiplication field or this field together with a cyclotomic field. The conductor is obtained from the primes in the relative discriminant and the relation between the ring or ray with equivalence in the wider sense is expressed by the theorem.

For all prime ideals of the same ring class or ray class we have the same factorization law in the super field.

Having thus determined the laws of factorization the author evaluates

$$\lim_{s\to 1}(s-1)\zeta_k(s)$$

and obtains a finite expression for the number of classes.

HECKE[47] shows that if K_1 is a field which is relative cyclic of degree l with respect to K_2 and has a relative discriminant which is prime to l and if H and h are the class numbers of K_1 and K_2 respectively, then H/h can be expressed by the logarithm of a certain transcendental function. By taking K_2 as the field of rational numbers he obtains the theorem: The class number of each solvable field whose discriminant is relative prime to its degree can be expressed as an integral of the logarithm of a transcendental function similar to the expression obtained by Kummer for cyclotomic fields by means of logarithms of cyclotomic units.

In a later work HECKE[48] states the theorem: If K is a totally real field (by a totally real field is meant one which with all its conjugates is real) and δ is a totally negative number of K then $\sqrt{\delta}$ generates an imaginary relative quadratic field K_1 and if H and h are the class numbers of K_1 and K respectively then H/h can be expressed as an elementary arithmetic function of δ. The proof of the theorem is given for the case when K is a quadratic field.

The case when K is a totally real cubic field has been studied by REIDEMEISTER[49] who obtains an expression similar to that obtained by Hecke.

Let $Q(\mathfrak{p})=0$ when $\mathfrak{p}$ is a divisor of δ, and $Q(\mathfrak{p})=\left(\frac{\delta^*}{\mathfrak{p}}\right)$ where $\mathfrak{p}$ and δ are relatively prime and δ^* is δ times the square of an integer of K, $\left(\frac{\delta^*}{\mathfrak{p}}\right)$ being the quadratic residue symbol. Furthermore, if

$$\mathfrak{a}=\mathfrak{p}_1^{a_1}\cdot\mathfrak{p}_2^{a_2}\ldots\mathfrak{p}_r^{a_r}$$
$$Q(\mathfrak{a})=Q(\mathfrak{p}_1)^{a_1}\cdot Q(\mathfrak{p}_2)^{a_2}\ldots Q(\mathfrak{p}_r)^{a_r}$$

and

$$L(S,Q)=\sum_j\frac{Q(j)}{N(j)^s}$$

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[47] Verhandlungen der Naturforschendengesellschaft in Basel, 28: 365-372 (1917).

[48] Göttingen Nachr. Math: Phys. Kl., 1921: 1-23.

[49] Abh. aus d. Math. Sem. Hamburg., 1921: 27-48.

where the summation extends over all ideals of K, then

$$\frac{H}{h} = \frac{\kappa}{\kappa_1} L(1, Q)$$

where

$$\kappa_1 = \frac{2^8 \pi^3 R(K_1)}{W|\sqrt{d^2 N(\overline{\Delta})}|}$$

and

$$\kappa = \frac{4R(K)}{|\sqrt{d}|}.$$

Here W is the number of roots of unity in K_1, $R(K_1)$ and $R(K)$ the regulators of K_1 and K respectively, $N(\Delta)$ the norm of the relative discriminant of K_1, taken as an ideal in K, and d the discriminant of K.

8. The Distribution of the Prime Ideals in the Classes

One of the interesting problems connected with the classification of ideals is that of the distribution of the prime ideals in the classes. This problem has been investigated by **Landau**.[50] If an algebraic number field K has the property that the sum $\sum \frac{1}{N(\mathfrak{p})}$ extending over all prime ideals of the principal class (classification in the restricted sense) diverges, and if $P(t)$ is the number of prime ideals of a given class whose norms do not exceed t, and m a certain real number, then

$$\lim_{t \to \infty} \frac{\log^m t}{t} \left(P(t) - \frac{1}{h} \int_2^t \frac{du}{u} \right) = 0.$$

From this it follows that asymptotically the number of prime ideals in a given class is independent of the class.

The same problem is considered by **Landau**[51] in connection with the problem of the primes in an arithmetic progression. The fact expressed by the last theorem may also be expressed by the equation

$$P(t) = \frac{1}{h} \int_2^t \frac{du}{u} + O(t e^{-\sqrt[d]{\log t}})$$

where by $O[g(x)]$ is meant a function having the property that for all x greater than some ξ the quotient $|O[g(x)]/g(x)$ is limited. **Landau**[51] shows that $d > 8$. See also **Hecke**.[52]

[50] Math. Ann., 63: 145-204 (1907).
[51] Sitzungsberichte Wien, 117: IIa, 1095-1107 (1908).
[52] Göttingen Nachr. Math. Phys. Kl., 1917: 299-318.

Another theorem proved by Landau[35] is the following: If K is a field such that the sum $\sum \frac{1}{N(\mathfrak{p})}$ diverges, and $R(t)$ is the number of ideals having an odd number of prime factors, and $S(t)$ the number of ideals having an even number of prime factors, and in each case norms not exceeding t, then

$$\operatorname*{Lim}_{t \to \infty} \frac{R(t)}{S(t)} = 1.$$

If $H(t, C)$ is the number of ideals in the class C whose norms do not exceed t, then

$$H(t, C) = at + O(t^{\theta})$$

where a is a positive constant dependent only on the field, and $\theta = n/(n+2)$ where n is the degree of the field.

A system S of numbers of an algebraic number field constitutes an ideal if with α and β also $\lambda\alpha + \mu\beta$ belongs to S when λ and μ are integers of the field and when there exists a fixed number $v \neq 0$ such that the product of v by any number of S is an algebraic integer. By this definition α and β are not necessarily integers and if α and β are fractions we have a fractional ideal. Hecke[36] proves the following generalization of Dirichlet's theorem regarding the primes in an arithmetic progression.

If f is an ideal of an algebraic number field, and r an arbitrary ideal prime to f, then in the class to which $\frac{1}{r}$ belongs there is an infinite number of prime ideals $\mathfrak{p}$ such that in the principal ideal $(\xi) = \mathfrak{p} \cdot r$ ξ is congruent to a unit mod f.

A further study of the distribution of the prime ideals in the classes is given by **Landau**[53] in connection with the study of the zeta function.

For further developments see **Landau**.[54]

In a series of three articles **Weber**[55] has studied groups of numbers and ideals of a group. Two ideals i_1 and i_2 of a group O are said to be equivalent with respect to a subgroup O', if $i_1 = i' i_2$ where i' is an ideal of O'. (Same definition as "ray," cf. **Fueter**.[55a]) The class number is the index of O' under O. He shows that under certain assumptions regarding the groups each class must contain an infinite number of prime ideals.

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[53] Math. Zeitschrift, 2: 52-154 (1918).

[54] Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale. Berlin, Teubner, 1918.

[55] Math. Ann., 48: 433-473 (1896); 49: 83-100 (1897); 50: 1-26 (1897).

[55a] Jahresber. Deut. Math. Verein, 20: 1-47 (1911).

Landau[50] has developed other theorems regarding this generalized idea of classification. He shows that if $P(t)$ is the number of prime ideals with norms not exceeding t in each class, then

$$P(t) = \frac{1}{h}\int_2^t \frac{du}{\log u} + O(te^{-\sqrt[d]{\log t}})$$

and further, that if $R(t)$ is the number of ideals with norms not exceeding t and having an odd number of prime factors, and $S(t)$ the number of ideals with norms not exceeding t and having an even number of prime factors, then

$$\lim_{t \to \infty} \frac{R(t)}{S(t)} = 1.$$

Landau makes the same assumptions regarding the groups and fields as does Weber.

Based on the general existence proof for class fields and the method of Weber,[55] FURTWÄNGLER[56] arrives at the theorem: If $\mathfrak{p}_i$ varies over all the prime ideals of a class in a field whose class number is h, then

$$\sum_{\mathfrak{p}_i} \frac{1}{N(\mathfrak{p}_i)^s} = \frac{1}{h} \log \frac{1}{s-1} + f(s)$$

where $f(s)$ is a function of the real variable s which remains finite as s approaches 1.

Ideas similar to those of Weber[55] are used by Fueter.[9, 45, 46]

9. SPECIAL PROBLEMS

Following an article by Reuschle on the factors of rational primes in certain cyclotomic fields, KUMMER[57] discusses the distribution of the factors of rational primes among the 8 classes of the field of the 29th roots of unity. The ideals are not principal ideals and hence do not belong to the principal class. He finds that each of the other seven classes contains four of the 28 conjugate prime ideals and in the same class are found those four whose corresponding roots of $x^{29}-1=0$ have their sum equal to a period of four terms. In this field the square of any ideal is a principal ideal.

For the field of the 31st roots of unity the first factor of the class number is 9. Reuschle has computed the 9th powers of the ideal factors of 2 and found them to be principal ideals. KUMMER[58] has shown that the cubes of the prime factors of 2 are not principal ideals.

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[56] Math. Ann., 63: 37 (1907).

[57] Monatsberichte, Berlin, 1860: 744.

[58] Monatsberichte, Berlin, 1870: 755-766.

H. H. **Mitchell**[50] shows that if h_1 is the first factor of the class number of $K(e^{\frac{2\pi i}{l}})$ where l is a prime such that $l-1$ is not a power of 2, and if $2e$ is an integer such that $(l-1)/2e$ is odd, and if q is a prime such that $q^{\frac{l-1}{2l}} \equiv 1 \bmod l$, then in the field of the $2e$ periods, q is the product of $2e$ prime ideals which constitute e pairs of conjugate imaginary ideals, and if an ideal be selected from each of the e pairs the h_1th power of their product is a principal ideal.

Mitchell also shows that if two conjugate imaginary prime factors of q be raised to the h_1th power the resulting ideals belong to the same class.

Further references on Chapter I:

Beeger. K. Akad. van Weten. Verslagen, Amsterdam, 28: 293-311, 427-446, 1021-1023 (1919); 29: 331-360 (1919); K. Akad. van Weten. Proceedings, Amsterdam, 21: 395-414 (1919).

Verebrussov. Mémoire sur les classes des nombres complexes idéaux conjuges avec l'application à la demonstration du dernier théorème de Fermat, Paris, Gauthier-Villars, 18 p. (1912).

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[50] Trans. Am. Math. Soc., 19: 119-126 (1918).

CHAPTER II

IRREGULAR CYCLOTOMIC FIELDS AND FERMAT'S LAST THEOREM

INTRODUCTION

The present chapter is concerned with the properties of irregular cyclotomic fields, which are those defined by a primitive p^n-th root of unity in which the class number of the field is divisible by the odd prime p, and the application of both regular and irregular cyclotomic fields to Fermat's last theorem. The latter topic has been reported on in Dickson's History of the Theory of Numbers, Volume 2, Chapter 26, pages 731-776. That chapter contains abstracts of papers which appeared prior to the year 1920. The present report is designed to cover all papers on Fermat's last theorem which employ the theory of cyclotomic fields as well as all papers on the subject which have appeared since January 1920. Also, for easy reference, abstracts of a few other papers on the theorem are included. Since important features of papers so far published are the particular devices employed to obtain criteria for the solution of Fermat's equation (1), considerable space has been devoted to descriptions of these devices provided that algebraic numbers are involved.

Papers treating the divisibility of the class number of the field defined by a primitive p^n-th root of unity by p^k are included here as well as in chapter one.

A number of results in the theory of irregular cyclotomic fields are contained in the papers by Kummer.[17, 20, 23, 39] Furtwängler,[43, 46] in investigations concerning the class fields and laws of reciprocity in a relative cyclic field obtained general results which include as special cases theorems concerning irregular cyclotomic fields. This was pointed out by Pollaczek,[76] who made important additions to the theory.

If

$$x^p + y^p + z^p = 0 \tag{1}$$

is satisfied in rational integers x, y and z prime to the odd prime p, this will be referred to as Case I of Fermat's last theorem; and if one of these integers is divisible by p with x, y and z prime to each other, this will be referred to as Case II of the theorem.

In this connection it may be noted that although many contributions have been made to the first case of the theorem no paper has been published on the second case since Kummer's 1857 memoir which is proved to represent an advance over his results of that paper.

LEGENDRE[1] published a proof of the following theorem due to Sophie Germain:

If there exists an odd prime p, such that

$$\xi^n+\eta^n+\zeta^n\equiv 0 \pmod p$$

has no set of integral solutions ξ, η, ζ each not divisible by p, and such that n is not the residue of the n^{th} power of any integer modulo p, then $x^n+y^n+z^n=0$ has no integral solutions each prime to n. Cf. Wendt,[32] Dickson,[33] Vandiver.[34]

LAMÉ[2] essayed to prove Fermat's last theorem but assumed without proof that integers in the field $\Omega(a)$, where a is a primitive n-th root of unity, n being an odd prime, decomposed uniquely into prime factors in the field. This error was pointed out by LIOUVILLE[3] in commenting on Lamé's first article. Lamé recognized this lacuna but showed that this theorem was true for the case $n=5$ and affirmed that it was also true for the cases 7, 11, 13, and he believed it true for n general. He proved that the equation

$$A^5+B^5+C^5=0$$

is impossible for integers A, B, C, in the field $\Omega(a)$, $a^5=1$, $a\neq 1$. KUMMER[4] pointed out in connection with Lamé's work that unique decomposition into prime factors did not hold in all cyclotomic fields defined by an n-th root of unity, n prime.

WANTZEL[5] proved that Euclid's g.c.d. process holds for complex integers $a+b\sqrt{-1}$ (already proved by C. F. GAUSS[6]) and for complex integers formed from an imaginary cube root of unity, and stated that a like result holds for complex integers $a_0+a_1r+a_2r^2+\ldots+a_{n-1}r^{n-1}$, where r is an imaginary n-th root of unity with n arbitrary, since the norm (or modulus) of this number is <1 when $a_0, \ldots, a_{n-1}$ are between 0 and 1 [erroneous, Cauchy[7]].

CAUCHY[7] showed that the final statement by Wantzel[5] is false for $n=7$ and for any prime $n=4m+1\geqq 17$. He pointed out lacunae in the proposed proof by Lamé[7] of Fermat's last theorem. He defined the factorial of a complex number in the form used by Wantzel[6] to be the product of itself and the complex numbers obtained from it by replacing r by the remaining primitive n-th roots of unity, and obtained upper limits for such factorials [norms]. He[8] proved that any common factor of $M_h = Ar^h + B$ and M_k divides M_0 if A and B are relatively prime.

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[1] Mem. Acad. Sci. Inst. France, 6: 1-60 (1823). Thèorie des nombres ed. 2, 1808, second supplement 1825, 1-40.

[2] Compt. Rend., 24: 310-315, 352, 569-572, 888 (1847). Jour. de Math. (1), 12: 137-184 (1847).

[3] Compt. Rend., 24: 315-316 (1847).

[4] Compt. Rend., 24: 899-900 (1847). Jour. de Math., (1), 12: 136 (1847).

[5] Compt. Rend., 24: 430-434 (1847).

[6] Comm. Soc. Sc. Göttingen Recentiores, 7, 1832, § 46; Werke, II, 1863, 117. German transl. by H. Maser, Gauss' Untersuchungen über höhere Arith., 1889, 556.

[7] Compt. Rend., 24: 469-481 (1847); Oeuvres (1), X, 240-254.

Cauchy[9] attempted to prove the false theorem that the norm of the remainder obtained on dividing one complex number by another can always be made less than the norm of the divisor. He concluded (falsely) that a product of complex integers can be decomposed into complex primes in a single manner, and that the other laws of divisibility of integers hold for these complex integers.

Cauchy[10] noted (erroneous) conclusions which follow from the assumption that his preceding theorems hold for a given number n; in particular, errors relating to the factors $A + r^i B$ of $A^n + B^n$. He promised to discuss later the objections which can be raised against proofs in his preceding paper.

Cauchy[11] further developed the subject and admitted at the end of his final paper that his basic theorem is false, failing for $n = 23$.

Cauchy[12] obtained results, most of which are included in Kummer's[13] general theorems on cyclotomic integers and Fermat's last theorem; in the fifth paper, p. 179 (Oeuvres, p. 362) he gives without proof the result that if $a^n + b^n + c^n = 0$ is satisfied in rational integers a, b, c, prime to each other, n a prime >2, then

$$u_{a_1} u_{a_2} \cdots u_{a_{\frac{n-1}{2}}} = [F(\rho)]^n \omega(\rho),$$

where

$$\left| \begin{array}{lll} u = a + b\rho, & \rho^n = 1, & \rho \neq 1, \\ u_k = a + b\rho^k, & & \\ l \cdot a_l \equiv 1 \pmod{n}, & & \end{array} \right.$$

and $F(\rho)$ is a polynomial in ρ with rational integral coefficients, $\omega(\rho)$ is a power of ρ (cf. Kummer,[20] p. 63, for $r = 1$). Also (p. 181, Oeuvres, p. 364) he stated that $a^n + b^n + c^n = 0$ is impossible in rational integers not divisible by the odd prime n if

$$1 + 2^{n-4} + 3^{n-4} + \ldots + \left(\frac{n-1}{2}\right)^{n-4}$$

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[8] Compt. Rend., 24: 347-348 (1847); Oeuvres (1), X, 224-226.

[9] Compt. Rend., 24: 516-528 (1847); Oeuvres (1), X, 254-268.

[10] Compt. Rend., 24: 578-584 (1847); Oeuvres (1), X, 268-275.

[11] Compt. Rend., 24: 633-636, 996-999, 1022-1030 (1847); Oeuvres (1), X, 276-285, 296-308.

[12] Compt. Rend., 25: 37-46, 46-54, 93-99, 132-136, 177-182, 242-243, 285-288 (1847); Oeuvres (1), X, 324-351, 354-371.

is not divisible by n, that is, if the Bernoulli number $B_{(n-3)/2}$ is not divisible by n. (Cf. Kummer,[20] p. 64.)

KUMMER[21] founded a theory of ideal numbers. Let λ be a prime integer and α a primitive λ-th root of unity, $ef=\lambda-1$. (Throughout the remainder of this report we shall use the term "integer in the field $\Omega(\alpha)$" in lieu of Kummer's expression "complex number".)

$$\begin{aligned}\eta_0&=\alpha^{\gamma^0}+\alpha^{\gamma^e}+\alpha^{\gamma^{2e}}+\dots+\alpha^{\gamma^{(f-1)e}},\\ \eta_1&=\alpha^{\gamma^1}+\alpha^{\gamma^{e+1}}+\alpha^{\gamma^{2e+1}}+\dots+\alpha^{\gamma^{(f-1)e+1}},\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \eta_{e-1}&=\alpha^{\gamma^{e-1}}+\alpha^{\gamma^{2e-1}}+\alpha^{\gamma^{3e-1}}+\dots+\alpha^{\gamma^{fe-1}},\end{aligned}$$

(γ denoting a primitive root of λ), are the roots of an irreducible equation of degree e having rational integral coefficients, which we shall symbolize by

$$F(y)=y^e+A_1y^{e-1}+A_2y^{e-2}+\dots+A_{e-1}y+A_e=0.$$

He then establishes the following theorem:

If q be a rational prime integer satisfying the congruence $q^f\equiv1$, mod λ, the congruence $F(y)\equiv0$, mod q, is completely resolvable, i. e., it is possible to establish an indeterminate congruence of the form

$$F(y)\equiv(y-u_0)(y-u_1)\dots(y-u_{e-1}),\ \text{mod}\ q,$$

$u_0, u_1, \dots, u_{e-1}$ denoting rational integers, congruent or incongruent, mod q.

Every equation which subsists between two functions of the periods will subsist as a congruence for the modulus q when we substitute for the periods the roots of the congruence $F(y)\equiv0$ taken in a certain order.

If we put, the c's being rational integers,

$$\begin{aligned}\psi(\eta_0)&=c_0\eta_0+c_1\eta_1+c_2\eta_2+\dots+c_{e-1}\eta_{e-1},\\ \psi(\eta_1)&=c_0\eta_1+c_1\eta_2+c_2\eta_3+\dots+c_{e-1}\eta_0,\\ \psi(\eta_2)&=c_0\eta_2+c_1\eta_3+c_2\eta_4+\dots+c_{e-1}\eta_1,\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \psi(\eta_{e-1})&=c_0\eta_{e-1}+c_1\eta_0+c_2\eta_1+\dots+c_{e-1}\eta_{e-2},\end{aligned}$$

we also have

(I.) The necessary and sufficient condition that $\psi(\eta)$ should be divisible by q (i. e., that the coefficients $c_0, c_1, \ldots, c_{e-1}$ should be all separately divisible by q) is that the e congruences

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[21] De numeris complexis qui radicibus unitatis et numeris realibus constant, Breslau, 1844. (Jour. de Math., I, 12: 185-212 (1847).) Ueber die Divisoren gewisser Formen der Zahlen, welche aus der Theorie der Kreistheilung entstehen. J. reine angew. Math., 30: 107-116 (1846). Zur Theorie der complexen Zahlen, Berlin Monatsberichte for March, 1845, or J. reine angew. Math., 30: 319-326 (1847). Ueber die Zerlegung der aus Wurzeln der Einheit gebildeten complexen Zahlen in ihre Primfactoren. J. reine angew. Math., 35: 327-367 (1847). Recherches sur les nombres complexes. Jour. de Math., I, 16: 377-498 (1851).

$$\begin{aligned} \psi(u_0) &= c_0u_0 + c_1u_1 + c_2u_2 + \ldots + c_{e-1}u_{e-1} \equiv 0, \bmod q, \\ \psi(u_1) &= c_0u_1 + c_1u_2 + c_2u_3 + \ldots + c_{e-1}u_0 \equiv 0, \bmod q, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \psi(u_{e-1}) &= c_0u_{e-1} + c_1u_0 + c_2u_1 + \ldots + c_{e-1}u_{e-2} \equiv 0, \bmod q, \end{aligned}$$

should be simultaneously satisfied.

(II.) The necessary and sufficient condition that the norm of $\psi(\eta)$, taken with respect to the periods, i. e., the number $\psi(\eta_0)\psi(\eta_1)\ldots\psi(\eta_{e-1})$, should be divisible by q, is that *one* of the e congruences

$$\psi(u_0) \equiv 0,\ \psi(u_1) \equiv 0,\ \ldots,\ \psi(u_{e-1}) \equiv 0, \bmod q,$$

should be satisfied.

If $f(\alpha)$ is any integer in the field $\Omega(\alpha)$, it is possible to find integers in $\Omega(\alpha)$ of the same form as $\psi(\eta_0)$, $\psi_0, \psi_1, \ldots, \psi_{f-1}$, so that

$$f(\alpha) = \psi_0(\eta_0) + \alpha\psi_1(\eta_0) + \alpha^2\psi_2(\eta_0) + \ldots + \alpha^{f-1}\psi_{f-1}(\eta_0),$$

the ψ's being in the field $\Omega(\alpha)$.

Kummer then proves the theorems:

(I.) The necessary and sufficient condition that $f(\alpha)$ should be divisible by q, is that the congruences

$$\psi_0(u_k) \equiv 0,\ \psi_1(u_k) \equiv 0,\ \ldots,\ \psi_{f-1}(u_k) \equiv 0, \bmod q,$$

should be simultaneously satisfied for every value of k.

(II.) And the condition that the norm of $f(\alpha)$ should be divisible by q, is that the same congruences should be satisfied for some one value of k.

When the congruences $\psi_0(u_k) \equiv 0$, $\psi_1(u_k) \equiv 0$, $\ldots$, $\psi_{f-1}(u_k) \equiv 0$ (mod q), are simultaneously satisfied, $f(\alpha)$ is said to be ***congruent to zero* (*mod* q), *for the substitution* $\eta_0 = u_k$.**

In order to lead up to Kummer's definition of ideal numbers, let q be a rational integer belonging to the exponent f and suppose first that it is possible to express it as the norm (where this norm is taken with respect to the periods) of an integer $\psi(\eta_0)$, which contains the periods of f terms only, so that

$$q = \psi(\eta_0) \cdot \psi(\eta_1) \ldots \psi(\eta_{e-1}).$$

In view of the above results we may distinguish the e factors of q by means of substitutions which render them congruent to 0, modulo q, so that if $\psi(\eta_{e-k}) \equiv 0$ (mod q), if we substitute therein u_k in place of η_0, then $\psi(\eta_{e-k})$ *is said to be the factor appertaining to the substitution* $\eta_0 = u_k$.

He then shows that if $f(\alpha)$ is congruent to 0 (mod q) for any substitution ($\eta_0 = u_0$), $f(\alpha)$ is divisible by the factor of q appertaining to that substitution. Further, it is shown that these factors of q in the field are incapable of resolution into two other factors in the field unless one of them is a unit, and if any one of them divides the product of two factors, it will divide one of the two factors separately.

In order to make use of these ideals in connection with primes q belonging to the exponent f which cannot be decomposed into the product of e integers in the field, he defines an ideal factor of q as a factor of $f(\alpha)$ where $f(\alpha)$ is congruent to 0 under the substitution $\eta_0=u_0$; that is, $f(\alpha)$ *is said to contain the ideal factor of* q *appertaining to the substitution* $\eta_0=u_0$. Kummer, using this definition, arrives at theorems equivalent to the statement that an integer in the field $\Omega(\alpha)$ may be decomposed into prime ideal factors in but one way; the case of divisors of the rational integer λ is treated separately.

He then defines classes of ideal numbers in a way which is quite analogous to similar definitions in Dedekind's theory of ideals. This being the case, we shall employ the modern terminology in connection with these aspects of Kummer's theory, in particular, if an ideal P is such that P^h is a principal ideal but no lower power of P has that property, then P is said to *belong to the exponent* h.

KUMMER [14] obtained results concerning Fermat's last theorem. These results, together with the methods by which they are derived are entirely included in Kummer.[18]

The second article is entitled "Beweis des Fermatschen Satzes die Unmöglichkeit von $x^\lambda-y^\lambda=z^\lambda$ für eine unendlichen Anzahl Primzahlen λ," and at the beginning of it Kummer states in effect that he proves that the relation is impossible for an infinity of primes λ but he does not know for just which λ's the assumptions hold. Kummer does not, however, in any of his papers prove that there exists an infinity of regular primes although he apparently believed in its existence. He made similar statements in letters to Kronecker (cf. Kummer,[39] pp. 75, 84) and in his Vorlesungen über Zahlentheorie, I, 1901, 23, Kronecker stated that Kummer proved the impossibility of $x^\lambda+y^\lambda=z^\lambda$ for an infinity of primes λ and at first believed that his proof applied to nearly all λ's, but later believed the contrary.

Kummer (Hensel,[40] p. 32) is elsewhere quoted as believing it probable that there are approximately as many regular primes as irregular (exceptional) primes. (Cf. Kummer,[32] Jensen.[58])

DIRICHLET [15] commented on the results and methods in Kummer's [14] article.

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[14] Berlin Monatsberichte, 1847: 132-139, 305-319.

[15] Berlin Monatsberichte, 1847: 139-141.

Kummer[16] determined the number of nonequivalent classes of ideals in the field $\Omega(a)$, where a is a primitive λ-th root of unity, λ an odd prime. This number H is found to be $H=\frac{P}{(2\lambda)^{\mu-1}}\times\frac{D}{\Delta}$. In this formula P is a quantity defined by the equations

$$P=\phi(\beta)\cdot\phi(\beta^3)\cdot\phi(\beta^5)\ldots\phi(\beta^{\lambda-2}),$$
$$\phi(\beta)=1+\gamma_1\beta+\gamma_2\beta^2+\gamma_3\beta^3+\ldots+\gamma_{\lambda-2}\beta^{\lambda-2},$$

β representing a primitive root of the equation $\beta^{\lambda-1}=1$, γ a primitive root of the congruence $\gamma^{\lambda-1}\equiv 1$, mod λ, and γ_1, γ_2, γ_3, …, the least positive residues of γ, γ^2, γ^3, … for the modulus λ; D is the logarithmic determinant of a particular system of independent but not necessarily fundamental units, $e(a)$, $e(a^\gamma)$, $e(a^{\gamma^2})$, …, $e(a^{\gamma^{\mu-2}})$ defined by the equation

$$e(a)=\sqrt{\frac{(1-a^\gamma)(1-a^{-\gamma})}{(1-a)(1-a^{-1})}}=\pm\frac{a^{\mu(\gamma-1)}(1-a^\gamma)}{1-a}=\pm\frac{\sin\frac{k\gamma\pi}{\lambda}}{\sin\frac{\pi}{\lambda}},$$

so that if

$$a=e^{\frac{2ik\pi}{\lambda}},$$

$$D=\begin{vmatrix} L.e(a), & L.e(a^\gamma), & L.e(a^{\gamma^2}), & \ldots, & L.e(a^{\gamma^{\mu-2}}) \\ L.e(a^\gamma), & L.e(a^{\gamma^2}), & L.e(a^{\gamma^3}), & \ldots, & L.e(a^{\gamma^{\mu-1}}) \\ L.e(a^{\gamma^2}), & L.e(a^{\gamma^3}), & L.e(a^{\gamma^4}), & \ldots, & L.e(a^{\gamma^\mu}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ L.e(a^{\gamma^{\mu-2}}), & L.e(a^{\gamma^{\mu-1}}), & L.e(a^{\gamma^\mu}), & \ldots, & L.e(a^{\gamma^{2\mu-}} \end{vmatrix}$$

Also,

$$\Delta=\begin{vmatrix} L.c_1(a), & L.c_2(a), & \ldots, & L.c_{\mu-1}(a) \\ L.c_1(a^\gamma), & L.c_2(a^\gamma), & \ldots, & L.c_{\mu-1}(a^\gamma) \\ \cdots & \cdots & \cdots & \cdots \\ L.c_1(a^{\gamma^{\mu-2}}), & L.c_2(a^{\gamma^{\mu-2}}), & \ldots, & L.c_{\mu-1}(a^{\gamma^{\mu-2}}) \end{vmatrix}$$

where $c_1(a)$, $c_2(a)$, …, $c_{\mu-1}(a)$ is any system of $\mu-1$ fundamental units and $\mu=(\lambda-1)/2$. Quantities defined by the symbols $L.c_1(a)$, $L.c_2(a)$, etc., are the arithmetical logarithms of the real units $c_1(a)$, etc., taken positively.

Each of the two factors $P/(2\lambda)^{\mu-1}$ and D/Δ, of which the value of H is composed, is separately an integral number.

Kummer[17] proved that the number of distinct classes H of integers in the field $\Omega(a)$, where a is a primitive λ-th root of unity, λ an odd prime, is divisible by λ if, and only if, λ is a divisor of one of the first $(\lambda-3)/2$ Bernoulli numbers.

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[16] J. reine angew. Math., 40: 93-116 (1850).

[17] J. reine angew. Math., 40: 117-129 (1850). J. de Math., (1), 16: 473-486 (1851). Abstract in Berlin Monatsberichte, 1847: 305-319.

For the demonstration, Kummer starts with the first factor of the class number which is

$$\frac{P}{(2\lambda)^{\mu-1}},$$

where

$$P=\phi(\beta)\cdot\phi(\beta^3)\cdot\phi(\beta^5)\ldots\phi(\beta^{\lambda-2}),$$
$$\phi(\beta)=1+\gamma_1\beta+\gamma_2\beta^2+\gamma_3\beta^3+\ldots+\gamma_{\lambda-2}\beta^{\lambda-2},$$

and β is a primitive root of $x^{\lambda-1}=1$. If we represent the product

$$(\gamma\beta-1)\phi(\beta)=(\gamma\gamma_{\lambda-2}-1)+(\gamma-\gamma_1)\beta+(\gamma\gamma_1-\gamma_2)\beta^2+\ldots$$
$$+(\gamma\gamma_{\lambda-3}-\gamma_{\lambda-2})\beta^{\lambda-2},$$

by

$$\lambda[b_0+b_1\beta+b_2\beta^2+\ldots+b_{\lambda-2}\beta^{\lambda-2}],\qquad\text{or}\qquad\lambda\psi(\beta),$$

(b_m denoting the quotient $\frac{\gamma\gamma_{m-1}-\gamma_m}{\lambda}$ or $I\,\frac{\gamma\gamma_{m-1}}{\lambda}$, if I represents the greatest integer contained in the fraction before which it is placed), we obtain by multiplication the equality

$$(\gamma^\mu+1)P=\lambda^\mu\psi(\beta)\cdot\psi(\beta^3)\ldots\psi(\beta^{\lambda-2});$$

or, since $\gamma^\mu+1$ is divisible by λ, and may be, by choice of γ, supposed not divisible by λ^2,

$$2^{\mu-1}G\cdot\frac{P}{(2\lambda)^{\mu-1}}=\psi(\beta)\cdot\psi(\beta^3)\ldots\psi(\beta^{\lambda-2}),$$

G denoting a coefficient prime to λ. The congruence $\frac{P}{(2\lambda)^{\mu-1}}\equiv 0,\ \text{mod}\ \lambda$, is therefore equivalent to the congruence

$$\psi(\beta)\cdot\psi(\beta^3)\ldots\psi(\beta^{\lambda-2})\equiv 0,\ \text{mod}\ \lambda,$$

which may, in its turn, be replaced by the following,

$$\psi(\gamma)\cdot\psi(\gamma^3)\ldots\psi(\gamma^{\lambda-2})\equiv 0,\ \text{mod}\ \lambda.$$

For, if there be an equation which, considered as a congruence for a given modulus λ, is completely resolvable for that modulus, any symmetric function of the roots of the congruence is congruent for the modulus λ, to the corresponding function of the roots of the equation. The symmetric function

$$\psi(\beta)\cdot\psi(\beta^3)\ldots\psi(\beta^{\lambda-2}),$$

of β, β^3, …, $\beta^{\lambda-2}$, which are the roots of the equation $x^\mu+1=0$, is therefore congruent to $\psi(\gamma)\cdot\psi(\gamma^3)\ldots\psi(\gamma^{\lambda-2})$ which is the same function of γ, γ^3, γ^5, …, $\gamma^{\lambda-2}$, the roots of the congruence $x^\mu+1\equiv 0$, mod λ.

Hence the necessary and sufficient condition for the divisibility of $P/(2\lambda)^{\mu-1}$ by λ is that one of the μ congruences included in the formula

$$\psi(\gamma^{2n-1}) \equiv 0, \text{ mod } \lambda, \; n=1, 2, 3, \ldots, \mu,$$

should be satisfied. Now

$$\gamma^{-(2n-1)}\psi(\gamma^{2n-1}) \equiv b_0\gamma_{\lambda-2}^{2n-1}+b_1{\gamma_0}^{2n-1}+b_2{\gamma_1}^{2n-1}+\ldots+b_{\lambda-2}\gamma_{\lambda-3}^{2n-1}.$$

The right-hand member of this congruence may be written in the form

$$\begin{aligned}&1[(t_1+1)^{2n-1}+(t_1+2)^{2n-1}+\ldots+{t_2}^{2n-1}],\\ &+2[(t_2+1)^{2n-1}+(t_2+2)^{2n-1}+\ldots+{t_3}^{2n-1}],\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &+(\gamma-1)[(t_{\gamma-1}+1)^{2n-1}+(t_{\gamma-1}+2)^{2n-1}+\ldots\\ &\qquad\qquad+(\lambda-1)^{2n-1}]\equiv 0 \pmod{\lambda},\end{aligned}$$

where t_s is the greatest integer contained in $(s\lambda)/\gamma$. Write

$$\chi(x)=\frac{x^{2n}}{\Pi_{2n}}-\frac{x^{2n-1}}{2\Pi_{2n-1}}+\frac{B_1x^{2n-2}}{\Pi_{2n-2}\Pi_2}-\frac{B_2x^{2n-4}}{\Pi_{2n-4}\Pi_4}+\ldots+\frac{(-1)^nB_{n-1}x^2}{\Pi_2\Pi_{2n-2}},$$

where $B_1=1/6$, $B_2=1/30$, etc., are the Bernoulli numbers and $\Pi_r=r!$ Then

$$1^{2n-1}+2^{2n-1}+3^{2n-1}+\ldots+(x-1)^{2n-1}=\Pi_{2n-1}\chi(x),$$

provided x is a positive integer. By means of these, the last congruence becomes

$$\chi(t_1+1)+\chi(t_2+1)+\chi(t_3+1)+\ldots+\chi(t_{\gamma-1}+1)\equiv 0 \pmod{\lambda}.$$

The integer t_s can be written in the form $(s\lambda-r_s)/\gamma$ where r_s is a positive integer less than γ. Using this in the last congruence and neglecting the multiples of λ, we obtain

$$\chi\left(\frac{\gamma-r_1}{\gamma}\right)+\chi\left(\frac{\gamma-r_2}{\gamma}\right)+\ldots+\chi\left(\frac{\gamma-r_{\gamma-1}}{\gamma}\right)\equiv 0 \pmod{\lambda}.$$

Since $r_1, r_2, \ldots, r_{\gamma-1}$ are the numbers $1, 2, 3, \ldots, \gamma-1$, in some order, we then obtain

$$\chi\left(\frac{1}{\gamma}\right)+\chi\left(\frac{2}{\gamma}\right)+\chi\left(\frac{3}{\gamma}\right)+\ldots+\chi\left(\frac{\gamma-1}{\gamma}\right)\equiv 0 \pmod{\lambda}.$$

To transform this congruence we use the following known expansion of $\chi(x)$ in the infinite series

$$\chi(x)=\frac{(-1)^nB_n}{\Pi_{2n}}-\frac{(-1)^n2}{(2\pi)^{2n}}\left(\frac{\cos 2x\pi}{1^{2n}}+\frac{\cos 4x\pi}{2^{2n}}+\frac{\cos 6x\pi}{3^{2n}}+\ldots\right),$$

where x is contained within the limits 0 and 1. Setting in this $x=0$, $1/\gamma, 2/\gamma, \ldots, (\gamma-1)/\gamma$ in turn, and adding, we obtain

$$\chi\left(\frac{1}{\gamma}\right)+\chi\left(\frac{2}{\gamma}\right)+\chi\left(\frac{3}{\gamma}\right)+\ldots+\chi\left(\frac{\gamma-1}{\gamma}\right)$$
$$=\frac{(-1)^n\gamma B_n}{\Pi_{2n}}-\frac{(-1)^n 2\gamma}{(2\pi)^{2n}}\left[\frac{1}{\gamma^{2n}}+\frac{1}{(2\gamma)^{2n}}+\frac{1}{(3\gamma)^{2n}}+\ldots\right],$$

and using

$$1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\ldots=\frac{(2\pi)^{2n}B_n}{2\Pi_{2n}},$$

we have

$$\chi\left(\frac{1}{\gamma}\right)+\chi\left(\frac{2}{\gamma}\right)+\ldots+\chi\left(\frac{\gamma-1}{\gamma}\right)=\frac{(-1)^n(\gamma^{2n}-1)B_n}{\gamma^{2n-1}\Pi_{2n}}.$$

The condition for the divisibility of H by λ is therefore that one of the μ congruences included in the formula $B_n(\gamma^{2n}-1)\equiv 0$, mod λ, should be satisfied. The last of these congruences, or $B_\mu(\gamma^{2n}-1)\equiv 0$, is never satisfied; for it is easily proved that the denominator of B_μ contains λ as a factor, while

$$\gamma^{2\mu}-1=(\gamma^\mu+1)(\gamma^\mu-1),$$

though divisible by λ, is not divisible by λ^2. Hence, since γ is a primitive root of λ, a necessary and sufficient condition that the first factor of the class number is divisible by λ is that one of the Bernoulli numbers B_s, $s=1, 2, \ldots, \mu-1$, is divisible by λ.

Kummer then examines the divisibility of the second factor of the class number by λ on the basis of certain theorems proved in Section II of his article now being discussed. It is possible to find an independent system of units $e(a), e(a^\gamma), \ldots, e(a^{\gamma^{\mu-2}})$, in which

$$e(a)=\sqrt{\frac{(1-a^\gamma)(1-a^{-\gamma})}{(1-a)(1-a^{-1})}},$$

such that a certain power of every unit in the field may be expressed as a product of integral powers of the elements of this independent system, and also

$$\epsilon_1(a)^{n_1}=e(a)^{r'_1}\cdot e(a^\gamma)^{r'_2}\ \ldots e(a^{\gamma^{\mu-2}})^{r'_{\mu-1}},$$
$$\epsilon_2(a)^{n_2}=e(a)^{r_1^{(2)}}\cdot e(a^\gamma)^{r_2^{(2)}}\ \ldots e(a^{\gamma^{\mu-2}})^{r_{\mu-1}^{(2)}},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\epsilon_{\mu-1}(a)^{n_{\mu-1}}=e(a)^{r_1^{(\mu-1)}}\cdot e(a^\gamma)^{r_2^{(\mu-1)}}\ldots e(a^{\gamma^{\mu-2}})^{r_{\mu-1}^{(\mu-1)}}$$

where $\epsilon_1(a), \epsilon_2(a), \ldots, \epsilon_{\mu-1}(a)$ designate a fundamental system of units and where the exponent n_h has no factor in common with all of the integral exponents $r_1^h, r_2^h, \ldots, r^h_{\mu-1}$; also the parentheses are omitted from the superscripts in the r's.

We obtain from Section II, p. 398, of Kummer's article

$$\frac{D}{\Delta}=\frac{n_1\cdot n_2\ldots n_{\mu-1}}{R}$$

in which R is the determinant,

$$\begin{vmatrix} r_1^1, & r_2^1, & \ldots, & r^1_{\mu-1}, \\ r_1^2, & r_2^2, & \ldots, & r^2_{\mu-1}, \\ \cdots & \cdots & \cdots & \cdots \\ r_1^{\mu-1}, & r_2^{\mu-1}, & \ldots, & r^{\mu-1}_{\mu-1}. \end{vmatrix}$$

From the expression we have just given for D/Δ, it follows that there exists a unit in $\Omega(\alpha)$ such that

$$\epsilon(\alpha)^n=e(\alpha)^{r_1}\cdot e(\alpha^\gamma)^{r_2}\ldots e(\alpha^{\gamma^{\mu-2}})^{r_{\mu-1}},$$

where n is an integer divisible by λ but $r_1, r_2, \ldots, r_{\mu-1}$ are integers not all divisible by λ. It is easily seen that the λth power of an integer in the field $\Omega(\alpha)$ is congruent to a rational integer (mod λ). Consequently,

$$e(\alpha)^{r_1}\cdot e(\alpha^\gamma)^{r_2}\ldots e(\alpha^{\gamma^{\mu-2}})^{r_{\mu-1}}\equiv c \pmod{\lambda}.$$

It is now noted that if we have two integers in the field $\Omega(\alpha)$, say $f(\alpha)$ and $f_1(\alpha)$ such that $f(\alpha)=f_1(\alpha)$, then we may replace this relation by an equation involving an indeterminate x as follows:

$$f(x)=f_1(x)+(1+x+x^2+\ldots+x^{\lambda-1})\psi(x),$$

where $\psi(x)$ is a polynomial in x with rational integral coefficients. Hence we may write

$$\begin{aligned} &e(x)^{r_1}\cdot e(x^\gamma)^{r_2}\ldots e(x^{\gamma^{\mu-2}})^{r_{\mu-1}} \\ &\qquad =c+\lambda\phi(x)+(1+x+x^2+\ldots+x^{\lambda-1})\psi(x). \end{aligned}$$

In this equality take the logarithms of each member, differentiate both members with respect to x, multiply each by x, substitute $x=\alpha$, and we have

$$\begin{aligned} &r_1\frac{\alpha e'(\alpha)}{e(\alpha)}+r_2\gamma\frac{\alpha^\gamma e'(\alpha^\gamma)}{e(\alpha^\gamma)}+\ldots+r_{\mu-1}\gamma^{\mu-2}\frac{\alpha^{\gamma^{\mu-2}}e'(\alpha^{\gamma^{\mu-2}})}{e(\alpha^{\gamma^{\mu-2}})} \\ &\qquad =\frac{\lambda\alpha\phi'(\alpha)+[\alpha+2\alpha^2+3\alpha^3+\ldots+(\lambda-1)\alpha^{\lambda-1}]\psi(\alpha)}{c+\lambda\phi(\alpha)}. \end{aligned}$$

From this relation we obtain easily

$$\begin{aligned} (2)\qquad &r_1F(\alpha)+r_2\gamma F(\alpha^\gamma)+\ldots+r_{\mu-1}\gamma^{\mu-2}F(\alpha^{\gamma^{\mu-2}}) \\ &\qquad \equiv M[\alpha+2\alpha^2+\ldots+(\lambda-1)\alpha^{\lambda-1}], \end{aligned}$$

where M is congruent to an integer modulo λ and

$$\frac{2e'(\alpha)}{e(\alpha)}=F(\alpha),$$

giving $e(a)$ its actual value. It is found after some transformations that

$$(3) \qquad F(a) = \gamma - 1 + 2b_0 a + 2b_1 a^{\gamma} + 2b_3 a^{\gamma^2} + \dots + 2b_{\lambda-2} a^{\gamma^{\lambda-2}}$$

in which

$$b_s = \frac{\gamma\gamma_{s-1} - \gamma_s}{\lambda}.$$

Writing

$$M(a + \gamma a^{\gamma} + \gamma^2 a^{2\gamma^2} + \dots + \gamma^{\lambda-2} a^{\gamma^{\lambda-2}})$$

for the second member of (2) and comparing the coefficients of the various powers of a we obtain from (3), after the new forms of the F's are substituted, the system of congruences

$$\left.\begin{array}{l}
r_1 b_0 + \gamma r_2 b_{\lambda-2} + \gamma^2 r_3 b_{\lambda-3} + \dots + \gamma^{\mu-2} r_{\mu-1} b_{\lambda-\mu+1} - K \equiv M \\
r_1 b_1 + \gamma r_2 b_0 + \gamma^2 r_3 b_{\lambda-2} + \dots + \gamma^{\mu-2} r_{\mu-1} b_{\lambda-\mu+2} - K \equiv \gamma M \\
r_1 b_2 + \gamma r_2 b_1 + \gamma^2 r_3 b_0 + \dots + \gamma^{\mu-2} r_{\mu-1} b_{\lambda-\mu+3} - K \equiv \gamma^2 M \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
r_1 b_{\lambda-2} + \gamma r_2 b_{\lambda-3} + \gamma^2 r_3 b_{\lambda-4} + \dots + \gamma^{\mu-2} r_{\mu-1} b_{\lambda-\mu} - K \equiv \gamma^{\lambda-2} M
\end{array}\right\} \pmod{\lambda},$$

where

$$r_1 + \gamma r_2 + \gamma^2 r_3 + \dots + \gamma^{\mu-2} r_{\mu-1} = K.$$

These congruences, multiplied respectively by 1, γ^{2n-1}, $\gamma^{2(2n-1)}$, ..., $\gamma^{(\lambda-2)(2n-1)}$, and added, give a sum which may be decomposed into the two factors

$$\left(\begin{array}{l} r_1 + \gamma^{2n} r_2 + \gamma^{4n} r_3 + \dots \\ \qquad\qquad + \gamma^{2(\mu-2)n} r_{\mu-1} \end{array}\right) \cdot \left(\begin{array}{l} b_0 + b_1 \gamma^{2n-1} + b_2 \gamma^{3(2n-1)} + \dots \\ \qquad\qquad + b_{\lambda-2} \gamma^{(\lambda-2)(2n-1)} \end{array}\right) \equiv 0,$$

modulo λ, for $n = 1, 2, 3, \dots, \mu - 1$. If we assume that the second factor is not divisible by λ for any of the given values of n, it follows that

$$\left.\begin{array}{l}
r_1 + \gamma^2 r_2 + \gamma^4 r_3 + \dots + \gamma^{2(\mu-2)} r_{\mu-1} \equiv 0, \\
r_1 + \gamma^4 r_2 + \gamma^8 r_3 + \dots + \gamma^{4(\mu-2)} r_{\mu-1} \equiv 0, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
r_1 + \gamma^{2(\mu-1)} r_2 + \gamma^{4(\mu-1)} r_3 + \dots + \gamma^{(\mu-2)(\mu-1)} r_{\mu-1} \equiv 0,
\end{array}\right\} \pmod{\lambda}.$$

The determinant

$$\left|\begin{array}{ccccc}
1 & \gamma^2 & \gamma^4 & \dots & \gamma^{2(\mu-2)} \\
1 & \gamma^4 & \gamma^8 & \dots & \gamma^{4(\mu-2)} \\
\dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots \\
1 & \gamma^{2(\mu-1)} & \gamma^{4(\mu-1)} & \dots & \gamma^{(\mu-2)(\mu-1)}
\end{array}\right|$$

is not divisible by λ since it is the product of factors of the form $\gamma^{2k} - \gamma^{2h}$, where k and h are distinct and each less than μ, hence

$$r_1 \equiv 0, \quad r_2 \equiv 0, \quad r_3 \equiv 0, \quad \dots, \quad r_{\mu-1} \equiv 0 \pmod{\lambda},$$

and this is contrary to the supposition that these exponents are not all divisible by λ, consequently

$$b_0+b_1\gamma^{2n-1}+b_2\gamma^{2(2n-1)}+\ldots+b_{\lambda-2}\gamma^{(\lambda-2)(2n-1)}$$

must be divisible by λ for some value of n if D/Δ is divisible by λ. But this condition is precisely a necessary and sufficient condition that the first factor of the class number be divisible by λ.

KUMMER[18] applied his methods to Fermat's last theorem. He considers the equation

$$u^\lambda+v^\lambda+w^\lambda=0 \tag{4}$$

where u, v and w are integers in the field $\Omega(\alpha)$ prime each to each, and proved that the relation did not hold for λ an odd prime greater than 3 and not dividing any of the first $(\lambda-3)/2$ Bernoulli numbers.

Kummer makes use of two theorems established by him on p. 447 and 487 of the second article cited, namely, under the assumption that λ does not divide any of the first $(\lambda-3)/2$ Bernoulli numbers, then if the λ-th power of the ideal equals an integer in the field, the ideal itself is an integer in the field. The second theorem states that under the same assumption every unit in the field which is congruent to a rational integer modulo λ equals the λ-th power of another unit.

In (4) we note that in place of integers u, v, w, we may write the same integers multiplied by α^k. Hence we can determine k so that $\alpha^k u$ has the form $a+(1-\alpha)^2P$ where a is a rational integer and P is an integer in the field $\Omega(\alpha)$, as may be easily shown. Hence we write

$$\begin{aligned} u&=a+(1-\alpha)^2P,\\ v&=b+(1-\alpha)^2Q,\\ w&=c+(1-\alpha)^2R, \end{aligned}$$

where a, b, c are rational integers, and we shall assume that none of them is divisible by λ, which is equivalent to the assumption that none of the integers u, v, w is divisible by $1-\alpha$. Factoring $u^\lambda+v^\lambda$, we obtain the equation

$$(u+v)\cdot(u+\alpha v)\cdot(u+\alpha^2v)\ldots(u+\alpha^{\lambda-1}v)=-w^\lambda.$$

It is then shown that since $1-\alpha$ is prime to u, v, w, then all of the foregoing factors are prime to each other. Hence, using the theorem established in Article V, Kummer concludes that

$$u+\alpha^r v=\alpha^\rho E_r(\alpha)t_r{}^\lambda, \tag{5}$$

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[18] J. reine angew. Math., 40: 130-138 (1850). J. de Math., (1), 16: 488-498 (1851).

where t_r is an integer in the field $\Omega(\alpha)$, $\alpha^\rho E_r(\alpha)$ is a unit in the field such that $E_r(\alpha)=E_r(\alpha^{-1})$, and obtains the congruence

$$u+\alpha^r v\equiv\alpha^\rho E_r(\alpha)\cdot m \pmod{\lambda},$$

where m is a rational integer. In (5) set α^{-1} in lieu of α, and we obtain a result which gives, as before, if $u'=u(\alpha^{-1})$ and $u=u(\alpha)$, etc.,

$$u'+\alpha^{-r}v'\equiv\alpha^{-\rho}E_r(\alpha)\cdot m \pmod{\lambda},$$

and comparison with the last congruence mentioned gives

$$\alpha^{-\rho}(u+\alpha^r v)\equiv\alpha^\rho(u'+\alpha^{-r}v') \pmod{\lambda}.$$

This relation reduces modulo $(1-\alpha)^2$ to

$$\alpha^{-\rho}(a+\alpha^r b)\equiv\alpha^\rho(a+\alpha^{-r}b) \quad [\mathrm{mod}\ (1-\alpha)^2].$$

Also, using the relation

$$\alpha^h\equiv 1-h(1-\alpha) \quad [\mathrm{mod}\ (1-\alpha)^2],$$

we obtain

$$(a+b)\rho\equiv br \pmod{\lambda}.$$

Determining k by the congruence

$$(a+b)k\equiv b \pmod{\lambda},$$

we have

$$\rho\equiv kr \pmod{\lambda},$$

and

$$\alpha^{-kr}(u+\alpha^r v)\equiv\alpha^{kr}(u'+\alpha^{-r}v') \pmod{\lambda},$$

for $r=0, 1, 2, \dots, \lambda-1$. This gives for $r=0$

$$u+v\equiv u'+v' \pmod{\lambda},$$

and since u, v, w are symmetric in the original equation (4), we have

$$\left.\begin{cases} u\equiv u' \\ v\equiv v' \\ w\equiv w' \end{cases}\right\} \pmod{\lambda}.$$

These relations give immediately

$$\alpha^{-kr}(u+\alpha^r v)\equiv\alpha^{kr}(u+\alpha^{-r}v) \pmod{\lambda}.$$

Taking $r=1$ and $r=2$ in turn, there are obtained relations from which we can infer that $k\equiv 1$ or $2k\equiv 1 \pmod{\lambda}$, noting that $\lambda>3$. These relations, combined with $(a+b)k\equiv b$ show that

$$a\equiv 0 \pmod{\lambda} \qquad \text{or} \qquad a\equiv b \pmod{\lambda},$$

hence $a \equiv c$ and $b \equiv c \pmod{\lambda}$. From (4) we also have

$$0 \equiv u^\lambda + v^\lambda + w^\lambda \equiv a + b + c \pmod{\lambda},$$

which gives

$$3a \equiv 3b \equiv 3c \equiv 0 \pmod{\lambda},$$

which is impossible since $\lambda > 3$.

We now proceed to the second part of the demonstration, in which it is supposed that one of the three integers u, v, w, is divisible by $1-a$. In this case it is convenient to write (4) in the generalized form

$$u^\lambda + v^\lambda = E(a)(1-a)^{m\lambda} w^\lambda, \tag{6}$$

in which the integers u, v, w, are prime each to each and where $1-a$ and $E(a)$ designate a unit in the field. We have immediately

$$(u+v) \cdot (u+av) \cdot (u+a^2v) \ldots (u+a^{\lambda-1}v) = E(a)(1-a)^{m\lambda} w^\lambda.$$

It is first shown that each of the factors on the left hand side of the equation is divisible by $1-a$ but none of them is divisible by $(1-a)^2$ except $u+v$, so that we have

$$u+v = (1-a)^{m\lambda-\lambda+1} \cdot \phi$$

and

$$u + a^r v = (1-a^r)\phi_r,$$

so that

$$\phi \cdot \phi_1 \cdot \phi_2 \ldots \phi_{\lambda-1} = E(a) w^\lambda,$$

where $\phi, \phi_1, \phi_2, \ldots, \phi_{\lambda-1}$ are integers in the field which are prime each to each and to $(1-a)$.

It is concluded from this that

$$u+v = e_0(a)(1-a)^{m\lambda-\lambda+1} \cdot w_1^\lambda,$$
$$u + a^r v = e_r(a)(1-a^r) t_r^\lambda$$

for $r = 1, 2, 3, \ldots, (\lambda - 1)$. Setting s in place of r we have

$$u + a^s v = e_s(a)(1-a^s) t_s^\lambda,$$

and eliminating u and v from these three equations, we find

$$t_r^\lambda - \frac{e_s(a)}{e_r(a)} t_s^\lambda = \frac{e_0(a)(a^r - a^s)(1-a)}{e_r(a)(1-a^r)(1-a^s)} (1-a)^{(m-1)\lambda} \cdot w_1^\lambda;$$

which we write in the form

$$t_r^\lambda + \epsilon(a) t_s^\lambda = E_1(a)(1-a)^{(m-1)\lambda} w_1^\lambda.$$

By considering this equation modulo λ, we find $\epsilon(a)$, which is a unit in the field, to be such that

$$\epsilon(a) \equiv c \pmod{\lambda},$$

hence

$$u_1^\lambda + v_1^\lambda = E_1(a)(1-a)^{(m-1)\lambda} w_1^\lambda,$$

where u_1, v_1, w_1, are integers in the field. This equation is of exactly the same type as (6) except that the exponent of $(1-\alpha)^\lambda$ is decreased by 1. We may proceed as before and still further decrease this exponent and finally obtain

$$u^\lambda_{m-1}+v^\lambda_{m-1}=E(\alpha)\,(1-\alpha)^\lambda w^\lambda_{m-1},$$

but this relation is easily seen to be impossible from the remarks made in the first part of the demonstration.

Kronecker[19] obtained in another way Kummer's[18] criterion for the divisibility of the first factor of the class number of the field $\Omega(\alpha)$, where $\alpha^\lambda=1$, $\alpha\neq 1$, λ an odd prime. Retaining Kummer's notation, the problem reduces to the question as to whether the congruence

$$\psi(\gamma^{2n-1})=b_0+b_1\gamma^{2n-1}+\ldots+b_{\lambda-2}\gamma^{(\lambda-2)(2n-1)}\equiv 0 \pmod{\lambda}$$

is or is not satisfied for one of the values $n=1, 2, 3, \ldots, \mu$. Setting

$$\lambda b_k=\gamma\cdot\gamma_{k-1}-\gamma_k,$$

the above gives

$$\lambda\cdot\psi(\gamma^{2n-1})=\sum(\gamma\gamma_{k-1}-\gamma_k)\gamma^{(2n-1)k}\equiv 0 \pmod{\lambda^2},$$

the summation extending over the values

$$k=0,\ 1,\ 2,\ 3,\ \ldots,\ (\lambda-2).$$

Using the identity

$$\gamma^k-(\gamma^k-\gamma_k)=\gamma_k$$

for k, and also $k-1$ in lieu of k, we obtain, after some transformations,

$$2n\cdot\sum(\gamma\gamma_{k-1}-\gamma_k)\gamma^{(2n-1)k}\equiv\gamma^{2n}\sum\gamma_{k-1}{}^{2n}-\sum\gamma_k{}^{2n} \pmod{\lambda^2},$$

which gives immediately

$$\begin{aligned}&2n\cdot\sum(\gamma\gamma_{k-1}-\gamma_k)\gamma^{(2n-1)k}\\ &\qquad\equiv(\gamma^{2n}-1)\,[1^{2n}+2^{2n}+\ldots+(\lambda-1)^{2n}] \pmod{\lambda^2}.\end{aligned}$$

In the case $n=\mu$ he shows that the right-hand member is never divisible by λ^2, since we may select γ so that $\gamma^{2\mu}-1$ is divisible by λ but not by λ^2; hence, since

$$1^{2n}+2^{2n}+\ldots+(\lambda-1)^{2n}\equiv\pm B_n\cdot\lambda \pmod{\lambda^2}$$

for $n=1, 2, \ldots, \mu-1$, we have the criterion desired.

Kummer[20] made a further investigation of Fermat's last theorem. For the purposes of this report his long discussion will be divided into four parts, each part being concerned with the proof of a certain theorem involving algebraic numbers. He attempts to prove that (1) for $p=\lambda$, is impossible for any rational integers x, y, z, under the three assumptions:

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[19] J. de Math., (2), 1: 396-398 (1856).

[20] Abhandlungen der Königlichen Akademie der Wissenschaften zu Berlin, Math.-Phys. Klasse, 1857: 41-74. Abstract in Berlin Monatsberichte, 1857: 275-282.

Assumption I: The first factor of the class number H is divisible by λ but not by λ^2.

Assumption II: If $B_\nu \equiv 0 \pmod{\lambda}$, $\nu<\frac{1}{2}(\lambda-1)$, there exists an ideal in the field $\Omega(\alpha)$, $\alpha^\lambda=1$, $\alpha \neq 1$, with respect to which as a modulus the unit

$$E_\nu(\alpha) = \prod_{k=0}^{\mu-1} e(\alpha^{\gamma^k})^{\gamma^{-2k\nu}}$$

is not congruent to the λ-th power of an integer in $\Omega(\alpha)$.

Assumption III: The Bernoulli number $B_{\nu\lambda}$ is not divisible by λ^3.

The first of the above-mentioned theorems is

Theorem (I): If there is but one of the first $\mu-1$ Bernoulli numbers divisible by λ, say $B_\nu \equiv 0 \pmod{\lambda}$, and the second factor of the class number of the field defined by the λ-th root of unity is also divisible by λ, then $E_\nu(\alpha)$ is the λ-th power of a unit in the field $\Omega(\alpha)$.

Here α is a primitive λ-th root of unity and when we speak of a Bernoulli number being divisible by an integer, we mean the numerator is divisible by the integer. Also, λ is an odd prime and $\mu=(\lambda-1)/2$. For the demonstration of this, he first remarked that under the assumption that the first factor of the class number is divisible by λ, it will follow from the investigations in Kummer[17] that there is but one of the first $\mu-1$ Bernoulli numbers divisible by λ. He calls this particular Bernoulli number B_ν. (Incomplete, cf. Vandiver.[64])

As from Kummer,[17] p. 480, if the second factor of the class number is divisible by λ then we have

$$(7) \qquad \epsilon(\alpha)^{h\lambda}=e(\alpha)^{r_1}\cdot e(\alpha^{\gamma})^{r_2}\cdot e(\alpha^{\gamma^2})^{r_3}\ldots e(\alpha^{\gamma^{\mu-2}})^{r_{\mu-1}}$$

γ being a primitive root of λ and

$$e(\alpha)=\sqrt{\frac{(1-\alpha^{\gamma})(1-\alpha^{-\gamma})}{(1-\alpha)(1-\alpha^{-1})}}$$

and the integers $r_1, r_2, \ldots, r_{\mu-1}$ are not all divisible by λ, $\epsilon(\alpha)$ being a fundamental unit. From this it follows that

$$r_1+\gamma^{2n}r_2+\gamma^{4n}r_3+\ldots+\gamma^{2(\mu-2)n}r_{\mu-1}\equiv 0, \bmod \lambda,$$

for all values of n in the series 1, 2, 3, ..., $\mu-1$, for which the n-th Bernoulli number B_n is not divisible by λ. For the case $n=\nu$ we have

$$r_1+\gamma^{2\nu}r_2+\gamma^{4\nu}r_3+\ldots+\gamma^{2(\mu-2)\nu}r_{\mu-1}\equiv \mu m, \bmod \lambda,$$

where m is a rational integer. From the last two sets of congruences we obtain by elimination

$$r_{k+1}=m\gamma^{-2k\nu}-m\gamma^{2\nu}+\lambda s_{k+1},$$

where s_{k+1} is an integer. Applying this to the relation (7) we have

$$\epsilon(a)^{h\lambda} = (e(a)e(a^{\gamma})^{\gamma^{-2n}}e(a^{\gamma^2})^{\gamma^{-4n}}\dots e(a^{\gamma^{\mu-1}})^{\gamma^{-2(\mu-1)n}})^m U(a)^{\lambda},$$

where $U(a)$ is some unit in the field $\Omega(a)$. Setting

$$E_n(a) = e(a)e(a^{\gamma})^{\gamma^{-2n}}e(a^{\gamma^2})^{\gamma^{-4n}}\dots e(a^{\gamma^{\mu-1}})^{\gamma^{-2(\mu-1)n}},$$

we have

$$\epsilon(a)^{h\lambda} = E_{\nu}(a)^m U(a)^{\lambda}.$$

Since m is prime to λ, we obtain easily

$$E_{\nu}(a) = \mathfrak{E}(a)^{\lambda}.$$

Theorem II. If $B_{\nu} \equiv 0 \pmod{\lambda}$ but $B_{\nu\lambda} \not\equiv 0 \pmod{\lambda^3}$, then a unit in $\Omega(a)$ which is congruent modulo λ^2 to a rational integer is the λ^{th} power of a unit in $\Omega(a)$.

To show this he takes any unit in the field $E(a)$ and sets

$$E(a)^t = \pm a^s e(a)^m e(a^{\gamma})^{m_1} e(a^{\gamma^2})^{m_2}\dots e(a^{\gamma^{\mu-1}})^{m_{\mu-1}}$$

where $s, m, m_1, \dots, m_{\mu-1}$ and t are rational integers, the latter integer being the least exponent to which $E(a)$ must be raised in order to be expressed as the product of integral powers of the e's. He then applies to this relation the congruence

$$(7a)\qquad (\lambda-1)l\left(\frac{\phi(a)}{\phi(1)}\right) \equiv l\left(\frac{N\phi(a)}{[\phi(1)]^{\lambda-1}}\right) + \sum_{i=0}^{\lambda-2} \frac{d_0{}^{i\lambda n} l\phi(e^{\nu})}{dv^{i\lambda n}}\,\chi_i(a)$$

modulo λ^{n+1}. In this formula $\phi(a)$ is an algebraic integer in the field $\Omega(a)$, and the symbol

$$l\left(\frac{\phi(a)}{\phi(1)}\right)$$

modulo λ^{n+1}, represents that part of the logarithmic series

$$\frac{\phi(a)-\phi(1)}{\phi(1)} - \frac{1}{2}\left(\frac{\phi(a)-\phi(1)}{\phi(1)}\right)^2 + \frac{1}{3}\left(\frac{\phi(a)-\phi(1)}{\phi(1)}\right)^3 \dots$$

none of whose terms are divisible by λ^{n+1}. The symbol $N\phi(a)$ stands for the norm of $\phi(a)$ and

$$\frac{d_0{}^s l\phi(e^v)}{dv^s}$$

indicates that $\log\phi(e^v)$ is to be differentiated s times and zero substituted for v in the result. Further,

$$\chi_i(a) = a + \gamma^{-i\lambda^n}a^{\gamma} + \gamma^{-2i\lambda^n}a^{\gamma^2} + \dots + \gamma^{-(\lambda-2)i\lambda^n}a^{\gamma^{\lambda-2}},$$

which KUMMER[20a] had attempted to prove in another article. (Error, cf. Vandiver.[64])

[20a] J. reine angew. Math., 44: 134 (1852).

We then proceed to

Theorem III. If $F(a)$ is an integer in the field $\Omega(a+a^{-1})$, and if

$$\frac{d_0{}^{2\nu} lF(e^v)}{dv^{2\nu}} \equiv 0 \quad (\operatorname{mod} \lambda),$$

then a unit $E(a)$ in $\Omega(a)$ can be found such that

$$E(a)F(a) \equiv a \quad (\operatorname{mod} \lambda),$$

where a is a rational integer.

In the argument covering this (pp. 48-50), the incorrect formula just mentioned is also used. He then considered

Theorem IV. Under Assumptions I and II, if P is an ideal such that P^λ is the actual integer $F(a)$ and $B_\nu \equiv 0 \pmod{\lambda}$, then P is or is not an actual integer according as

$$\frac{d_0{}^{\lambda-2\nu} lF(e^v)}{dv^{\lambda-2\nu}} \equiv \text{ or } \not\equiv 0 \quad (\operatorname{mod} \lambda).$$

To demonstrate this, Kummer uses first the following congruence

$$\text{(7b)} \quad \operatorname{Ind} \mathrm{E}_n(a) \equiv \frac{\gamma^{2n}-1}{2(1+r^{\lambda-2n}-(r+1)^{\lambda-2n})} \cdot \frac{d_0{}^{\lambda-2n} l\Psi_r(e^v)}{dv^{\lambda-2n}}, \quad \operatorname{mod} \lambda,$$

established in Kummer[20a] (p. 103), in which $f(a)$ is an ideal prime in the field $\Omega(a)$ and ind $E_n(a)$ is the integer i in the relation

$$(E_n(a))^{(q-1)/\lambda} \equiv a^i \quad (\operatorname{mod} f(a)),$$

q being the rational prime which $f(a)$ divides, $q \equiv 1 \pmod{\lambda}$,

$$\psi_r(a) = \Sigma a^{-(r+1)h+\operatorname{ind}(g^h+1)}$$

where g is a primitive root of $f(a)$ such that

$$g^{(q-1)/\lambda} \equiv a \quad (\operatorname{mod} f(a));$$

h ranges over the integers 0, 1, 2, ..., $q-2$, excepting $(q-1)/2$. (This relation is proved only for $f(a)$ an ideal of the first degree, whereas Kummer applies Theorem IV to ideals not necessarily of the first degree. Cf. Vandiver.[64])

He writes for the class number $H=H_1\lambda$ where H_1 is not divisible by λ, and we may put

$$f(a)^{H_1\lambda} = F(a)$$

where $F(a)$ is an integer in the field $\Omega(a)$. This gives by a known property of the function $\Psi_r(a)$

$$\Psi_r(a)^{H_1\lambda} = \pm \Pi F(a^{\gamma^k})$$

where the product runs over all those values of h in the series 0, 1, 2, $\ldots, \lambda-2$, for which

$$\gamma_{\mu-h}+\gamma_{\mu-h+\operatorname{Ind} r}>\lambda$$

where γ_i is the least positive integer which is congruent to $\gamma^i \pmod{\lambda}$.

Then using the relation, (not proved, cf. Vandiver[66]) where

$$\phi(a)\equiv\phi_1(a) \quad (\operatorname{mod} \lambda^{r+1}),$$

$$\frac{d_0{}^{m\lambda^r}l\phi(e^v)}{dv^{m\lambda^r}} \equiv \frac{d_0{}^{m\lambda^r}l\phi_1(e^v)}{dv^{m\lambda^r}}, \operatorname{mod} \lambda^{r+1},$$

he obtains

$$H_1\lambda \operatorname{Ind} E_n(a) \equiv \frac{\gamma^{2n}-1}{2(1+r^{\lambda-2n}-(r+1)^{\lambda-2n})} \cdot \frac{d_0{}^{(\lambda-2n)\lambda}lF(e^v)}{dv^{(\lambda-2n)\lambda}} \cdot \sum\gamma^{(\lambda-2n)\lambda h}$$

Comparing this with the congruence (7b) above, we obtain

$$H_1\lambda \frac{d_0{}^{(\lambda-2n)\lambda}l\Psi_r(e^v)}{dv^{(\lambda-2n)\lambda}} \equiv \sum \frac{d_0{}^{(\lambda-2n)\lambda}lF(e^{v\gamma^h})}{dv^{(\lambda-2n)\lambda}},$$

modulo λ^2, where the summation extends over all the values of h as defined above.

Kummer then goes through a special investigation involving rational integers only which leads him to

$$\sum\gamma^{(\lambda-2n)\lambda} \equiv (1+r^{(\lambda-2n)\lambda}-(r+1)^{(\lambda-2n)\lambda})(-1)^{\mu+\nu}B_{\nu\lambda-\mu}, \operatorname{mod} \lambda^2,$$

which yields

$$\operatorname{Ind} E_\nu(a) \equiv \frac{(-1)^{\nu+\mu}(\gamma^{2\nu}-1)B_{\nu\lambda-\mu}}{2H_1\lambda} \cdot \frac{d_0{}^{\lambda-2\nu}lF(e^v)}{dv^{\lambda-2\nu}}, \operatorname{mod} \lambda.$$

Kummer, in §5, employs the symbol

$$\left(\frac{\Phi(a)}{f(a)}\right)$$

and defines it to have the value a^i where

$$\Phi(a)^{Nf(a)-1/\lambda} \equiv a^i, \quad \operatorname{mod} f(a),$$

and we also write $i \equiv \operatorname{Ind} \Phi(a)$, modulo λ. If $\phi(a)$ is a composite ideal so that $\phi(a)=f(a)\cdot f_1(a)\cdot f_2(a)\ldots$, we write

$$\left(\frac{\Phi(a)}{\phi(a)}\right)=\left(\frac{\Phi(a)}{f(a)}\right)\cdot\left(\frac{\Phi(a)}{f_1(a)}\right)\cdot\left(\frac{\Phi(a)}{f_2(a)}\right)\ldots.$$

If

$$\left(\frac{E_\nu(a)}{\phi(a)}\right)=a^i$$

and $i=\operatorname{ind} E_\nu(a)$, then

$$\operatorname{Ind} E_\nu(a) \equiv \frac{(-1)^{\nu+\mu}(\gamma^{2\nu}-1)B_{\nu\lambda-\mu}}{2H_1\lambda} \cdot \frac{d_0{}^{\lambda-2\nu}\phi(e^v)^{H_1\lambda}}{dv^{\lambda-2\nu}}, \operatorname{mod} \lambda.$$

It is then noted that if $\phi(\alpha)$ is an actual integer in the field $\Omega(\alpha)$,

$$\left(\frac{E_\nu(\alpha)}{\phi(\alpha)}\right)=1.$$

Also, if $\phi(\alpha)$ and $\phi_1(\alpha)$ are two equivalent ideals, then

$$\left(\frac{E_\nu(\alpha)}{\phi(\alpha)}\right)=\left(\frac{E_\nu(\alpha)}{\phi_1(\alpha)}\right).$$

Using Assumption II, it follows that

$$\left(\frac{E_\nu(\alpha)}{X(\alpha)}\right)=\alpha^i,\quad \text{where } i\not\equiv 0,\quad \text{mod } \lambda,$$

where $X(\alpha)$ is a prime ideal in the field $\Omega(\alpha)$ which belongs to the exponent $h\lambda$ where h is not divisible by λ and therefore

$$\left(\frac{E_\nu(\alpha)}{\phi(\alpha)^m}\right)=\alpha^{him},\quad \text{Ind}\, E_\nu(\alpha)\equiv him,\quad \text{mod } \lambda,$$

where $\phi(\alpha)=X(\alpha)^h$ and m is any integer. If $f(\alpha)$ is any ideal which belongs to the exponent λ it is equivalent to one of the ideals

$$\phi(\alpha),\quad \phi(\alpha)^2,\quad \phi(\alpha)^3,\quad \ldots,\quad \phi(\alpha)^{\lambda-1}.$$

If $f(\alpha)$ is equivalent to $\phi(\alpha)^m$, then

$$\text{Ind}\, E_\nu(\alpha)\equiv him,\quad \text{mod } \lambda,$$

where the index is taken with respect to the ideal $f(\alpha)$. Since hi is not divisible by λ, then the index of $E_\nu(\alpha)$ is necessarily congruent to 0 or not congruent to 0 according as m is congruent to 0 or not congruent to 0, or, what is the same thing, according as $f(\alpha)$ is an integer or an ideal in the field. Setting $f(\alpha)^\lambda=F(\alpha)$, then

$$\text{Ind}\, E_\nu(\alpha)\equiv\frac{(-1)^{\nu+\mu}(\gamma^{2\nu}-1)B_{\nu\lambda-\mu}}{2\lambda}\cdot\frac{d_0{}^{\lambda-2\nu}lF(e^v)}{dv^{\lambda-2\nu}},\quad \text{mod } \lambda,$$

from which it follows that $\text{Ind}\, E_\nu(\alpha)\equiv 0 \pmod{\lambda}$ only if the second factor of the right-hand member of this congruence is divisible by λ, which completes the discussion of Theorem IV.

Kummer also derives from what has just been given the result that if the λ^{th} power of an ideal belonging to the field $\Omega(\alpha+\alpha^{-1})$ is an integer in that field, then the ideal is also an integer in the field.

He then applies these results to Fermat's last theorem, assuming that $x^\lambda+y^\lambda+z^\lambda=0$ where x, y, z, are rational integers prime to each other and to λ. Assume first that x, y, z are each prime to λ as in Kummer[18]

$$x+\alpha y=\epsilon(\alpha)f(\alpha)^\lambda$$

whence

$$x+\alpha^{\gamma^k}y=\epsilon(\alpha^{\gamma^k})f(\alpha^{\gamma^k})^\lambda$$

where h is not divisible by λ. Giving h all those values in the set 0, 1, 2, $\ldots$, $\lambda-2$, for which

$$\gamma_{\mu-h}+\gamma_{\mu-h+\mathrm{ind}\,r}>\lambda$$

we obtain by multiplication

$$\Pi(x+a^{\gamma^h}y)=\Pi\epsilon(a^{\gamma^h})(\Pi f(a^{\gamma^h}))^{\lambda}$$

where the products range over all the values of h just mentioned. The product $\Pi f(a^{\gamma^h})$ is an integer in the field $\Omega(a)$. Also, he finds that $\Pi\epsilon(a^{\gamma^h})$ is a power of a which gives

$$\Pi(x+a^{\gamma^h}y)=\pm a^k\phi(a)^{\lambda}.$$

Replacing this equality by an identity with e^v in lieu of a, and taking the $(\lambda-2n)$-th derivative of the logarithm of both members and substituting $v=0$, we have

$$\frac{d_0{}^{\lambda-2n}l(x+e^vy)}{dv^{\lambda-2n}}\sum\gamma^{(\lambda-2n)h}\equiv 0, \quad \mathrm{mod}\ \lambda,$$

where the summation extends over all those values of h used in the above product and n is any integer in the set 1, 2, 3, $\ldots$, $\mu-1$.

By previous results it is noted that

$$\sum\gamma^{(\lambda-2n)h}\equiv(-1)^n(1+r^{\lambda-2n}-(r+1)^{\lambda-2n})\frac{B_n}{2n}, \quad \mathrm{mod}\ \lambda,$$

and since we may choose r so that the factor in the above congruence involving $r\not\equiv 0 \pmod{\lambda}$, then for all values of n other than $n=\nu$, by using Assumption I, we have

$$\frac{d_0{}^{\lambda-2n}l(x+e^vy)}{dv^{\lambda-2n}}\equiv 0, \quad \mathrm{mod}\ \lambda.$$

Kummer then examines this congruence for $n=\mu-1$ and $n=\mu-2$, as well as those congruences obtained by substituting z in place of x, x in place of y, etc., and finds that all the congruences cannot be satisfied for either of these two values of n, which establishes a contradiction. These results are equivalent to the statement that if (1) for $p=\lambda$, is satisfied in Case I, then $B_{\mu-1}\equiv B_{\mu-2}\equiv 0 \pmod{\lambda}$. He notes that the first of these criteria is equivalent to that given by Cauchy.[13a]

Kummer then considers (1) for $p=\lambda$, in which x, y, z are prime to each other, $z=\lambda^k z_1$, where z_1 is prime to λ. He uses this relation in the more general form

$$U^{\lambda}+V^{\lambda}=E(a)(2-a-a^{-1})^{m\lambda}W^{\lambda}, \tag{8}$$

where U, V, W are integers in the field $\Omega(a+a^{-1})$ prime to each other and to $(1-a)$, $E(a)$ is a unit in the field $\Omega(a)$, and $m>1$. By a method analogous to that used in Kummer,[18] Case II, we obtain

$$U+a^r V=\epsilon_r(a)(1-a^r)\Theta_r(a)^\lambda, \tag{9}$$

where $r=1, 2, 3, \ldots, \lambda-1$, and

$$U+V=\epsilon(a)(2-a-a^{-1})^{m\lambda-\mu}T(a)^\lambda \tag{10}$$

where $\epsilon_r(a)$ and $\epsilon(a)$ are units in the field $\Omega(a)$, $\Theta_r(a)$ is an ideal in the field $\Omega(a)$, and $T(a)$ is an ideal in the field $\Omega(a+a^{-1})$.

Eliminating U from the last two relations we have

$$V=-\epsilon_r(a)\Theta_r(a)^\lambda+\frac{\epsilon(a)(2-a-a^{-1})^{m\lambda-\mu}T(a)^\lambda}{1-a^r}$$

whence we obtain

$$\frac{d_0{}^{\lambda-2\nu}lV(e^v)}{dv^{\lambda-2\nu}}\equiv\frac{d_0{}^{\lambda-2\nu}l\epsilon_r(e^v)}{dv^{\lambda-2\nu}}+\frac{d_0{}^{\lambda-2\nu}l(\Theta_r(e^v)^\lambda)}{dv^{\lambda-2\nu}}, \quad (\text{mod}\,\lambda),$$

which reduces to

$$\frac{d_0{}^{\lambda-2\nu}l(\Theta_r(e^v)^\lambda)}{dv^{\lambda-2\nu}}\equiv 0, \quad \text{mod}\,\lambda,$$

showing by the use of Theorem IV that $\Theta_r(a)$ is an integer in the field $\Omega(a)$. It also follows that $T(a)$ is an integer in the field $\Omega(a+a^{-1})$.

Setting a^{-1} in lieu of a in the last relation involving V, and comparing with the original relation, we obtain a result which reduces to

$$\Theta_r(a)^\lambda-\Theta_r(a^{-1})^\lambda=\epsilon'(a)(1-a)^{(2m-1)\lambda}T(a)^\lambda,$$

where $\epsilon'(a)$ is a unit in $\Omega(a)$. From this relation we find

$$\Theta_r(a)-a^t\Theta_r(a^{-1})=\epsilon''_t(a)(1-a^t)P_t(a)^\lambda$$

for $t=1, 2, 3, \ldots, \lambda-1$, and

$$\Theta_r(a)-\Theta_r(a^{-1})=\epsilon''(a)(1-a)^{(2m-2)\lambda+1}Q(a)^\lambda,$$

where $P_t(a)$ and $Q(a)$ are ideals in the field $\Omega(a)$, and $\epsilon''_t(a)$, $\epsilon''(a)$, are units in the field $\Omega(a)$. Eliminating $\Theta_r(a^{-1})$ we obtain

$$\Theta_r(a)\equiv\epsilon''_t(a)P_t(a)^\lambda, \quad \text{mod } (1-a)^{(2m-2)\lambda}$$

and

$$\Theta_r(a^{-1})\equiv\epsilon''_t(a^{-1})P_t(a^{-1})^\lambda, \quad \text{mod } (1-a)^{(2m-2)\lambda}$$

whence

$$\Theta_r(a)\Theta_r(a^{-1})\equiv\epsilon''_t(a)\epsilon''_t(a^{-1})(P_t(a)P_t(a^{-1}))^\lambda, \quad \text{mod } (1-a)^{(2m-2)\lambda}$$

From the last relation we obtain by setting $a=e^v$ and changing the relation into an identity, differentiating 2ν times,

$$\frac{d_0{}^{2\nu}l(\Theta_r(e^v)\Theta_r(e^{-v}))}{dv^{2\nu}}\equiv 0, \quad \text{mod}\,\lambda,$$

which shows by Theorem III that $\Theta_r(\alpha)$ and $\Theta_r(\alpha^{-1})$ may be multiplied by a certain unit so that the product is congruent to a rational integer modulo λ.

From relations (9) and (10) we obtain

$$U^2+(\alpha^r+\alpha^{-r})UV+V^2=\epsilon_r(\alpha)\epsilon_r(\alpha^{-1})(2-\alpha^r-\alpha^{-r})(\Theta_r(\alpha)\Theta_r(\alpha^{-1}))^\lambda,$$
$$U^2+(\alpha^s+\alpha^{-s})UV+V^2=\epsilon_s(\alpha)\epsilon_s(\alpha^{-1})(2-\alpha^s-\alpha^{-s})(\Theta_s(\alpha)\Theta_s(\alpha^{-1}))^\lambda,$$
$$U^2+2UV+V^2=\epsilon(\alpha)^2(2-\alpha-\alpha^{-1})^{2m\lambda-2\mu}T(\alpha)^{2\lambda}.$$

Eliminating the two quantities U^2+V^2 and UV, we obtain, after some reductions, and making use of the last mentioned property of $\Theta_r(\alpha)$ and $\Theta_r(\alpha^{-1})$,

$$\mathfrak{e}_r(\alpha)T_r(\alpha)^\lambda-\mathfrak{e}_s(\alpha)T_s(\alpha)^\lambda=\frac{\epsilon(\alpha)^2(2-\alpha-\alpha^{-1})^{2m\lambda-2\mu}(\alpha^r+\alpha^{-r}-\alpha^s-\alpha^{-s})T(\alpha)^{2\lambda}}{(2-\alpha^r-\alpha^{-r})(2-\alpha^s-\alpha^{-s})}$$

where $\mathfrak{e}_r(\alpha)$ and $\mathfrak{e}_s(\alpha)$ are units in the field $\Omega(\alpha)$, $T_r(\alpha)$ and $T_s(\alpha)$ are integers in the field $\Omega(\alpha+\alpha^{-1})$, which are congruent to rational integers modulo λ. It is then shown that the unit $\dfrac{\mathfrak{e}_s(\alpha)}{\mathfrak{e}_r(\alpha)}$ is congruent to a rational integer modulo λ^2, and using Theorem III, we infer that this unit is the λ^{th} power of a unit in the field $\Omega(\alpha)$, hence the last relation may be written

$$U_1^\lambda+V_1^\lambda=E_1(\alpha)(2-\alpha-\alpha^{-1})^{(2m-1)\lambda}W_1^\lambda,$$

where U_1, V_1, W_1 are integers in the field $\Omega(\alpha+\alpha^{-1})$, prime to each other, and $E_1(\alpha)$ is a unit in the field $\Omega(\alpha)$. This equation has the same form as (8) with $2m-1$ in lieu of m. Comparing the value of W_1 with the value of W we see that since $W_1=T(\alpha)^2$ then W_1 necessarily contains a lesser number of distinct prime ideal factors than does W, aside from the exceptional case when W is equal to $T(\alpha)$ times a unit in the field, in which event

$$\frac{U+\alpha^rV}{1-\alpha^r}$$

is a unit in the field $\Omega(\alpha)$ for $r=1, 2, 3, \ldots, \lambda-1$, whence

$$\frac{U+\alpha^rV}{1-\alpha^r}=\frac{\alpha^k(U+\alpha^{-r}V)}{1-\alpha^{-r}}$$

which gives, using $U+V\equiv 0 \pmod{\lambda}$,

$$(1-\alpha^k)(1-\alpha^r)\equiv 0, \quad \operatorname{mod} \lambda,$$

and this is impossible for $\lambda>3$. Hence we obtain by repetitions of our processes a series of integers in the field $\Omega(\alpha)$, W, W_1, W_2, W_3, $\ldots$, in each of which the number of distinct ideal factors is less than in the preceding, which is impossible.

Kummer then tests Assumptions I, II, III, for $\lambda=37$, 59 and 67, and finds all of them satisfied in each case; hence he concludes that (1) is impossible for any odd prime $p=\lambda<100$.

Smith[21] reported on Kummer's[17] derivation of the criterion of the divisibility of the first factor of the class number by λ. In the part of the demonstration where the expression

$$\gamma^{-(2n-1)}\psi(\gamma^{2n-1})$$

is transformed, Smith employs in lieu of Kummer's work the identity

$$\sum_{x=1}^{x=\lambda-1} I\frac{\gamma x}{\lambda}\cdot f(x)+\sum_{x=1}^{x=\gamma-1} F\left(I\frac{\lambda x}{\gamma}\right)=(\gamma-1)F(\lambda-1),$$

where $f(x)$ is any function of x and $F(x)=\sum\limits_{x=1}^{x} f(x)$. He establishes this relation by a geometric method and obtains therefrom Kummer's[18] relation

$$\sum_{x=1}^{x=\gamma-1} F_{2n-1}\left(-\frac{x}{\gamma}\right)=\frac{(-1)^n B_n(\gamma^{2n}-1)}{2n\gamma^{2n-1}}.$$

He also reproduced[21a] Kummer's demonstration that

$$x^\lambda+y^\lambda+z^\lambda=0$$

has no non-zero solutions in rational integers x, y, z, where λ is a regular prime >3.

Smith[22] stated that if λ is an exceptional prime satisfying the Assumptions I, II, III of Kummer,[20] then Kummer showed that the equation

$$x^\lambda+y^\lambda+z^\lambda=0$$

is insolvable for x, y, z integers in the field $\Omega(\alpha+\alpha^{-1})$, one of the integers x, y, z being divisible by $1-\alpha$, α being a primitive λ^{th} root of unity. Kummer's work as it stands, however, only gives this result under the assumption that there is no common factor of all the integers x, y, z.

Kummer[23] considers further the subject of irregular primes, observing: "Eine nähere Untersuchung der besonderen Eigenschaften derjenigen Primzahlen λ, für welche die Klassenzahl durch λ theilbar ist, hat mich schon oft und dauernd beschäftigt." He computes the numerical value of the first factor of the class number of the λ^{th} roots of unity, λ an odd prime, for $100<\lambda<164$. The method employed depends on expressing this first factor, multiplied by $(2\lambda)^{\mu-1}$, $\mu=(\lambda-1)/2$ in the form

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[21] Report on the Theory of Numbers, Collected Works, I, 1894, 114-117. Report of the British Association, Part II, 1860: 136-138.

[21a] L. c., 131-137; 148-152, respectively.

[22] Report on the Theory of Numbers, Collected Works, Vol. I, 134-135, footnote. Report of British Assn., Part II, 1860: 150-151.

[23] Berlin Monatsberichte, 1874: 239-248.

$$P=N\phi(\beta)\cdot N\phi(\beta^{m})\cdot N\phi(\beta^{m'})\ldots$$

where $1, m, m', m'', \ldots$ range over all the odd divisors of $\lambda-1$. It is found that the only irregular primes between the limits mentioned are 101, 103, 131, 149, 157. For $\lambda=157$, the first factor of the class number is divisible by 157^2 and not by 157^3, but in the other cases by the first power of λ only. For $\lambda=101$, the 34th Bernoulli number is divisible by λ; for $\lambda=103$, the 12th Bernoulli number; for $\lambda=131$, the 11th Bernoulli number; for $\lambda=149$, the 65th Bernoulli number; for $\lambda=157$, the 31st and 55th Bernoulli numbers.

Kummer concludes the article with some speculations (p. 248) on the frequency of irregular primes. Cf. Kummer,[30] Jensen,[58] Vandiver,[66] Hensel.[40]

MIRIMANOFF[34] gave a new criterion for the divisibility of the second factor of the class number of the field defined by θ where θ is a root of $x^\lambda=1$, $\theta\neq 1$, λ an odd prime, by λ. Set

$$\lambda=2\nu+1,\qquad e(\theta)=\frac{(1-\theta^{g})(1-\theta^{-g})}{(1-\theta)(1-\theta^{-1})},\qquad e_\kappa=e(\theta^{g^\kappa})$$
$$(\kappa=0,1,2,\ldots,\nu-1),$$

where g is a primitive root of λ. The second factor of the class number is divisible by λ if there exists a set of integral exponents $m_0, m_1, \ldots, m_{\nu-3}, m_{\nu-2}$, such that $e_0^{m_0}e_1^{m_1}\ldots e_{\nu-2}^{m_{\nu-2}}$ is equal to the λ-th power of a unit in the field $\Omega(\theta)$ and not all the m's are divisible by λ. Cf. Kummer,[17] pp. 480-481.

Using KRONECKER'S[34a] notation of symbolic powers, we put

$$e_0^{m_0}e_1^{m_1}\ldots e_{\nu-2}^{m_{\nu-2}}=e^{m_0+m_1x+\ldots+m_{\nu-2}x^{\nu-2}}=e^{m(x)};$$

whence

$$\left(e^{m(x)}\right)^{n(x)}=e^{m(x)\cdot n(x)},$$

where the exponents are reduced by the use of the formula

$$x^{\nu-1}+x^{\nu-2}+\ldots+x+1=0.$$

Let d be an arbitrary divisor of ν; p, q, etc., the distinct prime divisors of d; set $\nu=\delta d$. A unit ϵ is said to belong to the divisor d if

$$\epsilon_\kappa=\epsilon_{d+\kappa}=\ldots=\epsilon_{(\delta-1)d+\kappa},$$
$$\epsilon_\kappa\cdot\epsilon_{d/p+\kappa}\ldots\epsilon_{(p-1)d/p+\kappa}=1,$$
$$\epsilon_\kappa\cdot\epsilon_{d/q+\kappa}\ldots\epsilon_{(q-1)d/q+\kappa}=1,$$

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[34] J. reine angew. Math., 109: 82-88 (1892).

[34a] J. reine angew. Math., 93: 32 (1882).

in which ϵ_κ is put in place of $\epsilon(\theta^{g^\kappa})$. If ϵ belongs to the divisor d, the symbol $\epsilon^{m(x)}$ may be replaced by $\epsilon^{m(r_d)}$, r_d being a primitive root of the equation $z^d-1=0$. Set

$$\overset{(d)}{\epsilon}=e^{(1+x^d+\dots+x^{(\delta-1)d})(1-x^{d/p})(1-x^{d/q})}\dots$$

If $d_1, d_2, \dots, d_\kappa$ are the different divisors of ν, we have the theorem that if $e^{m(x)}$ is the λ-th power of a unit in the field, $m(x)$ being prime to λ, one or more of the units $\overset{(d_i)}{\epsilon}{}^{m(r_{d_i})}$ satisfies the condition

$$\overset{(d_i)}{\epsilon}{}^{m(r_{d_i})}=\lambda^{\text{th}} \text{ power}$$

provided $m(r_{d_i})$ is not divisible by λ, and conversely if the unit $\overset{(d_i)}{\epsilon}{}^{m(r_{d_i})}$ is the λ-th power of a unit in the field and λ does not divide $m(r_{d_i})$, there exists always an exponent $n(x)$ prime to λ and such that $e^{n(x)}$ is the λ-th power of a unit in the field. Hence, if the second factor b of the class number is divisible by λ, we have for a particular i

$$\overset{(d_i)}{\epsilon}{}^{m(r_{d_i})}=\lambda^{\text{th}} \text{ power},$$

the exponent $m(r_{d_i})$ being prime to λ. It is stated that this condition is necessary and sufficient. If $\overset{(d_i)}{\epsilon}{}^{m(r_{d_i})}$ is the λ-th power of a unit in the field, then the unit $\overset{(d_i)}{\epsilon}{}^{Nm(r_{d_i})}$ is also the λ-th power of a unit where $Nm(r_{d_i})$ is the norm of $m(r_{d_i})$ taken with respect to all the primitive roots of $z^{d_i}-1=0$.

Consider the decomposition of λ into prime factors in the field $\Omega(z)$ in Kummer's[11] sense. Further, suppose that these are also integers in the field and let $f_1, f_2, \dots, f_j$ be the factors of this type of λ which also divide $m(r_{d_i})$ and set

$$R(r)=f_1^{n_1}f_2^{n_2}\dots f_j^{n_j},$$

where f_k is contained exactly n_k times in $m(r_{d_i})$, $k=1, 2, \dots, j$, hence

$$\overset{(d_i)}{\epsilon}{}^{R(r)}=\lambda^{\text{th}} \text{ power},$$

and this gives, after some transformations, the result that if $\overset{(d_i)}{\epsilon}{}^{m(r_{d_i})}$ is the λ-th power of a unit in the field such that $m(r_{d_i})$ is not divisible by λ, it is necessary and sufficient that $\overset{(d_i)}{\epsilon}{}^{\frac{\lambda}{f_l}}$ be a λ-th power of a unit in the field for some value of l. Set $\nu=\delta_i d_i$ and let $c_0=1, c_1, c_2, \dots, c_{\psi(d_i)-1}$ be the numbers less than and prime to d_i, then

$$\Pi_j(g^{2\delta_i\kappa}-r^{c_j})\equiv 0 \pmod{\lambda}$$

where r is a primitive root of $x^{d_i}-1=0$ and the κ is prime to d_i.

Mirimanoff then considers the unit obtained by replacing the f's which occur in the relation

$$\overset{(d_i)}{\epsilon}{}^{f_1 f_2 \cdots f_l} = \lambda^{\text{th}} \text{ power}$$

by the corresponding factors $g^{2\delta_i\kappa} - r^{c_j}$. Let ϵ_0 be the unit $\overset{(d_i)}{\epsilon}$ or a symbolic power of this unit, develop ϵ_0 in ascending powers of $t = 1 - \theta$, so that

$$\epsilon_0 \equiv 1 + at^a + bt^{a+1} + \dots \pmod{\lambda}, \qquad a \not\equiv 0 \pmod{\lambda},$$

and we have

$$\epsilon_j = \epsilon_0(\theta^{g^j}) \equiv 1 + ag^{ja}t^a + \dots \pmod{\lambda}$$
$$(j = 1, 2, \dots, \phi(d_i) - 1).$$

These relations are then used to examine the congruence

$$\overset{(d_i)}{\epsilon}{}^{\frac{\lambda}{f_l}} \equiv 1 \pmod{\lambda}$$

which must hold if the left-hand member is the λ-th power of a unit. If the congruence does hold, he observes that if $\gamma_0, \gamma_1, \dots, \gamma_{d_i-1}$ are the periods of $2\delta_i$ terms, and setting

$$M_\kappa = \sqrt[\lambda]{\overset{(d_i)}{\epsilon}{}^{\frac{\lambda}{f_{c_j}}}_{d_i-\kappa}} \qquad (\kappa = 0, 1, 2, \dots, d_i - 1)$$

and

$$M_\kappa = a_0\gamma_{d_i-\kappa} + a_1\gamma_{d_i+1-\kappa} + \dots + a_{d_i-1}\gamma_{2d_i-1-\kappa};$$
$$(\kappa = 0, 1, \dots, d_i - 1)$$

then in order that the unit $\overset{(d_i)}{\epsilon}{}^{\frac{\lambda}{f_{c_j}}}$ is the λ-th power of a unit, it is necessary and sufficient that the coefficients $a_0, a_1, \dots, a_{d_i-1}$ are rational integers.

These criteria are then applied to the particular case $\lambda = 37$ and yield the result that the second factor of the class number of the field defined by a primitive 37th root of unity is prime to 37. Cf. Kummer,[20] p. 73. Other special results, p. 88, concerning units in this field are obtained.

MIRIMANOFF[35] applied the results of his[34] previous paper to the relation

$$x^{37} + y^{37} + z^{37} = 0$$

where x, y, z, are integers in the field defined by $(\theta + \theta^{-1})$, θ being a primitive root of $u^{37} - 1 = 0$ and where z, assumed prime to x and y, is divisible by $\beta = (1 - \theta)(1 - \theta^{-1})$. In lieu of this equation we consider the more general form

$$x^{37} + y^{37} = \epsilon\beta^\kappa z^{37}, \tag{11}$$

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[35] J. reine angew. Math., 111: 26-30 (1893).

where ϵ is some unit in the field $\Omega(\theta+\theta^{-1})$, κ is a multiple of 37, and $x, y, z,$ are defined as above and are also prime to β. Further, assume that the numbers $(x+y)^2$ and xy belong to the field $\Omega(\theta+\theta^{-1}+\theta^{g^9}+\theta^{-g^9})$, g being a primitive root of 37, and we have

$$x^{37}+y^{37}=(x+y)\prod_{i=1}^{i=18}(x^2+y^2+\delta_i xy),$$

where

$$\delta_i=\theta^i+\theta^{-i},$$

whence

$$(12)\qquad \begin{cases} x+y=e_0\beta^{\kappa-18}u_0^{37}, \\ (x+y)^2-(2-\delta_i)xy=e_i\beta u_i^{37}, \end{cases}$$

$$(i=1,\ 2,\ \ldots,\ 18)$$

where e_0, e_i are units and u_0, u_i are integers in the field $\Omega(\theta)$ prime to β. For $i=1$, this gives

$$(13)\qquad (x+y)^2-\beta xy=\epsilon^{36}\beta\nu_1^{37},$$

where ϵ is a unit in the field $\Omega(\theta+\theta^{-1})$ and $\nu_1=e_1u_1$. Putting

$$\frac{\epsilon^{36}}{\epsilon_9{}^6\epsilon_3{}^2}=E(\theta),$$

also, if ϵ_a belongs to the divisor a of 18,

$$x=\epsilon_9{}^3\epsilon_3x'; \qquad y=\epsilon_9{}^3\epsilon_3y'; \qquad z=\epsilon_9{}^3\epsilon_3z',$$

it is shown by means of the results of his previous[26] paper that $E(\theta)$ is the 37-th power of a unit so that (13) becomes

$$(x'+y')^2-\beta x'y'=\beta\xi^{37}.$$

In this relation set θ^{g^9} in place of θ and let β'' and ξ'' be the conjugates of β and ξ. These give, with (12),

$$\xi^{37}+\eta^{37}=\epsilon_0\frac{\beta''-\beta}{\beta''}\beta^{2k-37}\zeta^{37},$$

where $u_0{}^2=\zeta$ and $-\xi''=\eta$ and ϵ_0 is a unit in the field $\Omega(\theta+\theta^{-1})$. This equation belongs to the class of equations (11) and the numbers $(\xi+\eta)^2$ and $\xi\eta$ belong to the field $\Omega(\theta+\theta^{-1}+\theta^{g^9}+\theta^{-g^9})$.

It may be proved by this manner of descent that we reach a contradiction. We have

$$\text{Mod. } (x^2+y^2+\delta_i xy)>\text{Mod. }\frac{(x+y)^2}{2}, \text{ for } \delta_i>0$$

and

$$\text{Mod. } (x^2+y^2+\delta_i xy)>\text{Mod. }\frac{(x-y)^2}{2}, \text{ for } \delta_i<0,$$

and these give

$$\text{Mod. } (\epsilon\beta^{\kappa}z^{37})>\frac{1}{2^{18}}\text{ Mod. } (x+y)^{19}\text{ Mod. } (x-y)^{18}.$$

Similar inequalities are obtained by taking the equations conjugate to (11) and they give on multiplication

$$37^{\kappa}\text{ Mod. }N(z)^{37} > \frac{1}{2^{18.18}}\text{ Mod. }N(x+y)^{19}\text{ Mod. }N(x-y)^{18},$$

where N designates the norm. Then, since

$$x+y=e_0\beta^{\kappa-18}u_0^{37}, \qquad u_0^2=\zeta \qquad \text{and} \qquad \kappa\geqq 37,$$

we have

$$\text{Mod. }N(z^{37}) > \left(\frac{37}{2}\right)^{18.18}\text{Mod. }N(\zeta^{37}),$$

from which we infer that the modulus of the norm of ζ is less than the modulus of the norm of z, and the moduli of the norms of z, ζ, etc., are always greater than $\left(\frac{37}{2}\right)^{18.18}$. Using the method of descent we obtain finally a number ζ such that

$$\text{Mod. }N(\zeta) < \left(\frac{37}{2}\right)^{18.18},$$

which yields the result that the equation (11) is never satisfied if the numbers x, y, z belong to the field $\Omega(\theta+\theta^{-1})$ and if the numbers $(x+y)^2$ and xy belong to the field $\Omega(\theta+\theta^{-1}+\theta^{g^9}+\theta^{-g^9})$, x, y, z being prime to each other in the field $(\theta+\theta^{-1})$.

Next, Mirimanoff considers the equation $x^{37}+y^{37}+z^{37}=0$ in which x, y, z are prime to $a=1-\theta$. Let η_3 be the period $(\theta+\theta^{g^{12}}+\theta^{g^{24}})$ and η_6 the period $\eta_3(\theta)+\eta_3(\theta^{-1})$. It is assumed that the numbers x, y, z are prime to each other and prime to $t=1-\theta$, and belong to the field $\Omega(\eta_6)$. We then have, if a, b and c are rational integers,

$$\left.\begin{aligned} x&\equiv a\\ y&\equiv b\\ z&\equiv c \end{aligned}\right. \quad (\text{mod. } t^6),$$

and

$$x^{37}+y^{37}=(x+y)\prod_{i=1}^{i=11}(x^3+x^2y\eta_3(\theta^{g^i})+xy^2\eta_3(\theta^{-g^i})+y^3)$$

which yields

$$x^3+x^2y\eta_3(\theta)+xy^2\eta_3(\theta^{-1})+y^3=\epsilon u_0^{37},$$

where ϵ is a unit in the field $\Omega(\eta_3)$ and since the number of classes of ideals in this field is not divisible by 37, u_0 is an integer in the field. Replacing θ by θ^{-1} in this relation, we find

$$a^3+a^2b\eta_3(\theta)+ab^2\eta_3(\theta^{-1})+b^3\equiv a^3+ab^2\eta_3(\theta^{-1})+ab^2\eta_3(\theta)+b^3 \quad (\text{mod. } t^6)$$

and therefore

$$(a-b)(\eta_3(\theta)-\eta_3(\theta^{-1}))\equiv 0(\text{mod. } t^6),$$

which yields the result that $(a-b)$ is divisible by 37. In the same way we find

$$a-c\equiv 0 \pmod{37} \qquad \text{and} \qquad b-c\equiv 0 \pmod{37},$$

which are inconsistent with $a+b+c\equiv 0 \pmod{37}$, and Mirimanoff then states the conclusion that $x^{37}+y^{37}+z^{37}=0$ is never satisfied in integers x, y, z belonging to the field $\Omega(\eta_6)$ prime to each other and to $1-\theta$. In particular, these results show that (1) is not satisfied in rational integers for $p=37$.

WENDT[36] proved that if $m=2^{\nu}p^{k}$ can be so chosen that $mp+1$ is a prime not dividing D_m where ν is not divisible by the prime p, then (1) is impossible in Case I. Cf. Vandiver.[34] Here

$$D_m=\begin{vmatrix} 1 & \binom{m}{1} & \binom{m}{2}\cdots\cdots\binom{m}{m-1} \\ \binom{m}{m-1} & 1 & \binom{m}{1}\cdots\cdots\binom{m}{m-2} \\ & \cdots\cdots\cdots\cdots & \\ & \cdots\cdots\cdots\cdots & \\ \binom{m}{1} & \binom{m}{2} & \binom{m}{3}\cdots\cdots \quad 1 \end{vmatrix}$$

HILBERT[37] reproduced Kronecker's derivation of the criterion for the divisibility of the first factor of the class number of the field $k(\zeta)$ where $\zeta=e^{2i\pi/l}$.

He also proved that the second factor of the class number is divisible by l only if the first factor is divisible by l. Let $\gamma_1, \gamma_2, \ldots, \gamma_{l^*}$ be a system of real fundamental units of the field $k(\zeta)$, $s=(\zeta:\zeta^r)$ then we may set, if $l^*=(l-3)/2$,

$$s^t\epsilon=\gamma_1^{m_{1t}}\gamma_2^{m_{2t}}\ldots\gamma_{l^*}^{m_{l^*t}} \tag{14}$$

for $t=0, 1, 2, \ldots, l^*-1$, and the exponents $m_{1t}, m_{2t}, \ldots, m_{l^*t}$ are integers and ϵ is defined by

$$\epsilon=\sqrt{\frac{(1-\zeta^r)(1-\zeta^{-r})}{(1-\zeta)(1-\zeta^{-1})}}$$

whence

$$\log|s^t\epsilon|=m_{1t}\log|\gamma_1|+m_{2t}\log|\gamma_2|+\ldots+m_{l^*t}\log|\gamma_{l^*}|, \tag{14a}$$

where the symbol $\log|\gamma|$ stands for the real value of the logarithm.

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[36] J. reine angew. Math., 113: 335-347 (1894).

[37] Die Theorie der algebraischen Zahlkörper, Jahresber., deut. math. Verein, 1897: 429-432, 435-437.

Set also

$$\epsilon_t = \epsilon^{n_{1t}} (s\epsilon)^{n_{2t}} \ldots (s^{l^*-1}\epsilon)^{n_{l^*t}}$$
$$(t=1,\ 2,\ \ldots,\ l^*)$$

where

$$\epsilon_t = \eta^{(r^2-s)(r^4-s)(r^6-s)\ldots(r^{2t-2}-s)(r^{2t+2}-s)(r^{2t+4}-s)\ldots(r^{l-1}-s)}$$

then

$$\log \epsilon_t = n_{1t} \log |\epsilon| + n_{2t} \log |s\epsilon| + \ldots + n_{l^*t} \log |s^{l^*-1}\epsilon| \tag{15}$$
$$(t=1,\ 2,\ \ldots,\ l^*)$$

where the exponents are symbolic powers. Combining (14a) and (15) gives

$$\log \epsilon_t = M_{1t} \log |\gamma_1| + M_{2t} \log |\gamma_2| + \ldots + M_{l^*t} \log |\gamma_{l^*}|, \tag{15a}$$
$$(t=1,\ 2,\ \ldots,\ l^*)$$

where $M_{1t}, M_{2t}, \ldots, M_{l^*t}$ are known bilinear combinations of the $2l^{*2}$ rational integers $n_{11}, n_{21}, \ldots, n_{l^*, l^*}$; $m_{10}, m_{20}, \ldots, m_{l^*, l^*-1}$. From (14a) and (15) we obtain in each case l^*-1 other equations after we make the substitutions $s, s^2, \ldots, s^{l^*-1}$. Setting

$$R = \left| \begin{matrix} \log |\gamma_1|, & \ldots, & \log |\gamma_{l^*}| \\ \log |s\gamma_1|, & \ldots, & \log |s\gamma_{d^*}| \\ \cdots & \cdots & \cdots \\ \log |s^{l^*-1}\gamma_1|, & \ldots, & \log |s^{l^*-1}\gamma_{l^*}| \end{matrix} \right|$$

$$\Delta = \left| \begin{matrix} \log |\epsilon|, & \log |s\epsilon|, & \ldots, & \log |s^{l^*-1}\epsilon| \\ \log |s\epsilon|, & \log |s^2\epsilon|, & \ldots, & \log |s^{l^*}\epsilon| \\ \cdots & \cdots & \cdots & \cdots \\ \log |s^{l^*-1}\epsilon|, & \log |s^{l^*}\epsilon|, & \ldots, & \log |s^{2l^*-2}\epsilon| \end{matrix} \right|$$

$$\bar{\Delta} = \left| \begin{matrix} \log \epsilon_1, & \log \epsilon_2, & \ldots, & \log \epsilon_{l^*} \\ \log s\epsilon_1, & \log s\epsilon_2, & \ldots, & \log s\epsilon_{l^*} \\ \cdots & \cdots & \cdots & \cdots \\ \log s^{l^*-1}\epsilon_1, & \log s^{l^*-1}\epsilon_2, & \ldots, & \log s^{l^*-1}\epsilon_{l^*} \end{matrix} \right|$$

we obtain

$$\frac{\bar{\Delta}}{R} = \frac{\bar{\Delta}}{\Delta} \cdot \frac{\Delta}{R} = \left| \begin{matrix} M_{11}, & M_{21}, & \ldots, & M_{l^*1} \\ M_{12}, & M_{22}, & \ldots, & M_{l^*2} \\ \cdots & \cdots & \cdots & \cdots \\ M_{1l^*}, & M_{2l^*}, & \ldots, & M_{l^*l^*} \end{matrix} \right| \tag{16}$$

after using the theorem for the multiplication of determinants. Assuming that the first factor of the class number is prime to l, we can then show that the above determinant in the M's is prime to l, for if this determinant is divisible by l, then there exist l^* rational integers N_1, $N_2, \ldots, N_{l^*}$, not all divisible by l so that

$$\sum_{(t)} N_t M_{1t}, \quad \sum_{(t)} N_t M_{2t}, \quad \ldots, \quad \sum_{(t)} N_t M_{l^*t}$$
$$(t=1, 2, \ldots, l^*)$$

are each divisible by l. Using (15a) we have

$$N_1 \log \epsilon_1 + N_2 \log \epsilon_2 + \ldots + N_{l^*} \log \epsilon_{l^*} = l \log E$$

where E is a positive unit in the field $k(\zeta)$, and this gives immediately

$$\epsilon_1^{N_1} \epsilon_2^{N_2} \ldots \epsilon_{l^*}^{N_{l^*}} = E^l.$$

Writing this equation as an identity in an indeterminate x by adding the proper multiple of $(x^l-1)/(x-1)$, setting $x=e^v$, and finding the $2t$-th derivative of the logarithm of each side of the resulting equation with respect to v and setting $v=0$, we have

$$(-1)^{t+l^*} \frac{B_t}{4tr^{2t}} N_t \equiv 0 \pmod{l},$$
$$(t=1, 2, \ldots, l^*),$$

where the B's are the Bernoulli numbers. This gives a contradiction since not all the N's are divisible by l. The desired result then follows from (16). Cf. Mirimanoff,[26] Kummer.[20]

HILBERT[23] proved that if l is a regular prime and α, β, γ, are integers in the field $\Omega(\zeta)$ where

$$\zeta = e^{2i\pi/l},$$

none zero, then the equation

$$\alpha^l + \beta^l + \gamma^l = 0 \tag{17}$$

is never satisfied. He divides the discussion into two cases according as one of the three integers α, β, γ, is or is not divisible by $\mathfrak{l}$ where $\mathfrak{l}$ is the ideal (λ), $\lambda = 1-\zeta$.

In the first case he employs a simple argument to show the truth of the theorem for $l=3$ and 5. Assume $l \geqq 7$. Since any of the integers α, β, γ, may be multiplied by a power of ζ without altering (17), we may assume that α, β, γ, are semi-primary. By means of the decomposition

$$(\alpha+\beta)(\alpha+\zeta\beta)(\alpha+\zeta^2\beta)\ldots(\alpha+\zeta^{l-1}\beta) = -\gamma^l,$$

it is shown that

$$\alpha+\beta = \mathfrak{j}^l \mathfrak{a},$$
$$\alpha+\zeta\beta = \mathfrak{j}_1{}^l \mathfrak{a},$$
$$\alpha+\zeta^2\beta = \mathfrak{j}_2{}^l \mathfrak{a},$$
$$\ldots\ldots\ldots\ldots$$
$$\alpha+\zeta^{l-1}\beta = \mathfrak{j}^l{}_{l-1} \mathfrak{a};$$

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[23] Jahresber. Deut. Math. Verein., 1897: 517-525.

where $\mathfrak{a}$ is the greatest common ideal divisor of α, β and $\mathfrak{j}, \mathfrak{j}_1, \mathfrak{j}_2, \ldots, \mathfrak{j}_{l-1}$ are ideals in $\Omega(\zeta)$.

Since $\alpha+\zeta^{l-1}\beta$ is prime to $\mathfrak{l}$, we can determine a power of ζ, say ζ^*, so that $\zeta^*(\alpha+\zeta^{l-1}\beta)$ is semi-primary and we may set

$$\mu=\frac{\alpha}{\zeta^*(\alpha+\zeta^{l-1}\beta)}, \qquad \rho=\frac{\beta}{\zeta^*(\alpha+\zeta^{l-1}\beta)},$$

whence

$$\mu+\rho=\left(\frac{\mathfrak{j}}{\mathfrak{j}_{l-1}}\right)^l,$$
$$\mu+\zeta\rho=\left(\frac{\mathfrak{j}_1}{\mathfrak{j}_{l-1}}\right)^l,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\mu+\zeta^{l-2}\rho=\left(\frac{\mathfrak{j}_{l-2}}{\mathfrak{j}_{l-1}}\right)^l.$$

As l is a regular prime, it follows that

$$\frac{\mathfrak{j}}{\mathfrak{j}_{l-1}}\sim 1, \qquad \frac{\mathfrak{j}_1}{\mathfrak{j}_{l-1}}\sim 1, \ldots, \qquad \frac{\mathfrak{j}_{l-2}}{\mathfrak{j}_{l-1}}\sim 1,$$

hence

$$\mu+\zeta^u\rho=\zeta^{e_u}\epsilon_u\alpha_u^{\,l}, \qquad (u=0, 1, 2, \ldots, l-2),$$

where e_u is a rational integer, ϵ_u is a real unit in $\Omega(\zeta)$ and α_u is a fraction in $\Omega(\zeta)$ with numerator and denominator prime to $\mathfrak{l}$. Since the lth power of the number α_u is congruent to a rational integer a_u (modulo $\mathfrak{l}^l$), we then have

$$\mu+\zeta^u\rho=\zeta^{e_u}\epsilon_u a_u, \qquad (\mathfrak{l}^l), \qquad (u=0, 1, 2, \ldots, l-2).$$

In this relation use the substitution $(\zeta:\zeta^{-1})$. We obtain a result which, when taken in connection with the above, yields, noting that $\epsilon_u=\epsilon_{-u}$,

$$\mu+\zeta^u\rho\equiv\zeta^{2e_u}\mu'+\zeta^{2e_u-u}\rho' \qquad (\mathfrak{l}^l) \tag{17a}$$
$$(u=0, 1, 2, \ldots, l-2)$$

where μ' and ρ' are the conjugate imaginaries of μ and ρ respectively. Since μ and ρ are semi-primary, then

$$\mu\equiv m, \qquad \rho\equiv r \qquad (\text{mod } \mathfrak{l}^2),$$

where m and r are rational numbers and we have

$$m+\zeta^u r\equiv\zeta^{2e_u}m+\zeta^{2e_u-u}r, \qquad (\mathfrak{l}^2),$$

which gives

$$2e_u(m+r)\equiv 2ru, \qquad (l),$$

and also

$$e_u\equiv ru, \qquad (l), \qquad (u=0, 1, 2, \ldots, l-2).$$

Then (17a) gives, by elimination, a congruence involving ζ only (mod $\mathfrak{l}^l$). This is shown to be impossible for $l \geqq 7$.

For the proof in Case II, assume γ is divisible by $\mathfrak{l}$ so that γ is divisible $\mathfrak{l}^m$ but not by $\mathfrak{l}^{m+1}$, hence we may write

$$\alpha^l + \beta^l = \epsilon\lambda^{lm}\delta^l, \tag{18}$$

where δ is an integer in the field $\Omega(\zeta)$ prime to $\mathfrak{l}$ and $\epsilon = -1$. We shall generalize this relation and prove it is impossible for ϵ any unit in $\Omega(\zeta)$.

We take, as before, α and β as semi-primary, and by taking the above relation modulo $\mathfrak{l}^{l+1}$ it is seen that $m > 1$. Proceeding by a method similar to that used in connection with the first case, we obtain

$$\begin{cases} \alpha+\beta = \lambda^{l(m-1)+1}\mathfrak{j}^l\mathfrak{a}, \\ \alpha+\zeta\beta = \lambda\mathfrak{j}_1{}^l\mathfrak{a}, \\ \dots\dots\dots\dots\dots\dots \\ \alpha+\zeta^{l-1}\beta = \lambda\mathfrak{j}^l{}_{l-1}\mathfrak{a}, \end{cases}$$

where $\mathfrak{j}, \mathfrak{j}_1, \dots, \mathfrak{j}_{l-1}, \mathfrak{a}$, are ideals in $\Omega(\zeta)$ prime to $\mathfrak{l}$. If $l > 3$, we set

$$\mu = \frac{\alpha\lambda}{\alpha+\zeta^{l-1}\beta}, \qquad \rho = \frac{\beta\lambda}{\alpha+\zeta^{l-1}\beta};$$

and obtain from the above relations

$$\begin{cases} \mu+\rho = \lambda^{l(m-1)+1}\left(\dfrac{\mathfrak{j}}{\mathfrak{j}_{l-1}}\right)^l, \\ \mu+\zeta\rho = \lambda\left(\dfrac{\mathfrak{j}_1}{\mathfrak{j}_{l-1}}\right)^l. \\ \mu+\zeta^2\rho = \lambda\left(\dfrac{\mathfrak{j}_2}{\mathfrak{j}_{l-1}}\right)^l, \end{cases}$$

and as in Case I, we have

$$\frac{\mathfrak{j}}{\mathfrak{j}_{l-1}} \sim 1, \qquad \frac{\mathfrak{j}_1}{\mathfrak{j}_{l-1}} \sim 1, \dots, \qquad \frac{\mathfrak{l}_{l-2}}{\mathfrak{j}_{l-1}} \sim 1.$$

The case $l = 3$ in Case II is treated separately and we find for any odd l the relation

$$\begin{cases} \mu+\rho = \dfrac{\epsilon^*\lambda^{l(m-1)+1}\gamma^{*l}}{\nu}, \\ \mu+\zeta\rho = \dfrac{\lambda\alpha^{*l}}{\nu}, \\ \mu+\zeta^2\rho = \dfrac{\epsilon\lambda\beta^{*l}}{\nu}, \end{cases}$$

where $\nu, \alpha^*, \beta^*, \gamma^*$, are integers prime to $\mathfrak{l}$ and ϵ and ϵ^* are units in the field $\Omega(\zeta)$. Elimination of μ and ρ gives

$$\alpha^{*l} + \eta\beta^{*l} = \eta^*\lambda^{l(m-1)}\gamma^{*l},$$

where η and η^* are units in $\Omega(\zeta)$.

Using the theorem that if l is a regular prime and E is a unit in the field $\Omega(\zeta)$ which is congruent to a rational integer modulo l, then E is the l^{th} power of a unit in $\Omega(\zeta)$, we obtain an equation of the same form as (18) with the number $m-1$ occurring in the exponent of λ in lieu of m. Repeating our operations as from the beginning of our treatment of Case II, we obtain ultimately equation (18) with $m=1$, and this states a contradiction.

Note that in Kummer's[18] proof he assumed that the three integers α, β, γ in (17) had no common factor, a restriction which does not occur in Hilbert's argument. Also, Kummer excepts the case $l=3$, which is included in Hilbert's treatment.

Hilbert also proves that the relation

$$\alpha^4+\beta^4=\gamma^2$$

is impossible in integers α, β, γ in the field $\Omega(i)$, $i=\sqrt{-1}$.

Cellerier[19] obtained results which are transformations of the Kummer[13] criteria. He also showed that if

$$x^p+y^p+z^p=0$$

where p an odd prime then

$$B_{\mu-3}\equiv B_{\mu-4}\equiv 0 \pmod{p},$$

where the B's are Bernoulli numbers with the exception of a finite number of specified values of p. Cf. Kummer,[20] Mirimanoff.[22]

Maillet[20] proved that if λ^{t-1} is the highest power of the prime λ which divides the class number H of ideals in the field defined by a primitive λ-th root of unity, then the equation

$$x^{\lambda^t}+y^{\lambda^t}+z^{\lambda^t}=0$$

is impossible in integers, in the field formed by this root, prime to each other and to λ for $\lambda>3$. The demonstration of this result is an immediate extension of the argument employed by Kummer for $t=1$.

Maillet[21] obtained a number of results concerning the equation

$$x^\lambda+y^\lambda=cz^\lambda$$

where λ is a regular prime, using in the main Kummer's[20] methods. He first establishes the lemma that the equation

$$(19)\qquad u^\lambda+v^\lambda=E(\alpha)(1-\alpha)^{\mu\lambda-\beta}Aw^\lambda \qquad (\mu>0,\ \beta=0 \text{ or } 1)$$

is impossible if u, v are integers in the field $\Omega(\alpha)$, and w an ideal in the field, prime each to each and to λ, where α is a primitive λ-th root of unity, $E(\alpha)$ is a unit in the field $\Omega(\alpha)$, A is an ideal in the field $\Omega(\alpha)$ prime to λ and equal to 1, or of the form $q_1^{a_1}q_2^{a_2}\ldots q_i^{a_i}$, where q_1, q_2, $\ldots$, q_i are distinct ideal primes with $i\leqq\lambda-3$. We have

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[19] Mem. soc. phys. d'Hist. nat. Geneve, 32, No. 7: 16-42 (1894-1897).

[20] Association Francaise pour l'avancement des sciences, 26th Session, 1897: 162-166.

[21] Acta Math., 24: 247-256 (1901). Compt. Rend., 129: 198-199 (1899).

$$(20)\qquad u^\lambda+v^\lambda=(u+v)(u+\alpha v)\ldots(u+\alpha^{\lambda-1}v)\,;$$

and if we put

$$u=a+(1-\alpha)^2Q,$$
$$v=b+(1-\alpha)^2R,$$

where a, b are rational integers, which is always possible, we have

$$u+v\equiv 0 \mod (1-\alpha)^2.$$

Now $u+\alpha^r\cdot v$ and $u+\alpha^n\cdot v$, $(r\neq n)$, have for their greatest common divisor $(1-\alpha)$, hence we can find among the factors in the second member of (20) $u+\alpha^r\cdot v$, $u+\alpha^s\cdot v$ $(r>0,\ s>0,\ r\neq s)$, which are not divisible by any of the numbers q_1, q_2, $\ldots$, q_i, such that

$$(21)\qquad \begin{cases} u+\alpha^r v=(1-\alpha)e_r(\alpha)t_r^\lambda(\alpha),\\ u+\alpha^s v=(1-\alpha)e_s(\alpha)t_s^\lambda(\alpha),\end{cases}$$

where $e_r(\alpha)$ and $e_s(\alpha)$ are units in the field $\Omega(\alpha)$, $t_r^\lambda(\alpha)$ and $t_s^\lambda(\alpha)$ are the λ-th powers of integers in the field prime to each other. We also have

$$(22)\qquad u+v=(1-\alpha)^{\mu\lambda-\lambda+1-\beta}E'(\alpha)A_1w_1(\alpha)^\lambda,$$

where $E'(\alpha)$ is a unit in the field, $A_1w_1(\alpha)^\lambda$ is an integer in the field $\Omega(\alpha)$, t_r, t_s, w_1 are prime each to each and to λ. The relation (19) is impossible for $\mu=1$: suppose $\mu>1$, then (21) and (22) give

$$\begin{vmatrix} 1 & \alpha^r & e_rt_r^\lambda \\ 1 & \alpha^s & e_st_s^\lambda \\ 1 & 1 & (1-\alpha)^{\mu\lambda-\lambda-\beta}E'A_1w_1^\lambda \end{vmatrix}=0,$$

or

$$t_r^\lambda-\epsilon t_s^\lambda=E_1(\alpha)A_1(1-\alpha)^{\mu\lambda-\lambda-\beta}w_1^\lambda,$$

where

$$\epsilon=\frac{e_s(1-\alpha^r)}{e_r(1-\alpha^s)}$$
$$E_1=\frac{E'(\alpha^r-\alpha^s)}{e_r(1-\alpha^s)}$$

ϵ and E_1 being units in the field. This gives

$$t_r^\lambda-\epsilon t_s^\lambda\equiv 0 \quad (\mathrm{mod}\ \lambda),$$

and we also have

$$t_r^\lambda\equiv c,\qquad t_s^\lambda\equiv c' \quad (\mathrm{mod}\ \lambda)$$

where c and c' are rational integers, and these give

$$c-\epsilon c' \equiv 0 \pmod{\lambda},$$

hence ϵ is the λ-th power of a unit ϵ' in the field. Setting

$$t_r = u_1, \qquad -\epsilon' t_s = v_1,$$

we obtain

$$u_1^\lambda + v_1^\lambda = E_1 A_1 (1-a)^{\mu\lambda-\lambda-\beta} w_1^\lambda,$$

and this equation is of the same form as (19) except that the power of $(1-a)$ has been diminished by λ. The repetition of this process leads to a contradiction.

The theorem just proved is then applied to various equations of the type $x^\lambda + y^\lambda = cz^\lambda$, which is shown to be impossible for a number of different rational integral values of c; x, y and z being rational integers.

GAMBIOLI[32] gave an abstract of a paper by Kummer,[12] a list of references on Bernoulli numbers and ideal numbers.

MIRIMANOFF[33] showed that the result of Kummer[20] (p. 65) is equivalent to the following theorem:

If (1) is satisfied in Case I, then

$$(23) \qquad \begin{cases} B_{\frac{(p-i)}{2}} f_i(t) \equiv 0 \pmod{p} \\ \qquad f_{p-1}(t) \equiv 0 \pmod{p} \end{cases}$$

$$i = 3, 5, \ldots, p-2; \qquad -t = x/y,\ y/x,\ x/z,\ z/x,\ y/z,\ z/y.$$

Also

$$f_i(t) = \sum_{k=1}^{p-1} k^{i-1} t^k,$$

$i = 2, 3, \ldots, p-1$.

He also derived the criteria (equivalent to (23))

$$(23a) \qquad f_{p-1}(t) \equiv f_{p-i}(t) f_i(t) \equiv 0 \pmod{p},$$

$i = 2, 3, \ldots, \nu$; $\nu = \dfrac{p-1}{2}$, and

$$B_{\nu-3} \equiv B_{\nu-4} \equiv 0 \pmod{p}.$$

Cf. Cellerier,[39] Wieferich,[37] Mirimanoff,[41] Frobenius,[38] Vandiver,[53, 63, 73, 77, 83, 86] Furtwängler,[46] Pollaczek.[61]

DICKSON[34] discussed the problem: Given an odd prime n to find the odd prime moduli p for which $\xi^n + \eta^n + \zeta^n \equiv 0 \pmod{p}$ has no solutions each prime to p. He also proved that the above relation, for n and p odd primes, has integral solutions each prime to p if

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[32] Periodico di Mat., 16: 145-192 (1901); 17: 48-50 (1902).

[33] J. reine angew. Math., 128: 45-68 (1905).

[34] J. reine angew. Math., 135: 134-141, 181-188 (1909).

$$p \geqq (n-1)^2(n-2)^2+6n-2.$$

Cf. Legendre,[1] Wendt,[28] Furtwängler,[50] Vandiver.[84]

DICKSON[35] showed that (2) is not satisfied for various values of n and p, and using Sophie Germain's theorem (Legendre[1]), he proved that (1) is not satisfied in Case I for any prime exponent <7000. Cf. Beeger.[79]

SCHÖNBAUM[36] reproduced Hilbert's proof of Fermat's last theorem for regular primes and reviewed some of the results in the theory of cyclotomic fields which lead up to it. He noted that it follows from Kummer,[23] that there are five irregular primes >100 and <167. Cf. Vandiver.[86]

WIEFERICH[37] proved by means of the criteria (23) that if (1) is satisfied in Case I, then $2^{p-1} \equiv 1 \pmod{p^2}$.

LINDEMANN[38] attempted a proof of Fermat's last theorem, using algebraic numbers. Assume $x^n-y^n-z^n=0$, n an odd prime. On page 74, the relation 302 is proved only if n is a regular prime as it is obtained from

$$\mathfrak{p}^n \equiv \mathfrak{p} \pmod{1-\zeta}\,; \qquad \zeta = e^{2i\pi/n},$$

and this cannot be given any meaning unless $\mathfrak{p}$ is a principal ideal.

KUMMER[39] mentioned results which he had obtained concerning irregular primes and Fermat's last theorem, in letters to Kronecker. All of said results are included in Kummer,[17, 18, 20] with the exception of the following statements which occur without proof in the letter dated December 28, 1849:

If a unit exists which is congruent to a rational integer c_1, modulo λ and B_n, $B_{n'}$, $B_{n''}$, ... are the Bernoulli numbers in the set B_1, B_2, ..., $B_{\mu-1}$ whose numerators are divisible by λ, where

$$\mu = \frac{\lambda-1}{2},$$

then

$$e(a)^{rm(\beta)}e(a)^{r'm'(\beta)}e(a)^{r''m''(\beta)} \ldots \equiv c \pmod{\lambda},$$

where c, r, r', r'', etc., are certain integers and

$$m(\beta) = 1+\gamma^{-2n}\beta+\ldots+\gamma^{-2n(\mu-1)}\beta^{\mu-1},$$
$$m'(\beta) = 1+\gamma^{-2n'}\beta+\ldots+\gamma^{-2n'(\mu-1)}\beta^{\mu-1},$$

and similarly for $m''(\beta)$.

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[35] Messenger of Math., (2), 38: 14-32 (1908). Quar. J. Math., 40: 27-45 (1908).

[36] Casopsis, Prag, 37: 384-506 (1908).

[37] J. reine angew. Math., 136: 272-292 (1909).

[38] Über den sogenannten letzten Fermatschen Satz. Leipzig, Veit and Co., 1909. 83 p.

[39] Abhandlungen zur Geschichte der Mathematik. 29: 75, 84-85, 88, 91 (1910).

In this formula Kummer is using Kronecker's[34a] notation of symbolic powers and β stands for the substitution (a/a^{γ}), γ being a primitive root of λ. The unit above which is congruent to c, modulo λ, is not the λ-th power of a unit unless each of the units

$$e(a)^{rm(\beta)},\ e(a)^{r'm'(\beta)},\ e(a)^{r''m''(\beta)},\ \ldots,$$

is the λ-th power of a unit. Kummer refers to this result as follows: "Sehr wichtiger satz, welcher eine grosse Schwierigkeit hebt!"

Hensel[40] commented on Kummer's various results concerning Fermat's last theorem.

Mirimanoff[41] making use of the criteria (23) proved that if (1) is satisfied in Case I, then

$$\text{(23b)} \qquad \prod_{i=1}^{m-1}(a_i - t)\sum_{i=1}^{m-1}\frac{R_i}{a_i - t} \equiv 0 \pmod{p},$$

$$R_i = \frac{f_{p-1}(a_i)}{(1-a_i)^{p-1}}$$

where $a_1, a_2, \ldots, a_{m-1}$ are the roots $\neq 1$ of $z^m = 1$. For m = 2 and 3 we obtain $2^{p-1} \equiv 3^{p-1} \equiv 1 \pmod{p^2}$.

Got[42] reproduced with explanations Kummer's memoir of 1857 on Fermat's last theorem. No errors are noted, but compare pages 43-44 with Vandiver,[74] page 404.

Furtwängler[43] gave many properties of ideals and units in relative irregular cyclotomic fields in connection with investigations concerning the class field and laws of reciprocity. Full reports on these papers will be given in later sections of the report of Algebraic Numbers and we shall state here only those results of Furtwängler which have been used by various writers in developing properties of the class number and results on Fermat's last theorem.

In the following definitions and theorems, k is a field containing the field defined by a primitive l-th root of unity ζ, as a sub-field, where l is an odd prime and $\lambda = (1-\zeta)$.

Definition 1. An integral or fractional number a of k whose numerator and denominator are prime to l is called primary when the congruence $x^l \equiv a$ modulo $l\lambda$ has a solution x in the field k.

Definition 2. An integral or fractional number a of k whose numerator and denominator are prime to l is called hyper-primary if the congruence

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[40] Abhandlungen zur Geschichte der Mathematik, 29: 18-32 (1910).

[41] Compt. Rend., 150: 204-206 (1910). J. reine angew. Math., 139: 309-324 (1911).

[42] Annales de la faculté des sciences de Toulouse, 25: 21-62 (1911).

[43] Math. Ann., 58: 1-50 (1904); 63: 1-37 (1906); 67: 1-29 (1909); 72: 346-386 (1912). Göttingen Nachr., Math.-Physik. Kl., 1911: 293-317. (These papers are cited as F. I-V respectively.)

$x^l \equiv \alpha$ modulo $l\lambda\mathfrak{L}$ has a solution x in k. The $\mathfrak{L}$ is defined as the product $\mathfrak{L}=\mathfrak{L}_1\mathfrak{L}_2\ldots\mathfrak{L}_z$ where $(\lambda)=\mathfrak{L}_1^{a_1}\mathfrak{L}_2^{a_2}\ldots\mathfrak{L}_z^{a_z}$ is the decomposition of the ideal (λ) in the field k, the $\mathfrak{L}$'s being distinct prime ideals in k. (F. I., §13, Def. 12.)

Definition 3. An integral or fractional number ω in k is called a singular primary number of k when it is primary and in addition the principal ideal (ω) is the l-th power of an ideal in k without ω being the l-th power of a number in k. (F. II, §3, p. 9.)

Let the class number of the field k be $H=l^h q\,(q \not\equiv 0$, modulo $l)$. Let also a normal basis for the group of the q-th powers of the ideal classes of k be $c_1, c_2, \ldots, c_e$ whose orders are respectively $l^{h'_1}, \ldots, l^{h'_e}$, that is, the q-th power of any ideal class in the field can be represented in the form $c_1^{j_1}c_2^{j_2}\ldots c_e^{j_e}$ where

$$c_s^{l^{h_s}} \sim 1$$
$$(s=1, 2, \ldots, e).$$

We then have the theorems:

Theorem A: In the field k there are exactly e singular primary numbers $\omega_1, \omega_2, \ldots, \omega_e$ which are, aside from l-th powers, independent under multiplication, that is, from any equation

$$\omega_1^{x_1}\omega_2^{x_2}\ldots\omega_e^{x_e}=\alpha^l,$$

where α is a number in k we have $x_1 \equiv x_2 \equiv \ldots \equiv x_e \equiv 0$, modulo l. (F. II, §7, p. 18.)

Theorem B: Let ω_0 be a singular primary number in k, then there belongs to ω_0 a uniquely determined subgroup $\mathfrak{G}_{\omega_0}$ of order H/l of the complete class group $\mathfrak{H}$ of k with the property that for all ideals $\mathfrak{j}_{\omega_0}$ of the group $\mathfrak{G}_{\omega_0}$

$$\left(\frac{\omega_0}{\mathfrak{j}_{\omega_0}}\right)=1$$

is satisfied while for all other ideals $\mathfrak{j}$ of $\mathfrak{H}$

$$\left(\frac{\omega_0}{\mathfrak{j}}\right)\neq 1$$

will hold. All singular primary numbers which belong to $\mathfrak{G}_{\omega_0}$ in this sense may be expressed in the form $\omega_0^x\alpha^l$, α being a number in k. (F. III, §10, Th. 10, p. 26); (F. III, §4, Th. 6, p. 364). The symbols in the last relation represent l-th power characters.

Theorem C: When c is an ideal class that does not belong to the subgroup $\mathfrak{G}_{\omega_0}$ of Theorem B, and if $\mathfrak{H}$ is expressed as the totality of accessory groups (Nebengruppen) with respect to $\mathfrak{G}_{\omega_0}$, that is,

$$\mathfrak{H}=\mathfrak{G}_{\omega_0}+c\mathfrak{G}_{\omega_0}+\ldots+c^{l-1}\mathfrak{G}_{\omega_0}$$

then $\left(\frac{\omega_0}{\mathfrak{j}}\right)$ assumes the same value for all ideals belonging to the same accessory groups; in particular $\left(\frac{\omega_0}{\mathfrak{j}}\right)$ has the same value for two equivalent ideals $\mathfrak{j}$.

Theorem D: If the number μ of k is not a singular primary number and not an l-th power in k, then in every ideal class of k there exist prime ideals $\mathfrak{j}_p$, for which $\left(\frac{\mu}{\mathfrak{j}_p}\right)=1$.

Let K be the field obtained by adjoining to k the number $\sqrt[l]{\mu}$ where μ is a number defined as in theorem D. Let S be the substitution of the relative group of K, $S=(\sqrt[l]{\mu}:\zeta_1\sqrt[l]{\mu})$. If $\mathfrak{j}_p$ is an ideal in k, prime to l then it follows from $\left(\frac{\mu}{\mathfrak{j}_p}\right)=1$ that $\mathfrak{j}_p$ decomposes into the product of l prime ideals in $K(\sqrt[l]{\mu}, k)$ and from $\left(\frac{\mu}{\mathfrak{j}_p}\right)\neq 1$ we infer that $\mathfrak{j}_p$ is prime in the field $K(\sqrt[l]{\mu}, k)$, hence Theorem B gives

Theorem E: If ω_0 is a singular primary number in k, then only those prime ideals $\mathfrak{j}_p$ in k decompose into l distinct prime ideals in $K(\sqrt[l]{\omega_0}, k)$ which belong to the ideal classes of the subgroup $\mathfrak{G}_{\omega_0}$. (F. V, §1, Th. III.)

From Theorem D we obtain

Theorem F: If the number μ in k is not a singular primary number and not the l-th power of a number in k, then there exist prime ideals in each ideal class of k which are composite in K.

If $c_1, c_2, \ldots, c_h$ are those classes of a relative cyclic field $K(\sqrt[l]{\mu}, k)$ which contain the ideals of k and let C be an arbitrary ideal class of K, a system of ideal classes $P=\{Cc_1, \ldots, Cc_{h'}\}$ will be designated as the class-complex generated by C and the system $I=\{c_1, \ldots, c_{h'}\}$ will be termed the principal complex. The class C_1C_2 defines the complex P_1P_2.

Definition 4. The complex P_A which is identical with the relative conjugate complex $P_A{}^S$ is called an ambiguous complex. A complex is evidently ambiguous if and only if for one of its generated classes C_A we have $C_A{}^{1-S}=c_i$ where c_i is a class of I. From

$$C_A^{1-S}=c_i \qquad \text{and} \qquad C_A^{1+S+\ldots+S^{l-1}}$$

we have

$$C_A{}^l=c_i \qquad \text{and} \qquad P_A{}^l=I.$$

We call any ambiguous complex $P_{A,1}, P_{A,2}, \ldots, P_{A,n}$, independent if from any relation $P_{A,1}^{x_1}P_{A,2}^{x_2}\ldots P_{A,n}^{x_n}=I$ we have

$$x_1\equiv x_2\equiv \ldots \equiv x_n\equiv 0, \text{ mod } l$$

The greatest number of distinct independent complexes in K will be indicated by a. We then have, if e has the same meaning as in Theorem B,

Theorem G: If ω_0 is a singular primary number in k, then in $K(\sqrt[l]{\omega_0}, k)$,

$$a = e - 1$$

(F. II, §8, Th. 6, p. 19).

Theorem H: If the number μ is a non-singular primary number in k, then in $K(\sqrt[l]{\mu}, k)$,

$$a \geqq e.$$

(F. III, §2, pp. 8, 9.)

Theorem J: If the relative discriminant of $K(\sqrt[l]{\mu}, k)$ does not contain two distinct ideal prime factors in k, then

$$a = e.$$

(F. III, §2, pp. 8, 9.)

Let the class number H_K of $K(\sqrt[l]{\mu}, k) = l^m Q$ $(Q \not\equiv 0, \bmod l)$, then we have

Theorem K: The system of the Q-th powers of the ideal classes of $K(\sqrt[l]{\mu}, k)$ can be expressed uniquely in the form

$$C_1^{F_1(S)} C_2^{F_2(S)} \dots C_a^{F_a(S)} c$$

where $C_1, C_2, \dots, C_a$ constitute a certain set of a classes of K and c is the Q-th power of a class included in the principal complex. For these classes C_i we have

$$C_i^{(1-S)^{p_i}} = c_i, \qquad C_i^{(1-S)^{p_i-1}} \neq c$$
$$(i = 1, 2, \dots, a)$$

for rational integers $p_i > 0$. The expressions $F_i(S)$, $i = 1, 2, \dots, a$ are polynomials in S with rational integral coefficients. (F. II, §6, Th. 4, p. 15.)

Hecke[44] considered the properties of irregular fields and their applications to Fermat's last theorem. Let H be the class number of the field K defined by a primitive l-th root of unity, l being an odd prime, $H = h_1 \cdot h$ where h is the class number of the real sub-field k of K of degree $(l-1)/2$. Let l^a be the highest power of l dividing h and l^{a_1} the highest power of l dividing h_1, and set

$$h = l^a q, \qquad h_1 = l^{a_1} q_1.$$

Consider the group of all ideal classes in the field k which are q-th powers so that for each class c in k we have the equivalence

$$c^q \sim c_1^{x_1} c_2^{x_2} \dots c_e^{x_e} \qquad (x_i = 0, 1, \dots, l^{b_i} - 1).$$

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[44] Göttingen Nachr., Math.-Physik. Kl., 1910: 420-424.

Also, if G is the class group of those ideals in K whose relative norms with respect to k belong to the principal class, then G is of degree h_1, hence there exists classes $C_1, C_2, \ldots, C_e$ so that for any class C in the group G we have

$$C^{q_1} \sim C_1^{y_1} C_2^{y_2} \ldots C_{e_1}^{y_{e_1}} \qquad (y_i = 0, 1, \ldots, l^{b_i} - 1).$$

The group of all classes C^{q_1} is called the group G_1. It is then shown that

$$e_1 \geqq e.$$

If $q_1 \cdot q = q_2$, then the q_2-th power of all classes in K may be expressed in the form

$$C \cdot c_1^{x_1} c_2^{x_2} \ldots c_e^{x_e}$$

where C is a class in G_1.

Hecke then constructs with respect to k for $e > 0$ the relative cyclic enlarged field of relative degree l that is indecomposable (unverzweigt), (cf. Furtwängler[43] II, 2, 21-2), which leads to the proof of the theorem. Using this result it is then proved that if $e_1 = e$, then the relation $x^l + y^l + z^l = 0$ is not satisfied with integers in K prime to l, and also, this relation is not satisfied under the conditions mentioned if the first factor of the class number is divisible by l but not by l^2. Cf. Hecke[33] of Chapter I of this report; Furtwängler,[46] Bernstein,[45] of the present chapter.

Bernstein[45] proved that (1) is impossible in Case I under the assumptions

I. The second factor $h_2 = l^\mu h'_2$ of the class number, $h = h_1 h_2$, of the cyclotomic field $k(\zeta)$ defined by a primitive l-th root of unity, l being an odd prime, is divisible by l.

II. In the sub-class field (Teilklassenkörper) of l^μ-th degree which belongs to the factor l^μ of h_2, all ideals of $k(\zeta)$ whose l-th powers are principal ideals in $k(\zeta)$ are themselves principal ideals.

It was shown, however, by Pollaczek[76] that this result is equivalent to that of Hecke[44] (cf. also Vandiver[86]).

Furtwängler[46] examined the equation

$$\alpha^l + \beta^l + \gamma^l = 0 \tag{24}$$

where l is an odd prime and α, β, γ are three integers in the field $k(\zeta)$, where ζ is a primitive l-th root of unity. An ideal $\mathfrak{q}$ in the field $k(\zeta)$

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[45] Göttingen Nachr., Math.-Physik. Kl., 1910: 482-488.

[46] Göttingen Nachr., Math.-Physik. Kl., 1910: 554-562.

is said to belong to the exponent n when it is prime to the ideal $\mathfrak{I}=(\zeta-1)$ and a number k exists in the field such that

$$\kappa=\mathfrak{q}^l$$

and

$$\kappa\equiv r_1(\mathfrak{I}^n),$$

and if there is no unit η in $k(\zeta)$ such that

$$\eta\kappa\equiv r_2(\mathfrak{I}^{n+1})$$

where r_1, r_2, are rational numbers. (This use of the phrase "belongs to the exponent n should not be confused with the other use made of this term in the present report; cf. Kummer.[13])

The equation (24) gives, if $\mathfrak{a}$ is the greatest common ideal divisor of α and β,

$$\alpha+\zeta^i\beta=\mathfrak{q}_i{}^l\mathfrak{a} \qquad (i=0,\ 1,\ \ldots,\ l-1)$$

and setting (cf. Hilbert [38])

$$\alpha+\zeta^{l-1}\beta=\rho,$$

we have, if ϵ_i is a unit in $k(\zeta)$, $\alpha^*=\alpha/\rho$, $\beta^*=\beta/\rho$, $\kappa_i=\mathfrak{q}_i{}^l$,

$$\alpha^*+\zeta^i\beta^*=\epsilon_i\kappa_i \qquad (i=0,\ 1,\ \ldots,\ l-1),$$

and to simplify the notation we write this as

$$\alpha+\zeta^i\beta=\epsilon_i\kappa_i \qquad (i=0,\ 1,\ \ldots,\ l-1).$$

Assume first that in the field $k(\zeta)$ no ideal exists which belongs to the exponent 3. If r is a primitive root of l and s is the substitution (ζ/ζ^r), then

$$(\alpha+\zeta^i\beta)^{(s-r)(s-r^3)}\equiv a_0(\mathfrak{I}^4)$$
$$(i=0, 1, \ldots, l-1)$$

where the exponent in the left-hand member is to be interpreted symbolically and a_0 is a rational number. From (24) we have

$$(s\alpha)^l+(s\beta)^l+(s\gamma)^l=0,$$

and writing α and β as functions of ζ, we have

$$[\alpha(\zeta^m)+\zeta^i\beta(\zeta^m)]^{(s-r)(s-r^3)}\equiv a_0(\mathfrak{I}^4)$$
$$(i=0, 1, \ldots, l-1)$$
$$(m=1, 2, \ldots, l-1).$$

This relation is simplified by expressing the left-hand member in ascending powers of λ where $\lambda=\zeta-1$, giving

$$(a+\zeta^i b)^{(s-r)(s-r^3)}\equiv a_0(\mathfrak{I}^4)$$
$$(i=0, 1, \ldots, l-1)$$

for $\alpha \equiv a$, $\beta \equiv b$, modulo $\mathfrak{l}$; a, b being rational numbers, and this yields

$$\left[\frac{d^3 \log (a+\zeta^i b)^{(s-r)(s-r^2)}}{(d\zeta)^3}\right]_{\zeta=1} \equiv 0 \quad (l)$$

$$(i=0, 1, \ldots, l-1).$$

Setting $\zeta^i = e^v$, we have

$$\frac{d_0{}^3 \log (a+e^v b)}{(dv)^3} \equiv 0 \quad (l),$$

and this relation is impossible if we replace the number pair (a, b) by (a, c) and (b, c).

This work is then extended, and if we assume that in the field $k(\zeta)$ no ideal exists which belongs to the exponent $2j+1\left(j \leqq \frac{l-3}{2}\right)$, then

$$(a+\zeta^i b)^{g(s)} \equiv a_0(\mathfrak{l}^{2j+2}) \qquad (i=0, 1, \ldots, l-1),$$

which gives

$$\frac{d_0{}^{2j+1} \log (a+e^v b)^{g(s)}}{(dv)^{2j+1}} \equiv 0 \quad (l),$$

where

$$g(s) = (s-r)(s-r^2)\ldots(s-r^j)$$

whence

$$\frac{d_0{}^{2j+1} \log (a+e^v b)}{(dv)^{2j+1}} \equiv 0 \quad (l).$$

According to Kummer and Mirimanoff this congruence is not satisfied for $j=1, 2, 3, 4$, which proves that if in the field $k(\zeta)$ there exists no ideal belonging to the exponents 3, 5, 7, 11, modulo $\mathfrak{l}$, then (24) is not satisfied for any integers in the field $k(\zeta)$ prime to (l). Using this result it is then shown, through a consideration of the group of classes of $k(\zeta)$, that if the number class sets (Klassenverbände) of $k(\zeta)$ is l^f and $k'(\zeta+\zeta^{-1})$ is l^e, and if $f-e \leqq 3$, then (24) cannot be satisfied by integers in the field $k(\zeta)$ prime to l and in particular if the class number of the field $k(\zeta)$ is divisible by l^3 but not by l^4, then (24) is not satisfied by integers in the field $k(\zeta)$ prime to l. Cf. Hecke.[44]

Bernstein[47] investigated Case II of Fermat's last theorem. He quotes (p. 508) Kummer's results concerning the last theorem and states Kummer's three assumptions. He then remarks that the first assumption is equivalent to the statement that the first factor of the class number of the field $k(\zeta)$, ζ being a primitive l-th root of unity, l an odd prime, is divisible by l but not by l^2. Concerning this statement, cf. Vandiver, Bachmann,[40] review.

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[47] Göttingen Nachr., Math.-Physik. Kl., 1910: 507-516.

Bernstein then states

Theorem I. The class field of the l-th degree of $k(\zeta)$ is obtained from a unit ϵ.

For proof, he refers to HILBERT.[43] He proves Theorem II: Every primary unit is real. He then gives Theorem III: Under the assumption that the class number of the cyclotomic field $k(Z)$ formed by a primitive l^2-th root of unity, is divisible by l but not by l^2, then if

$$\epsilon \equiv \alpha^{l^2} \pmod{l^2\mathfrak{l}}, \tag{25}$$

where $\mathfrak{l} = (1-\zeta)$ and α is an integer or fraction in the field $k(\zeta)$, then ϵ is the l-th power of a unit.

For proof it is observed that ϵ is a primary number, so that $E=\sqrt[l]{\epsilon}$ defines an indecomposable (unverzweigten) field. From (25) we may write

$$\epsilon \equiv a^{l^2} \pmod{l^2\mathfrak{l}},$$

where a is a rational number and we may take ϵ and E real, and then from (25) we have the decomposition

$$\mathfrak{l} = \mathfrak{L} \cdot S\mathfrak{L} \ldots S^{l-1}\mathfrak{L}$$

in the field $k(\sqrt[l]{\epsilon})$ where $\mathfrak{L}$ is defined as in Hilbert (l. c., p. 400). We also have

$$E - \zeta^u a^l \equiv 0 \qquad \left(\operatorname{mod} \mathfrak{l} \cdot (S^u\mathfrak{L})^{l-1}\right)$$

or

$$E \equiv (Z^u a)^l \qquad \left(\operatorname{mod} (S^u\mathfrak{L})^l\right) \qquad (u=0, 1, \ldots, l-1).$$

It is then shown from this that the class number of the field $k(Z)$ is divisible by l^2, which is contrary to the assumption and the theorem follows.

Using these theorems, Bernstein, following in the main the method of Kummer,[30] proved that under the assumption that the class number of the field $k(Z)$ is divisible by l and not by l^2, then

$$a^l + b^l + c^l = 0 \tag{26}$$

is not possible in rational integers a, b, c, none zero, and one of them divisible by l.

He then proves

Theorem IV. Under the assumption that the field $k(\zeta)$ contains no class belonging to the exponent l^2, and the class number h_2 of the field $k(\zeta+\zeta^{-1})$ is prime to l, then every primary number which is the l-th power of an ideal in the field is the product of a primary unit and the l-th power of a number in the field.

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[43] Jahresber. deut. math. Verein, 4: 432 *et seq.* (1894).

For proof we note that the classes whose l-th powers give the principal class may be expressed in the form

$$C_1^{X_1}\dots C_G^{X_G}\dots c_1^{x_1}\dots c_g^{x_g}D^l \qquad (X_i, x_i=0, 1, \dots, l-1)$$

and this reduces to the simplified form

$$C_1^{X_1}\dots C_G^{X_G}D^l,$$

using the assumption that the class number of the field $k(\zeta+\zeta^{-1})$ is prime to l.

Corresponding to the congruence

$$X_i \equiv 0 \quad (l)$$

there is a singular primary number M and the conjugate imaginary $\overline{M}$ of M belongs to the same congruence. From the properties of singular primary numbers (cf. Furtwängler[43, 44]) we have

$$M=\overline{M}^a\beta^l$$

and

$$M=M^{a\cdot a}(\beta\overline{\beta}^a)^l,$$

giving

$$a^2-1\equiv 0 \quad (l).$$

It is then shown that

$$a\equiv -1 \quad (l)$$

is impossible so that we have

$$a\equiv +1 \quad (l)$$

and this is shown to yield, if η is a unit in k,

$$M=\eta\cdot\kappa^l.$$

It is then observed that following the method used in the proof of the first theorem concerning Fermat's last theorem, and by the use of the theorem just proved, we obtain the result that under the assumption that the field $k(Z)$ contains no class belonging to the exponent l^2 and the class number h_2 of the field $k(\zeta+\zeta^{-1})$ is prime to l, then (26) is impossible, with $a, b, c,$ restricted as before.

Vandiver[64] (p. 420) showed, however, that Bernstein's first theorem concerning Fermat's last theorem constitutes no advance over Kummer's[18] result. Pollaczek[76] (p. 37) also pointed this out and in addition arrived at the conclusion that Bernstein's second result did not include Kummer's[20] as it does not exclude the exponents 37, 59 and 67.

Furtwängler[40] proved that if l is any prime, then the class numbers of the field defined by a primitive l^κ-th root of unity is divisible by l if, and only if, the class number of the field defined by a primitive l-th root of unity is divisible by l.

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[40] J. reine angew. Math., 140: 29-32 (1911).

For a proof of this theorem the field defined by the l^κ-th roots of unity is compared with the field defined by the $l^{\kappa+1}$-th roots of unity.

We have the theorem: If the class number of the field of the primitive l^κ-th roots of unity is prime to l, then every unit of this field is the relative norm of a unit of the field defined by a primitive $l^{\kappa+1}$-th root of unity.

For proof, we note that if $l' = l^{\kappa-1}\left(\frac{l-1}{2}\right)$ and k is defined by a primitive l^κ-th root of unity ζ_κ, then there are $l'-1$ fundamental units in k. Also, there exist l' independent unit systems (Einheitenverbände) in k, and therefore the number of distinct unit systems in k is $l^{l'}$. In the field K defined by a primitive $l^{\kappa+1}$-th root of unity, there are l' relative fundamental units with respect to k. Calling these $H_1, H_2, \ldots, H_{l'}$, and designating a substitution of the relative group of K with respect to k by S, then a unit of the form

$$H_1^{F_1(S)}\ldots H_{l'}^{F_{l'}(S)}\zeta^e_{\kappa+1}\eta$$

is the symbolic $(S-1)$-th power of a unit in K only if

$$F_1(1) \equiv F_2(1) \equiv \ldots \equiv F_{l'}(1) \equiv e \equiv 0 \quad (l).$$

Here the function F is a polynomial in S with rational integral coefficients, $\zeta_{\kappa+1}$ is a primitive $l^{\kappa+1}$-th root of unity and η is an arbitrary integer in k. Since the relative norm of $\zeta^e_{\kappa+1}\eta = 1$, we have $e \equiv 0$ (modulo l). It follows from the property of the fundamental units just mentioned that among the unit systems in k

$$(27) \qquad H_1^{e_1}\ldots H_{l'}^{e_{l'}}\zeta^e_{\kappa+1}H^l$$
$$(e_i,\ e=0,\ 1,\ \ldots,\ l-1)$$

there are at most l' units having the relative norm 1 taken with respect to k. If H' and H'' are two such units which are not l-th powers of a unit, then

$$H' = A^{S-1}, \qquad H'' = B^{S-1}$$

where A and B are integers in K. It is then shown that

$$A = \alpha\Lambda^m H_a, \qquad B = \beta\Lambda^n H_b,$$

where H_a and H_b are units in K, m and n are integers prime to l, α and β are integers in k, and $\Lambda = 1 - \zeta_{\kappa+1}$. From this it is shown that

$$H'' = (H')^f\bar{H}^l$$

where f is prime to l, and $\bar{H}$ is a unit in K. It follows from this that there are at most l' unit systems in (27) with the relative norm 1 with respect to k. Forming the relative norm of the unit (27) with respect to k we have, from what precedes, that these relative norms in k yield $l^{l'}$ distinct unit systems. As this is the exact number of unit systems in k, the theorem is proved. It is then shown that if J is an ideal in K, and

$$J^{S-1} \sim j$$

where j is an ideal in k, then J is equivalent to an ideal in k.

For the proof of the first theorem mentioned in this paper, take an arbitrary ideal J in K, and it is always possible to find an exponent q prime to l so that J^q is equivalent to some ideal in k. This follows from the fact that $J^{q(1-S)}$ is equivalent to an ideal in k since an $(S-1)^l$-th symbolic power of an ideal is always actually the l-th power of an ideal; hence by the preceding theorem, J^q is itself equivalent to an ideal in k. Hence, if the class number of k is prime to l, then the class number of K is prime to l. The converse of this result, namely, that if the class number of the field k is divisible by l, then the class number of K is divisible by l is a consequence of Furtwängler's (§29 of Chap. I, p. 91) theorem.

FURTWÄNGLER[50] considered the Fermat relation in the form

$$x_1^l + x_2^l + x_3^l = 0 \tag{27a}$$

where the x's are rational integers prime to each other, whence, if x_3 is prime to l,

$$x_1 + \zeta^i x_2$$

is the l^{th} power of an ideal in the field $k(\zeta)$ where $\zeta = e^{2i\pi/l}$. Also

$$\left(\frac{a}{x_1 + \zeta^i x_2}\right) = 1$$

where a is an arbitrary integer in $k(\zeta)$ prime to $(x_1 + \zeta^i x_2)$ and the symbol in the left hand member of the equation indicates the l^{th} power character of a with respect to $(x_1 + \zeta^i x_2)$. Using Eisenstein's law of reciprocity between the power characters involving a rational integer r and a semi-primary integer in the field $k(\zeta)$, we obtain

$$\left(\frac{r}{x_1\zeta^{x_2} + x_2\zeta^{-x_1}}\right) = \left(\frac{x_1\zeta^{x_2} + x_2\zeta^{-x_1}}{r}\right) = 1.$$

If r is a factor of x_1, then

$$\left(\frac{x_2\zeta^{-x_1}}{r}\right) = \left(\frac{\zeta^{-x_1}}{r}\right) = 1$$

which gives if x_1 is prime to l,

$$r^{l-1} \equiv 1 \quad (l^2).$$

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[50] Wiener Berichte, Abt. IIa, 121: 589-592 (1912).

This proves that if x_1, x_2, x_3, are three rational integers none zero, and prime to each other, connected by the relation (27a) and l is an odd prime, then

$$r^{l-1} \equiv 1 \quad (l^2)$$

where r is any factor of x_i $(i=1, 2, 3)$, if x_i is prime to l. Setting $r=2$, which is evidently a factor of $x_1x_2x_3$, we obtain

$$2^{l-1} \equiv 1 \quad (l^2).$$

Taking r as a factor of $x_1 \pm x_2$ in the above relations, we find that x_1, x_2, x_3, are three rational integers none zero, prime to each other and connected by the relation (27a) where l is an odd prime, then

$$r^{l-1} \equiv 1 \quad (l^2)$$

where r is any factor of $x_i \pm x_k$ $(i, k=1, 2, 3)$ and x_i+x_k and x_i-x_k are prime to l. These results are used to prove that

$$3^{l-1} \equiv 1 \quad (l^2)$$

and to derive Legendre's theorem that the Fermat relation with the exponent l and in case I will not hold for any l for which $2l+1$ is a prime.

BOHNICEK[51] proved if n is an integer >2 and α, β, γ are integers in the field defined by a primitive 2^n-th root of unity, none zero, then

$$\alpha^{2^{n-1}}+\beta^{2^{n-1}}+\gamma^{2^{n-1}}=0$$

is impossible. For $n=3$ this reduces to a result given by Hilbert.[28] The proof is obtained by the use of the theorem that the class number of the field defined by a primitive 2^n-th root of unity is odd (cf. Weber, Algebra II, Edition 2, 818) also the theorem that if k is a sub-field of the field defined by a primitive 2^n-th root of unity, m is an integer $\leqq n$ and ξ is a unit in k such that

$$\xi \equiv a^{2^m} \qquad (2^{m+1}),$$

where a is an integer in k, then ξ is the 2^m-th power of a unit in k, together with an extension of a method employed by Hilbert[28] to establish the theorem for the case $n=3$.

VÉRÉBRUSSOV[52] considered, using Kummer's methods, the classes of ideals to which two conjugate imaginary ideals belong in the field defined by a where a is a primitive λ-th root of unity, λ being an odd prime, and concludes by a meaningless argument that if $f(a)$ is an ideal in the field $\Omega(a)$, then $f(a)\cdot f(a^{-1})$ is an integer in the field. Based on this result, he attempts a demonstration of Fermat's last theorem.

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[51] Wiener Berichte, Abt. IIa, 121: 727-742 (1912).

[52] Mémoire sur les classes des nombres complexes ideaux conjugés avec l'application à la démonstration du dernier théorème der Fermat, Paris and Leipzig, 1912.

FUETER[52a] investigated the solutions of $\xi^3+\eta^3+\zeta^3=0$ in the quadratic imaginary field $k(\sqrt{m})$. If m is congruent to $2 \pmod 3$, then solutions exist only when the class number of the quadratic field is divisible by 3. In particular we have,

$$\left(\frac{9+\sqrt{-31}}{2}\right)^3+\left(\frac{9-\sqrt{-31}}{2}\right)^3+3^3=0.$$

Analogous results are obtained for the case of real quadratic fields.

FABRY[53] attempted to prove that if $xyz\not\equiv 0 \pmod\lambda$,

$$x^\lambda+y^\lambda+z^\lambda=0$$

is impossible in rational integers, none zero, where λ is an odd prime.

As Kummer had shown, so he shows that

$$(x+a^{n_1}y)(x+a^{n_2}y)\ldots(x+a^{n_\mu}y)=a^rQ^\lambda(a)$$

where a is a primitive λ-th root of unity, $Q(a)$ is an integer in the field defined by a,

$$h\cdot n_h\equiv 1 \quad \text{modulo } \lambda,$$

the h's being the $(\lambda-1)/2=\mu$ integers defined by

$$\frac{q+1}{k+1}\,\lambda<h<\frac{q+1}{k}\,\lambda, \qquad q=0,1,2,\ldots,k-1.$$

Here k is a rational integer,

$$0<k<\lambda-1.$$

He then expresses the left-hand member of the above equation in ascending powers of $1-a$ and obtains therefrom complicated congruences involving multinomial coefficients. MIRIMANOFF,[53a] however, pointed out that these congruences do not cover all possible cases which arise from the development and consequently Fabry's attempted proof is insufficient.

BURNSIDE[53b] extended the results of Fueter[52a] and showed that the equation

$$x^3+y^3+z^3=0$$

has non-trivial solutions in the field $k(\sqrt{n})$ if, and only if, the equation

$$nl^2+12k^3+3=0$$

has solutions in rational numbers l and k, and if this condition holds then the first equation has an infinite number of solutions.

[52a] Sitzungsber. Heidelberger Akad. Wiss., 1913: 25.

[53] Compt. Rend., 156: 1814 (1913).

[53a] Compt. Rend., 157: 491-492 (1913).

[53b] Proc. London Math. Soc., (2), 14: 1-4 (1914).

Vandiver[54] gave without proof the criteria for (1) in Case I:

$$\sum_{r=1}^{p-1} y_r^{(n)} r^{p-2} t^r \equiv 0 \pmod{p},$$

where $y_r^{(n)} \equiv -r/p \pmod{n}$; $0 \leqq y_r^{(n)} < n$, with n any integer prime to p, and t is defined as in next reference.

Vandiver[55] proved that if (1) is satisfied in Case I, then $5^{p-1} \equiv 1 \pmod{p^2}$, and $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{[\frac{p}{5}]} \equiv 0 \pmod{p}$, where $[p/5]$ is the greatest integer in $p/5$. These results were obtained by transformations of the Kummer Criteria (23) and with the use of the identical congruence in x and y,

$$(28) \quad \left\{ \begin{array}{l} (y^2-x)\sum_{n=1}^{p-1} 2^{p-n-1}\binom{p-2}{n-1} f_n(y) f_{p-n}(x) \\ \quad \equiv x f_{p-1}(y) - x^p y^2 f_{p-1}(y) \\ \qquad + 2^{p-2} K(x)\,(y^p x + y^{p+1} x^{\mu+1} - y^2 - y x^{\mu+1}) \\ \qquad + M(x)\,(y^{p+1} + y^p x^{\mu+1} - y^2 x^{\mu} - y), \\ \qquad\qquad \mu = \dfrac{p-1}{2}, \end{array} \right.$$

modulo p, where $K(x)$ and $M(x)$ are polynomials in x with integral coefficients, and $f_n(x)$ is defined as in Mirimanoff.[32]

This leads to the congruences

$$(28a) \qquad \frac{t^m-1}{t-1} \sum{}' \frac{f(a)}{1-a^p} \cdot \frac{1-a^2}{t-a^2} \equiv 0 \pmod{p}$$

where the summation extends over every distinct value of a given by $a^m = 1$, $a \neq 1$, $-t = \frac{x}{y}, \frac{y}{x}, \frac{x}{z}, \frac{z}{x}, \frac{y}{z}, \frac{z}{y}$.

Frobenius[56] proved the identity

$$(y^k-x)F(x,\ y) + (y^k-1)f_1(x)f(y) = x(y^p-1)\sum_{l=0}^{k-1} h_l^{(k)}(x) y^l$$

where k is any integer, $h_l^{(k)}(x) = \sum_{r=0}^{p-1} (kr+l)^{p-2} x^r$,

$$F(x,y) = \sum_{n=1}^{p-2} \binom{p-2}{n-1} k^{p-n-1} f_{p-n}(x) f_n(y)\,;\ f(y) = f_{p-1}(y)\,;\ f_1(x) = \frac{x^p-1}{x-1}.$$

For $k=2$ we have another form of the relation (28).

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[54] Bull. Am. Math. Soc., 21: 68 (1914).

[55] J. reine angew. Math., 144: 314-318 (1914).

[56] Sitzungsber. Akad. Wiss., Berlin, 1914: 653-681.

Using the relations

$$\frac{m^n-1}{n}\, b_n \equiv -{\sum}' \frac{f_n(\rho)}{1-\rho^p} \pmod{p},$$

and

$$\frac{m^{p-1}-1}{p} = q(m) \equiv -{\sum}' \frac{f(\rho)}{1-\rho^p} \pmod{p},$$

where m is an integer prime to p; $n=1, 2, \ldots, p-1$; the summation extends over all the values ρ other than unity such that $\rho^m=1$ and $b_0=1$, $b_1=-1/2$, $b_2=1/6$, $b_3=0$, $b_4=-1/30$, $\ldots$, are the Bernoulli numbers, he derives the identity in x,

$$(x^p-1)G_m^{(k)}(x) - (x^m-1)F_{m:k}(x) \equiv H_m^{(k)}(x) \pmod{p},$$

where

$$G_m^{(k)}(x) = G(x) = x\,\frac{x^m-1}{x-1} {\sum}' \frac{f(\rho)}{1-\rho^p}\cdot\frac{1-\rho^k}{x-\rho^k},$$

$$F_{m:k}(x) \equiv -\sum_{n=0}^{p-2}\left(-\frac{m}{k}\right)^n b_n f_{p-n}(x),$$

and

$$H_m^{(k)}(x) = x(x^m-1)\sum_{l=0}^{k-1}\sum_{\rho}\frac{h_l^{(k)}(x)\rho^l}{x-\rho^k},$$

the ρ summation ranging over all values $\rho^m=1$ including unity, m and k being any integers prime to p. Making use of (23) gives

$$(29) \qquad (t^p-1)G_m^{(k)}(t) \equiv H_m^{(k)}(t) \pmod{p},$$

which is satisfied by

$$-t \equiv x/y,\ y/x,\ x/z,\ z/x,\ y/x,\ z/y \pmod{p}$$

from (1), or the congruence (29) is also satisfied, if in lieu of t we take,

$$t,\ \frac{1}{t},\ 1-t,\ \frac{1}{1-t},\ \frac{t-1}{t},\ \frac{t}{t-1}.$$

By taking various values of m and k in (29) and eliminating the h functions appearing in H, he finds for various small prime values of m

$$G_m^{(k)}(t) \equiv 0 \pmod{p}$$

for $k=1, 2, \ldots, (m-1)/2$. For $k=1$, m arbitrary, this gives (23b). For $k=2$, m arbitrary, we have (28a).

It is also proved that (p. 664),

$$(29\text{a}) \qquad G_m^{(1)} + G_m^{(2)} + \ldots + G_m^{(\nu)} = -\frac{m}{2}\, q(m)\, t\,\frac{t^{m-1}-1}{t-1};$$

$\nu=(m-1)/2$, m prime. With the aid of these relations it is shown that if (1) holds in Case I, then $q(2) \equiv q(3) \equiv q(5) \equiv q(11) \equiv q(17) \equiv 0 \pmod p$, and if $p \equiv 1 \pmod 6$, $q(7) \equiv q(13) \equiv q(19) \equiv 0 \pmod p$. Cf. Vandiver.[55]

VANDIVER[57] examined the theory of Fermat's quotient $(m^{p-1}-1)/p$ and called the least positive residues of the integers

$$1^{p-1}, 2^{p-1}, 3^{p-1}, \ldots, (p-1)^{p-1}$$

proper residues, modulo p^2, and proved that there are not more than $\left[p - \frac{1+\sqrt{2p-5}}{2}\right]$ and not less than $[\sqrt{p}]$ incongruent proper residues modulo p^2, where p is a prime >2.

It is then noted that the criteria

$$2^{p-1} \equiv 3^{p-1} \equiv 1 \pmod{p^2}$$

in connection with Fermat's last theorem limits the number of incongruent proper residues modulo p^2.

JENSEN[58] proved that there is an infinity of irregular primes of the form $4k+3$. For proof, assume that the distinct irregular primes of this form are restricted to $p_1, p_2, \ldots, p_q$. There exists a prime integer K such that

$$K \equiv 1 \pmod{2p_1(p_1-1)p_2(p_2-1)\ldots p_q(p_q-1)},$$

and therefore by a known formula (cf. Bachmann, Niedere Zahlentheorie, II, 41, relation (107)).

$$-B_1 \equiv \frac{(-1)^K B_K}{K} \pmod p,$$

for p any of the numbers p_1, p_2, etc., $\neq 3$, $B_1 = 1/6$, $B_2 = 1/30$, etc., being the Bernoulli numbers. Hence all prime factors of the numerator of B_K are of form $4n+1$. Also, by the von-Staudt-Clausen theorem, Bachmann, l. c., p. 43, the denominator of $B_K = 6$. Hence

$$2B_K \equiv -1 \pmod 4. \tag{30}$$

But the formula

$$\frac{(-1)^{n-1}B_n}{n} \equiv \frac{S_{2n}(2^\rho - 1)}{n \cdot 2^\rho} \pmod{2^{\rho+1}}$$

where $S_{2n}(2^\rho - 1) = 1^{2n} + 2^{2n} + \ldots + (2^\rho - 1)^{2n}$, and $\rho > 1$, $n > 1$, gives for $n = K$, $\rho = 2$,

$$(-1)^{K-1} \cdot 2B_K \equiv 1 \pmod 4,$$

which contradicts (30).

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[57] Bull. Am. Math. Soc., 22: 64-67 (1915).

[58] Nyt Tidsskrift für Matematik, 1915 (Afdeling B): 82.

Bachmann[59] gave an almost complete reproduction of the papers by Dickson,[54] Kummer,[55] Mirimanoff,[56] Wieferich[57] and Furtwängler.[58]

Dickson[60] gave a history of Fermat's last theorem with abstracts of papers published before 1917. He discussed the origin of the theory of algebraic numbers, in particular the history of the introduction by Kummer of the concept of ideal numbers. The relation of this theory to Fermat's last theorem is brought out.

Pollaczek[61] used the Kummer criteria (23) in the form

$$B_i\left[\frac{d^{l-2i}\log(1-e^{v}t)}{dv^{l-2i}}\right]_{v=0}\equiv 0 \pmod{l},$$

$i=1, 2, \ldots, (l-3)/2$, and

$$x^l+y^l+z^l=0, \tag{31}$$

l an odd prime, $-t=x/y, y/x, y/z, z/y, x/z, z/x$. Also

$$\left[\frac{d^{l-1}\log(1-e^{v}t)}{dv}\right]_{v=0}\equiv 0 \pmod{l}.$$

Here B_i is the i^{th} Bernoulli number. He also employed the relations

$$\left[\frac{d^{2i}\log\dfrac{e^{ax}-1}{e^{x}-1}}{dx^{2i}}\right]_{x=0}=(-1)^{i+1}\frac{B_i}{2i}(a^{2i}-1)$$

and its equivalent

$$\sum_{\eta}\left[\frac{d^{2i-1}}{dx^{2i-1}}\left(\frac{1}{1-e^{x}\eta}\right)\right]_{x=0}=(-1)^{i}\frac{B_i}{2i}(a^{2i}-1)$$

η ranging over all the values given by $\eta^a=1$, $\eta\neq 1$. He proved that if (31) is satisfied in Case I, then

$$\frac{a}{h}\sum_{\tau}\left[\frac{1}{1-e^{x}\tau}\right]_0^{(l-2)}\frac{1}{1-\dfrac{1}{\tau^a}}$$
$$\equiv\sum_{\eta}\left[\frac{1}{1-e^{x}\eta}\right]_0^{(l-2)}\frac{1}{\dfrac{t}{\eta^h}-1}-\frac{q(a)}{1-t}\pmod{l},$$

where the τ summation ranges over all values, τ defined by $\tau^h=t$; a and h are positive integers, a prime to l; the symbol $[X]_0^{(l-2)}$ indicating that X is to be differentiated $(l-2)$ times with respect to x, and $x=0$ substituted in the result. This relation may be shown to be another form of

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[59] Das Fermatproblem in seiner bisherigen Entwicklung, Berlin and Leipzig, 1916, 160 p. Reviewed by Vandiver, Bull. Am. Math. Soc., 27: 373-376 (1921).

[60] Annals of Math., (2), 18: 161-187 (1917).

[61] Sitzungsber. Akad. Wiss., Wien 126, IIa: 45-59 (1917).

Frobenius' (29), if $a=m$, $h=k$, $l=p$. By means of it and (29a) Pollaczek, extending Frobenius' methods, shows that $q(a)\equiv 0 \pmod{l}$ for $a \leqq 31$, except for a finite number of values of l, if (31) is satisfied in Case I. It is also proved that $x^2+xy+y^2 \not\equiv 0 \pmod{l}$, or $t^2-t+1 \not\equiv 0 \pmod{l}$, if (31) is satisfied in Case I.

VANDIVER[63] proved that if $\mathfrak{P}$ is a prime ideal of the first degree in the algebraic field defined by $\alpha=e^{2i\pi/p}$ then

$$\prod_{\nu=1}^{k-1}\prod_{r=1}^{[\nu p/k]} \mathfrak{P}_{[1:r]} \sim 1,$$

$$k=2, 3, \ldots, p-1,$$

or the ideal on the left is a principal ideal, where $\mathfrak{P}_a$ is the ideal obtained from $\mathfrak{P}$ by the substitution (α/α^a); $[x]$ is the greatest integer in x; and $[1:r]$ is the least positive solution m of $mr\equiv 1 \pmod{p}$. Using the results from (1) that $(x+\alpha y)$ is the p^{th} power of an ideal it is shown that:

If (1) is satisfied in Case I, then it is necessary and sufficient that rational integers x, y and z exist which satisfy

$$(32) \qquad \prod_{\nu=1}^{k-1}\prod_{r=1}^{[\nu p/k]} (x+\alpha^{[1:r]}y)=\alpha^{g}\omega^{p},$$

$$x+y=v^p; \qquad v \text{ a rational integer,}$$

where k is an integer, $1<k<p$; ω is an integer in $\Omega(\alpha)$ and g is an integer satisfying

$$g\equiv -\frac{kyq(k)}{x+y} \pmod{p}.$$

In Kummer's derivation of his criteria (23) there were a number of steps not explained. Also, he used his own theory of algebraic numbers, definition of ideal and theorems obtained therefrom, and not all of Kummer's processes along these lines are accurate, as pointed out by Dedekind. In the paper now being reported on, a complete derivation of the Kummer criteria is attempted, based on Dedekind's theory of ideals and using (32). The argument begins with the relation

$$\prod_{q}\prod_{s}\mathfrak{P}_{[1:s]} \sim 1,$$

where s ranges over the values satisfying

$$\frac{qp}{k+1}<s<\frac{qp}{k}$$

and q ranges over the set $1, 2, \ldots, k$; k being any integer subject to the conditions $0<k<p-1$. This relation was given by Kummer but the

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[63] Annals of Math., 21: 72-80 (1919).

modern form of the proof was given by Weber, Lehrbuch der Algebra, II, ed. 2, 749-750.

In this connection Bachmann[59] gave a reproduction of Kummer's[30] derivation and supplied some but not all of the missing steps. (Cf. Vandiver, Bull. Am. Math. Soc., 27: 375 (1921).)

From (32) another proof of Furtwängler's[60] theorem is obtained. (Cf. Fueter.[62])

Vandiver[63] considered the first factor h of the class number of the field defined by $\zeta = e^{2i\pi/l}$ where

$$h = \frac{f(Z)f(Z^3)\ldots f(Z^{l-2})}{(2l)^{\frac{1}{2}(l-3)}},$$

$$f(x) = r_0 + r_1 x + r_2 x^2 + \ldots + r_{l-2} x^{l-2},$$

$Z = e^{2i\pi/(l-1)}$, r is a primitive root of l, and r_i is the least positive residue of r^i, modulo l. The decomposition of (l) into prime ideal factors in the field $\Omega(\zeta)$ shows that one of the prime factors is

$$\mathfrak{P} = (Z - r, l),$$

whence

$$Z^{kl^{a'}} \equiv r^{kl^{a'}} \pmod{\mathfrak{P}^{a'+1}}; \qquad a' > \frac{l-3}{2}.$$

This leads to

$$(2l)^{\frac{1}{2}(l-3)} h \equiv \prod_s f(r^{sl^{a'}}) \pmod{\mathfrak{P}^{a'+1}},$$

where $s = 1, 3, \ldots, l-2$, whence

$$h \equiv \frac{\prod_s f(r^{sl^{a'}})}{(2l)^{\frac{1}{2}(l-3)}} \pmod{l^{a+1}},$$

where $a = a' - \frac{1}{2}(l-3)$. Using $r_i \equiv r^i \pmod{l}$, and Fermat's theorem, this reduces to

$$(33) \qquad h \equiv \frac{\prod_s \sum_{n=1}^{l-1} n^{sl^a+1}}{(2l)^{\frac{1}{2}(l-3)}} \pmod{l^a}.$$

Employing the symbolic method of Blissard concerning Bernoulli numbers, it is then shown that

$$\sum_{n=1}^{l-1} n^{sl^a+1} \equiv (-1)^{(sl^a-1)/2} l B_{(sl^a+1)/2} \pmod{l^{a+1}},$$

where $B_1 = 1/6$, $B_2 = 1/30$, etc., are the Bernoulli numbers; together with (33) this gives

$$h \equiv \frac{l \prod_s (-1)^{(sl^a-1)/2} B_{(sl^a+1)/2}}{2^{\frac{1}{2}(l-3)}} \pmod{l^a}$$

$s = 1, 3, \ldots, l-2$. For $a = 1$, this yields a result due to Kummer.[17]

[63] Bull. Am. Math. Soc., 25: 458-461 (1919).

Vandiver[64] pointed out that the proofs of the results given in Kummer[30] are inaccurate and incomplete in several respects. Under this Assumption I, Kummer proceeds to show that only one of the numbers, B_i, $i=1, 2, \ldots, \mu-1$, is divisible by λ. In this connection he states (p. 42, 5th line from bottom) "...denn der erste Faktor der Klassenanzahl muss, wie aus meiner Untersuchung der Teilbarkeit der Klassenanzahl durch λ (Liouville's Journal, 16, 1851 (473)) unmittelbar folgt, den Faktor λ mindestens so viel mal enthalten, als wie viele der $(\lambda-3)/2$ Bernoullischen Zahlen durch λ theilbar sind." On examining the paper referred to by Kummer we find that he reduces h_1 modulo λ and obtains a residue which may be expressed in the form

$$m \prod_{i=1}^{\mu-1} B_i,$$

where m is an integer prime to λ. At no place in this work, however, does he refer to any reduction of h_1, modulo λ^n, $n>1$. Hence, he is not warranted in making, without additional argument, the statement given above.

Kummer's proofs of Theorems II and III are vitiated because of his use of the formula (7a) of this report, which was shown to be inaccurate by F. Mertens.[65]

In connection with Theorem IV, note the formula (A) which he gives on page 53 of his article, and (7b) of this report. It was proved by him only for ideals $f(\alpha)$ of the first degree, whereas he applies Theorem IV to ideals not necessarily of the first degree.

Vandiver[66] examined the divisibility of the first factor of the class number of the field $\Omega(e^{2i\pi/p^n})$ by p, and considered the relation of the results obtained to certain statements made by Bernstein.[47]

Westlund[67] reduced the first factor of the class number of the field mentioned for $n>1$ to the form

$$h_1 = k \times \frac{\prod_s \phi(\theta^s)}{2^{\frac{p^{n-2}(p-1)^2}{2}}\; p^{\frac{np^{n-2}(p-1)^2}{2}-1}} = kk_1,$$

where k is the first factor of the class number of $\Omega(e^{2i\pi/m'})$, $\theta=e^{2i\pi/\mu}$, $\mu=\phi(m)=p^{n-1}(p-1)$, $m=p^n$, $m'=p^{n-1}$; r_i is the least positive residue of r^i, modulo p^n, r being a primitive root of p^n. The integer s takes on all odd values $<\mu$ except multiples of p, and the function $\phi(\theta)$ defined by Westlund may be put in the form

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[64] Proc. Nat. Acad. Sci., 6: 266-269 (1920).

[65] Wiener Berichte, 126, Abt., IIa: 1337-1343 (1917).

[66] Proc. Nat. Acad. Sci., 6: 416-421 (1920).

[67] Trans. Am. Math. Soc., 4: 201-212 (1903).

$$\phi(\theta)=r_0+r_1\theta+\ldots+r_{\mu-1}\theta^{\mu-1}.$$

The integer k_1 is reduced modulo p by a modification of the method used by Kronecker[19] in reducing the first factor of the class number of $\Omega(e^{2i\pi/p})$. Set

$$(r-\theta^{-1})\phi(\theta)=p^n g(\theta),$$

where

$$g(\theta)=q_0+q_1\theta+\ldots+q_{\mu-1}\theta^{\mu-1}$$

and

$$q_i=\frac{rr_i-r_{i+1}}{p^n}.$$

From an argument used by Westlund

$$\prod_s(r-\theta^{-s})=\frac{r^{\frac{\mu}{2}}+1}{r^{\frac{\mu'}{2}}+1},\quad \mu'=\phi(m'),$$

which leads after a few transformations to

$$k_1\equiv\prod_s g(\theta^s)\pmod{p}.$$

We have the identity

$$\prod_s g(x^s)=a+V(x)W(x),$$

where $V(x)=\Pi(x-\theta^i)$, i ranging over the $\phi(\mu)$ integers less than and prime to μ, since $V(x)$ is irreducible in the rational field and $W(x)$ is a polynomial in x with integral coefficients. It is then shown that $V(r)\equiv 0$ modulo p, and this gives

$$k_1\equiv\prod_s g(r^s)\pmod{p}.$$

Now

$$\prod_s g(r^s)\equiv\prod_t\left(g(r^{2t-1})\right)^{p^{n-2}(p-1)}\pmod{p},$$

where $t=1, 2, \ldots, (p-1)/2$. To reduce the right-hand member, set

$$rr_i=r_{i+1}+rr_i-r_{i+1},$$

which gives easily

$$2tr_{i+1}{}^{2t-1}\cdot(rr_i-r_{i+1})\equiv r^{2t}r_i{}^{2t}-r_i{}^{2t}\pmod{p^{2n}}$$

and this yields

$$2tp^n r^{2t-1}\sum_i q_i r^{i(2t-1)}\equiv r^{2t}\sum_i r_i{}^{2t}-\sum_i r^{2t}{}_{i+1}\pmod{p^{2n}}$$

and

$$\prod_t g(r^{2t-1})\equiv\prod_t\frac{(r^{2t}-1)S_{2t}}{2tp^n r^{2t-1}}\pmod{p},$$

where $S_{2t} = a_1^{2t} + a_2^{2t} + \ldots + a_\mu^{2t}$, where the a's are the integers less than p^n and prime to it. The quantity S_{2t} is then reduced modulo p^{n+1} and it is shown that

$$S_{2t} \equiv R_{2t} \pmod{p^{n+1}}$$

for $t < (p-1)/2$ and

$$S_{2t} \equiv R_{2t} \pmod{p^n}$$

for $t = (p-1)/2$ where $R_{2t} = 1^{2t} + 2^{2t} + \ldots + (p^n - 1)^{2t}$. Using the Bernoulli summation formula we then obtain

$$\frac{(r^{2t}-1)S_{2t}}{2tp^n r^{2t-1}} \equiv \frac{(r^{2t}-1)R_{2t}}{2tp^{2t-1}} \equiv \frac{(-1)^{t+1}B_t(r^{2t}-1)}{2tr^{2t-1}} \pmod{p},$$

the B's being the Bernoulli numbers, this congruence holding for any t less than $(p-1)/2$. The expression

$$\frac{(r^{p-1}-1)S_{p-1}}{(p-1)p^n r^{p-2}}$$

may be reduced modulo p to an integer prime to p, hence we have

$$\prod_s g(r^s) \equiv \Big(\prod_{t=1}^{(p-3)/2} B_t\Big)^{p^{n-2}(p-1)} \pmod{p}.$$

Hence, $\prod_s g(r^s)$ and, therefore, k, is divisible by p if and only if at least one of the numbers B_t, $(t = 1, 2, \ldots, (p-3)/2)$ is divisible by p.

Now Bernstein[47] gave the result that under the assumption that the class number of the field $\Omega(e^{2i\pi/p^2})$ is divisible by p but not by p^2, then $x^p + y^p + z^p = 0$ is impossible in integers prime to each other for $xyz \equiv 0$ modulo p. In view of the results just obtained, this criterion constitutes no advance over Kummer's[18] result that the Fermat equation does not hold for rational integers if p is a regular prime. Taking this in connection with the errors in Kummer's memoir of 1857, it is noted that Fermat's last theorem is not rigorously proved for all irregular primes less than 100. These irregular primes are 37, 59 and 67. Mirimanoff[35] gave an adequate proof for the case $p = 37$, which leaves the cases of 59 and 67 still to be disposed of. Cf. Vandiver.[57]

It is then observed that Bernstein's[45] conclusions concerning the first case of the last theorem do not include Kummer's[30] (p. 65) results on that case but at most supplement them.

It is pointed out that Kummer's[22] computations show that the last theorem is proved for all exponents p such that $100 < p < 167$, excepting $p = 101, 103, 131, 149, 157$. Cf. Schönbaum.[36]

DICKSON[67a] gave the history of Fermat's last theorem with detailed abstracts of papers published prior to 1920. Cf. the remarks concerning this history in the introduction to present chapter of this report.

[67a] History of the Theory of Numbers, Vol. II, Ch. XXVI, 731-776 (1920).

Mordell[68] reproduced part of the work of Kummer.[18, 20] On p. 25 he remarks that it seems that a new application of the laws of reciprocity may be expected to lead to interesting results on Fermat's last theorem. **Fueter**[69] using methods due to Hilbert, proved that if (1) is satisfied in Case I, then employing the latter's notation of symbolic powers (**Hilbert**[70]), we have

$$(x+ay)^{r_0+r_{-1}s+\cdots+r_{-p+2}s^{p-2}}=a^g\omega^p$$

where s refers to the substitution $(a:a^r)$, r is a primitive root of p, $a=e^{2i\pi/p}$, r_n is the least positive residue of r^n, modulo p. Here, also, ω is an integer in $\Omega(a)$ and

$$g\equiv-\frac{y}{x+y}\pmod{p}.$$

By means of this relation the theorems of Furtwängler[50] are proved. He obtained a relation II, p. 15, which is a transformation of (32) of this report, deducing therefrom the Kummer criteria (23). He also derived the following transformations of the Kummer criteria:

$$\sum_{n=1}^{l-1}\left[\frac{an}{l}\right]\frac{\sigma^n}{n}\equiv 0 \pmod{l},$$
$$(a=1,\ 2,\ \ldots,\ l-1),$$

where $[an/l]$ is the greatest integer in an/l and $-\sigma=x/y,\ y/x,\ x/z,\ z/x,\ y/z,\ z/y$. Also

$$\text{(33a)}\qquad \sum_{i=0}^{l-2}q(r_{i+k})\sigma^{r_i}\equiv 0 \pmod{l},$$
$$(k=0,\ 1,\ \ldots,\ l-1),$$

where r_s is the least positive residue of r^s, modulo l, and

$$q(n)=\frac{n^{p-1}-1}{p}.$$

Cf. Vandiver.[62]

Takagi[71] considered the conditions under which a principal ideal is the l-th power of an ideal, using the norm residue symbol $(\mu,\ \nu)$. Let l be an odd prime; $\zeta=e^{2i\pi/l}$; k the cyclotomic field generated by ζ; r a primitive root of l; s the substitution $(\zeta|\zeta^r)$; $\lambda=1-\zeta$, the prime divisor of l in k; $\kappa_1, \kappa_2, \ldots, \kappa_l$ are bases such that

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[68] Three Lectures on Fermat's Last Theorem, Cambridge Univ. Press, 1921.

[69] Math. Ann., 85: 11-20 (1922).

[70] Bericht über die theorie der Algebraischen Zahlkörper, Jahresber. deut. Math. Verein, 1894: 271.

[71] Proc. Physico-Math. Soc. Japan, (3), 4: 170-182 (1922).

$$\mu\equiv\mu^l\kappa_1^{u_1}\kappa_2^{u_2}\ldots\kappa_l^{u_l} \pmod{\lambda^{l+1}},$$
$$\nu\equiv\nu^l\kappa_1^{v_1}\kappa_2^{v_2}\ldots\kappa_l^{v_l} \pmod{\lambda^{l+1}},$$

the κ's being further chosen so that

$$\kappa_a\equiv1-\lambda^a \pmod{\lambda^{a+1}},$$
$$\kappa_a^{s-r^a}\equiv1 \pmod{\lambda^{l+1}}.$$

It is sufficient to take

$$\begin{aligned}\kappa_1&=\zeta,\\ \kappa_{l-1}&=1+l,\\ \kappa_l&=1+\lambda^l,\\ \kappa_a&\equiv(1-\lambda^a)^{-r^a(s-1)(s-r)\ldots(s-r^{a-1})(s-r^{a+1})\ldots(s-r^{l-2})}\\ &\qquad(a=2, 3, \ldots, l-2)\end{aligned}$$

where the exponent in the value of κ_a refers to Hilbert's symbolic powers. For any ideal J in the field k we have

$$J^{Q(s)}\sim1$$

where

$$Q(s)=q_0s^{l-2}+q_1s^{l-3}+\ldots+q_{l-3}s+q_{l-2}.$$

If $(\omega)=J^l$, then

$$\omega^{Q(s)}=\epsilon a^l$$

where ϵ is a unit of k. Set

$$\left.\begin{aligned}\omega&\equiv\omega^l\kappa_1^{a_1}\kappa_2^{a_2}\ldots\kappa_{l-1}^{a_{l-1}}\\ \epsilon&\equiv\epsilon^l\kappa_1^{e_1}\kappa_2^{e_2}\ldots\kappa_{l-1}^{e_{l-1}}\end{aligned}\right\} \pmod{\lambda^{l+1}},$$

so that

$$e_t=0, \quad (t=3, 5, \ldots, l-2, l-1)$$

then

$$\omega^{Q(s)}\equiv\omega^{lQ(s)}\kappa_1^{a_1Q(r)}\kappa_2^{a_2Q(r^2)}\ldots\kappa_{l-1}^{a_{l-1}Q(r^{l-1})} \pmod{\lambda^l},$$

which gives

$$\left.\begin{aligned}a_1Q(r)&\equiv e_1,\\ a_tQ(r^t)&\equiv e_t, \quad (t=2, 4, \ldots, l-3)\\ a_tQ(r^t)&\equiv 0, \quad (t=3, 5, \ldots, l-2)\\ a_{l-1}Q(r^{l-1})&\equiv0,\end{aligned}\right\} \pmod{l},$$

yielding

$$(34)\qquad \left\{\begin{aligned}B_{\frac{l-n}{2}}a_n&\equiv0, \quad (n=3, 5, \ldots, l-2)\\ a_{l-1}&\equiv0,\\ e_t&\equiv0, \quad (t=2, 4, \ldots, l-1)\end{aligned}\right\} \pmod{l}.$$

This is another form of a result obtained by Kummer[20] (p. 65) for the case where ω is of the form $\omega = x + y\zeta$, x, y being rational integers, but whose method may be immediately applied to any integer in the field k.

He derives also another criterion for the existence of l-th powers of ideals, which criterion was obtained by MIRIMANOFF[72] for a special case.

If μ, ν are the l-th powers of any ideals or units in the field k, then

$$(\mu^{s^n}, \nu) = 1, \quad (n = 0, 1, \ldots, l-2). \tag{35}$$

Set

$$\left.\begin{aligned} \mu &\equiv \mu^l \kappa_1^{e_1} \kappa_2^{e_2} \ldots \kappa_{l-1}^{e_{l-1}} \\ \nu &\equiv \nu^l \kappa_1^{e'_1} \kappa_2^{e'_2} \ldots \kappa_{l-1}^{e'_{l-1}} \end{aligned}\right\} \pmod{\lambda^l},$$

whence

$$\mu^{s^n} \equiv \mu^{l s^n} \mu^{e_1 r^n} \kappa_2^{e_2 r^{2n}} \ldots \kappa_{l-1}^{e_{l-1} r^{(l-1)n}} \pmod{\lambda^l}$$

and from (35) we obtain

$$\begin{aligned} r^n e_1 e'_{l-1} + 2r^{2n} e_2 e'_{l-2} + \ldots + a r^{an} e_a e'_{l-a} & \\ + \ldots + (l-1) r^{(l-1)n} e_{l-1} e'_1 \equiv 0 & \pmod{l}. \\ (n = 0, 1, \ldots, l-2). & \end{aligned}$$

These $l-1$ congruences give by elimination

$$e_a e'_{l-a} \equiv 0 \pmod{l},$$
$$(a = 1, 2, \ldots, l-1).$$

Taking

$$\nu = \frac{1-\zeta^r}{1-\zeta}$$

these relations give another proof of (34).

VANDIVER[73] stated without proof a number of results concerning (1). The proofs of most of these theorems were published later. Cf. Vandiver,[80, 82, 85, 88] Fueter,[69] formula (33a) for $k = 0$.

VANDIVER[74] proved Theorems I and IV of Kummer.[20] To complete Kummer's proof of Theorem I it is necessary to prove only the statement that if h_1 is divisible by λ but not by λ^2, then one and only one Bernoulli number B_ν in the set B_i, $i = 1, 2, \ldots, \mu - 1$, is divisible by λ. Using the relation (cf. Vandiver[82])

$$h_1 \equiv \lambda \prod_s \frac{(-1)^{(s\lambda^2-1)/2} B_{(s\lambda^2+1)/2}}{2^{(\lambda-3)/2}} \pmod{\lambda^2}, \tag{36}$$

$$(s = 1, 3, \ldots, \lambda - 2).$$

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[72] J. reine angew. Math., 128: 67 (1905). Cf. (23a) of this report.

[73] Bull. Am. Math. Soc., 28: 258-260 (1922).

[74] Bull. Am. Math. Soc., 28: 400-407 (1922).

with the assumption that $\lambda>5$, $B_a\equiv B_b\equiv 0 \pmod{\lambda}$ where a, b are each included in the set $1, 2, \ldots, \mu-1$, we obtain, after using

$$\frac{B_c}{c}\equiv(-1)^{k\mu}\frac{B_{c+k\mu}}{c+k\mu} \pmod{\lambda},$$

k being an integer and c not a multiple of μ (cf. Kummer[30]) for $a=(s+1)/2$,

$$B_{(s\lambda^2+1)/2}\equiv 0 \pmod{\lambda}.$$

Similarly,

$$B_{(s_1\lambda^2+1)/2}\equiv 0 \pmod{\lambda},$$

where $(s_1+1)/2=b$. Applying these relations to (36), we obtain $h_1\equiv 0 \pmod{\lambda^2}$.

Kummer's Theorem IV is then stated but without the inclusion of Assumption II, as is necessary. For the proof the formula a on page 53 of Kummer's memoir is examined and also the corresponding formula for the generalized function $\psi_r(\alpha)$,

$$\operatorname{ind} E_n(\alpha)\equiv\frac{\gamma^{2n}-1}{2(1+r^{\lambda-2n}-(r+1)^{\lambda-2n})}\cdot\frac{d_0{}^{\lambda-2n}\log\psi_r(e^v)}{dv^{\lambda-2n}} \pmod{\lambda},$$

where r is an integer, $1<r<\lambda-1$, $E_n(\alpha)$ and γ are defined as before, and ind $(E_n(\alpha))$ is i in the relation

$$(E_n(\alpha))^{(q^t-1)/\lambda}\equiv\alpha^i \pmod{\mathfrak{P}},$$

$\mathfrak{P}$ being an ideal prime factor of the odd prime q, and t the exponent to which q belongs modulo λ. Further

$$\psi_r(\alpha)=\sum_h\alpha^{-(r+1)h+\operatorname{ind}(g^h+1)},$$

where g is a primitive root of $\mathfrak{P}$ such that $g^{(q^t-1)/\lambda}\equiv\alpha \pmod{\mathfrak{P}}$, h ranges over the integers $0, 1, 2, \ldots, q^t-2$, excepting $(q^t-1)/2$; $\operatorname{ind}(g^h+1)$ being defined as i in the relation $(g^h+1)\equiv g^i \pmod{\mathfrak{P}}$.

The case $q=2$ is exceptional and MITCHELL's[75] definition of $\psi_r(\alpha)$ is used in that case. Using the latter's decomposition of $\psi_r(\alpha)$ into ideal factors, it is shown that

$$\psi_r(\alpha)^{H_1\lambda}=\pm\alpha^d\Pi F(\alpha^{\gamma^h}) \tag{37}$$

where $H_1\lambda$ is the class number of $\Omega(\alpha)$, H_1 prime to λ and h ranges over the values in the set $0, 1, 2, \ldots, \lambda-2$, such that

$$\gamma_{\mu-h}+\gamma_{\mu-h+\operatorname{ind} r}>\lambda,$$

using Kummer's notation.

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[75] Trans. Am. Math. Soc., 17: 165 (1916).

Since Kummer employs (p. 54 of his article) without proof or reference the relation

$$(38)\qquad \frac{d_0^{m\lambda^r}\log\phi(e^v)}{dv^{m\lambda^r}} \equiv \frac{d_0^{m\lambda^r}\log\phi_1(e^v)}{dv^{m\lambda^r}} \pmod{\lambda^{r+1}},$$

m not being a multiple of $\lambda-1$, and $\phi(a)\equiv\phi_1(a) \pmod{\lambda^{r+1}}$, a special case of this theorem is here proved. Using

$$\phi(e^v)=\phi_1(e^v)+W(e^{vp}-1)+cV,$$

where c is a rational integer and $V=(e^{vp}-1)(e^v-1)$, differentiating $m\lambda$ times with respect to V, then the relation (38) is obtained under the assumptions that $\phi(1)=\phi_1(1)$ and $\phi(a)$ is prime to λ. Applying this to (37) after raising to the power $\lambda-1$, we obtain Kummer's relation

$$\text{ind}\, E_\nu(a)\equiv -\frac{(-1)^{\lambda+\mu}(\gamma^{2\nu}-1)B_{\nu\lambda-\mu}}{2H_1\lambda}\cdot\frac{d_0^{\lambda-2\nu}lG(e^v)}{dv^{\lambda-2\nu}} \pmod{\lambda}.$$

(Cf. Kummer,[30] p. 57.)

Bliss[75a] commented on Fermat's last theorem stating that one may with justice feel very uncertain of its validity.

Pollaczek[76] considered at length the properties of irregular fields defined by primitive l-th roots of unity and primitive l^2-th roots of unity and considered the relation of these properties to Fermat's last theorem. He quotes the theorems A to K of Furtwängler.[48] The first field referred to he designates by $K(\zeta)$ and the second field by $K(\epsilon)$. He notes first that it follows from Furtwängler, VI, p. 29, that the class number of $K(\epsilon)$ is divisible by l if, and only if; l is an irregular prime. He indicates two other methods for proving this, the last of which involves the use of Furtwängler's theorem H. He then notes that the method of proof can also be carried over and yields

Theorem I. When the class number H_1 of the field $K(\zeta)$ is prime to l, then also the class number of the field obtained by adjoining the l-th root of η to $K(\zeta)$ where η is a unit in $K(\zeta)$, is also prime to l.

Pollaczek then considers the canonical expressions for units and the ideal classes in both $K(\zeta)$ and $K(\epsilon)$. In what follows, η defines a unit in $K(\zeta)$, H is a unit in $K(\epsilon)$; the ideals and classes belonging to $K(\zeta)$ will be designated by small German letters and the ideals and classes in $K(\epsilon)$ with large German letters. Further, r is a primitive root modulo l^2 and s stands for the substitution $(\zeta:\zeta^r)$ in $K(\zeta)$ and also for the substitution $(\epsilon:\epsilon^r)$ in $K(\epsilon)$. He then proves

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[75a] Bull. Am. Math. Soc., 29: 161-162 (1923).

[76] Math. Zeit., 21: 1-38 (1924).

Theorem II. In the field $K(\zeta)$ there exists a system of fundamental real units η_i which have the property

$$\eta_i^{s-r^{2i}}=\eta'^{\,l}_i \qquad \left(i=1, 2, \ldots, \frac{l-3}{2}\right).$$

The proof of this theorem will be given in full as the method used is similar to a number of others used in this paper. Select a fundamental system of real units $\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_{\frac{l-3}{2}}$, and apply to them the substitution s. We may then write

$$(39) \qquad \begin{cases} \tilde{\eta}_1^{s}=\pm\tilde{\eta}_1^{a_{11}}\tilde{\eta}_2^{a_{12}}\cdots\tilde{\eta}_{\frac{l-3}{2}}^{a_{1,\frac{l-3}{2}}}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \tilde{\eta}^{s}_{\frac{l-3}{2}}=\pm\tilde{\eta}_1^{a_{\frac{l-3}{2},1}}\tilde{\eta}_2^{a_{\frac{l-3}{2},2}}\cdots\tilde{\eta}_{\frac{l-3}{2}}^{a_{\frac{l-3}{2},\frac{l-3}{2}}} \end{cases}.$$

We then apply to each of these equations the substitution s and obtain the units $\tilde{\eta}_i^{s^2}$ as products of the $\tilde{\eta}_i^{s}$ which in turn may be replaced as the products of powers of the $\tilde{\eta}_i$. We proceed further in this manner but since

$$\tilde{\eta}_i^{s^{\frac{l-1}{2}}}=\tilde{\eta}_i$$

it follows that the matrix A of the rational integers a_{ik} satisfies the relation

$$A^{\frac{l-1}{2}}=E_{\frac{l-3}{2}}$$

where $E_{\frac{l-3}{2}}$ is the unit matrix of degree $\dfrac{l-3}{2}$.

The characteristic equation of A will then have as roots $\left(\dfrac{l-1}{2}\right)$-th roots of unity. We shall assume temporarily that the characteristic equation of A is

$$(40) \qquad \frac{t^{\frac{l-1}{2}}-1}{t-1}=0.$$

Considering this relation as a congruence modulo l, we will have all the distinct quadratic residues of l as roots with the exception of unity. We may therefore select a matrix T' with rational integers as elements t'_{ik}, whose determinant $|T'|\equiv 1 \bmod l$ so that in $T'A(T')^{-1}$ the elements of the principal diagonal are the $\dfrac{l-3}{2}$ quadratic residues not equal to 1 and the other elements congruent to zero, that is $T'A(T')^{-1}$ is congruent to a diagonal matrix modulo l. Corresponding to a matrix T^{v} for which

$|T'| \equiv 1 \pmod{N}$ there exists, because of a theorem of Minkowski's, a matrix whose determinant $|T| = 1$ and $T \equiv T'$ (mod N) where N is an arbitrary rational integer. Applying this result to the case $N = l$, there will therefore exist a system of fundamental units η_i so that

$$\eta_i = \tilde{\eta}_i^{\,t_{11}} \tilde{\eta}_2^{\,t_{12}} \dots \tilde{\eta}_{\frac{l-3}{2}}^{\,t_{1,\frac{l-3}{2}}},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\eta_{\frac{l-3}{2}} = \tilde{\eta}_1^{\,t_{\frac{l-3}{2},1}} \tilde{\eta}_2^{\,t_{\frac{l-3}{2},2}} \dots \tilde{\eta}_{\frac{l-3}{2}}^{\,t_{\frac{l-3}{2},\frac{l-3}{2}}}$$

where the determinant of the rational integers $t_{ik} = 1$. Applying the substitution s to the foregoing we obtain

$$\eta_1^{s} = \pm \eta_1^{b_{11}} \eta_2^{b_{12}} \dots \eta_{\frac{l-3}{2}}^{b_{1,\frac{l-3}{2}}},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\eta^{s}_{\frac{l-3}{2}} = \pm \eta_1^{b_{\frac{l-3}{2},1}} \eta_2^{b_{\frac{l-3}{2},2}} \dots \eta_{\frac{l-3}{2}}^{b_{\frac{l-3}{2},\frac{l-3}{2}}}$$

and if B designates the matrix of the b_{ik} then $B = (TA(T')^{-1})$ and by the known property of this matrix we have by a properly selected notation $b_{ii} \equiv r^{2i}$, $b_{ik} \equiv 0 \pmod{l}$; $i \neq k$. Setting these values in the last set of equations we obtain our theorem under the assumption made as to the characteristic equation.

To complete the demonstration we show that (40) is the characteristic equation of A. Assume that this is not true, then the equation of lowest degree with rational integral coefficients which A satisfies,

$$G(x) = g_0 + g_1 x + \dots + g_n x^n = 0,$$

is such that $n < \dfrac{l-3}{2}$.

Consider the expressions $\tilde{\eta}_1^{G(s)}, \left(i = l,\ 2,\ \dots,\ \dfrac{l-3}{2}\right)$ and transform them by means of the equations (39), so that

$$\tilde{\eta}_1^{G(s)} = \pm \tilde{\eta}_1^{\,n_{11}} \tilde{\eta}_2^{\,n_{12}} \dots \tilde{\eta}_{\frac{l-3}{2}}^{\,n_{1,\frac{l-3}{2}}},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\tilde{\eta}_{\frac{l-3}{2}}^{G(s)} = \pm \tilde{\eta}_1^{\,n_{\frac{l-3}{2},1}} \tilde{\eta}_2^{\,n_{\frac{l-3}{2},2}} \dots \tilde{\eta}_{\frac{l-3}{2}}^{\,n_{\frac{l-3}{2},\frac{l-3}{2}}}.$$

For the matrix $N = (n_{ik})$ we have

$$N = g_0 + g_1 A + \dots + g_n A^n = G(A) = 0$$

because of the hypothesis concerning $G(A)$.

We therefore have for the units $\tilde{\eta}_i$, as well as for all real units of the field $k(\zeta)$,

$$\tilde{\eta}^{G(s)} = \pm 1.$$

Since $G(x)$ is a polynomial of degree $n < \frac{l-3}{2}$, it follows that the regulator of the field $k(\zeta)$ vanishes showing that our assumption that (40) was not the characteristic equation of A, leads to a contradiction. An extension of this method yields a proof of:

Theorem IIa. In the field $K(\zeta)$, there exists for every integer $c \geqq 1$, a fundamental system of real units $\eta_{i,c}$ with the property

$$\eta_{i,c}^{s-r^{a_i}l^{c-1}} = \eta'^{\,l^c}_{i,c}$$

$$(i=1, 2, \ldots, (l-3)/2).$$

Since $K(\zeta)$ is irregular, the class number $H_l = l^h q$, $q \not\equiv 0$ modulo l. In considering the group of classes in this field it is sufficient for the purposes of this paper to consider only the q-th powers of the ideal classes of $K(\zeta)$, which will be termed the irregular class group. In this sense, two ideal classes $\mathfrak{j}$, $\mathfrak{j}'$ are the same if $\mathfrak{j}^q = \mathfrak{j}'^q$. For the class of an ideal $\mathfrak{j}$ contained in an irregular group of ideal classes, we may take the class to which the ideal $\mathfrak{j}^{q \cdot q'}$ $(q \cdot q' \equiv 1$, modulo $l^h)$ belongs. With these definitions we have

Theorem III. In $K(\zeta)$ we may select for the group of the q-th powers of the ideal classes a normal basis

$$\mathfrak{j}_1, \ldots, \mathfrak{j}_e,\ \mathfrak{j}_i^{\,l^{h_i}} = 1,\ \mathfrak{j}_i^{\,l^{h_i-1}} \neq 1, \qquad (41)$$

$$(i=1, 2, \ldots, e)$$

such that

$$\mathfrak{j}_1^{\,s-c_1} = 1, \ \ldots, \ \mathfrak{j}_e^{\,s-c_e} = 1, \qquad (41a)$$

where the c's are rational integers.

The demonstration of this theorem is analogous to that of Theorem II, depending on the symbolic matrix method as there employed. The proof is carried out for the case $e=4$, but the method is general.

Theorem IV is then stated and proved by Pollaczek. He states a result for the field $K(\epsilon)$ which is analogous to the result given in Theorem II for the case $K(\zeta)$.

In the same way Theorem V is stated and proved and it contains a result for the field $K(\epsilon)$ which is analogous to that stated in Theorem III for the field $K(\zeta)$.

Pollaczek next examines the connection between the irregular class group as expressed in Theorem III with the singular primary numbers of the field $K(\zeta)$. Using Theorems IIa and III of this paper, together with Theorems B and C of Furtwängler,[43] there is obtained

Theorem VI. If among the elements of a normal basis of the irregular class group of $K(\zeta)$ there exists for a certain quadratic non-residue b_0 exactly z_{b_0} classes

$$\mathfrak{p}_{m_1}, \mathfrak{p}_{m_2}, \ldots, (\mathfrak{p}_{m_i}^{s-b_{m_i}}=1)$$

such that $b_{m_i} \equiv b_0$, modulo l, then there are either z_{b_0} or $z_{b_0}-1$ basis classes

$$\mathfrak{q}_{n_1}, \mathfrak{q}_{n_2}, \ldots, (\mathfrak{q}_{n}^{s-a_{n_i}}=1)$$

where $a_{n_i} \equiv r/b_0$ modulo l. In particular, if the second case holds, the unit $\eta_{\frac{1}{2}\,\mathrm{ind}\,r/b}$ is a singular primary unit of the type given in Theorem II. In the above, we designate by $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, (\mathfrak{p}_i^{s-b_i} \sim 1)$ the ideals of those classes $\mathfrak{j}$ in (41), (41a), in which the c_i's are quadratic non-residues b_i; we designate also by $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, (\mathfrak{q}_i^{s-a_i} \sim 1)$ the ideals of those classes $\mathfrak{j}_i$ in (41), (41a), in which the c_i's are quadratic residues a_i.

In the above theorem it follows that for the classes $\mathfrak{p}$ we have

$$\mathfrak{p}_i^{s^{\frac{l-1}{2}}-b_i^{\frac{l-1}{2}}} = \mathfrak{p}_i^{s^{\frac{l-1}{2}}+1} = 1.$$

Also, for the $\mathfrak{q}$-classes we have from Theorem VI

$$\mathfrak{q}_i^{s^{\frac{l-1}{2}}-1} = 1, \quad \mathfrak{q}_i^{s^{\frac{l-1}{2}}+1} = \mathfrak{q}_i^2$$

in $K(\zeta)$ and $K(\zeta+\zeta^{-1})$. It then follows that the $\mathfrak{q}$-classes contain only real ideals in $K(\zeta)$ and the $\mathfrak{p}$-classes only imaginary ideals. Since the second factor of the class number $K(\zeta)$ is equal to the class number of $K(\zeta+\zeta^{-1})$, it follows that the order of the group of the $\mathfrak{q}$-classes is equal to the highest power of l in the second factor of the class number of $K(\zeta)$, and the order of the $\mathfrak{p}$-class group is equal to the highest power of l in the first factor.

Pollaczek then notes that, analogous to Theorem VI, we have a theorem concerning what he terms $\mathfrak{P}$-basis-classes containing only imaginary ideals in the field $K(\epsilon)$, and $\mathfrak{Q}$-classes which contain only real ideals and are therefore in the field $K(\epsilon+\epsilon^{-1})$. Analogous to the results for the field $K(\zeta)$ we then have the theorem that since the second factor of the class number of $K(\epsilon)$ is equal to the class number of $K(\epsilon+\epsilon^{-1})$, (cf. Weber, Algebra, II: 784), then the order of the $\mathfrak{P}$-class group and the $\mathfrak{Q}$-class group are equal to the highest powers of l contained in the first and second factors of the class number of $K(\epsilon)$, respectively.

By means of Furtwängler's[43] theorem "F," it is then shown that the number of $\mathfrak{P}$-classes is not less than the number of $\mathfrak{p}$-basis-classes in

(41), and also that the number of $\mathfrak{Q}$-classes is not less than the number of $\mathfrak{q}$-classes in the irregular class field of $K(\zeta)$. In particular, we have (cf. Furtwängler[40])

Theorem Ia: If the class number of the field $K(\zeta+\zeta^{-1})$ is prime to l then the class number of the field $K(\epsilon+\epsilon^{-1})$ is prime to l.

The following theorem is then proved:

Theorem VII: An ideal of the $\mathfrak{p}$-class group which does not belong to the principal class in $K(\zeta)$, also does not belong in the principal class in the field $K(\epsilon)$.

Pollaczek then proceeds to the investigation of necessary conditions for the existence of a $\mathfrak{p}$-class of given order as well as a $\mathfrak{P}$-class of given order. We have the equivalence

$$\mathfrak{j}^{q_0+q_1 s^{-1}+\dots+q_{l-2}s^{-(l-2)}}\sim 1 \tag{42}$$

(cf. Hilbert, Bericht., p. 360) where $\mathfrak{j}$ is any ideal in the field $K(\zeta)$,

$$q_k=\frac{rr_k-r_{k+1}}{l}$$
$$(k=0, 1, \dots, l-2)$$

and r_k is the least positive residue of r^k modulo l. If an ideal $\mathfrak{p}_\iota$ of the $\mathfrak{p}$-basis-class of order l^{h_ι} exists, then

$$\mathfrak{p}_\iota^{s-b_\iota}\sim 1 \tag{43}$$

where

$$b_\iota\equiv r^{-Tl^{h_\iota-1}}, \quad \text{mod } l^{h_\iota}$$

and $T\leqq l-2$, which follows from Theorem III.

Combination of (42) and (43) gives

$$\mathfrak{p}_\iota^{q_0+q_1 b_1^{-1}+\dots+q_{l-2}b_\iota^{-(l-2)}}\sim 1$$

and

$$\sum_{k=0}^{l-2} q_k b_\iota^{-k}\equiv\sum_k q_k r^{kTl^{h_\iota-1}}\equiv 0, \quad \text{mod } l^{h_\iota}.$$

By extension of a method employed by Kronecker[19] in connection with obtaining the criterion that the first factor of the class number of $K(\zeta)$ is divisible by l (cf. Hilbert, Bericht., p. 431), we have

$$\sum_{k=0}^{l-2} q_k r^{kTl^{h_\iota-1}}\equiv\frac{r^{l^{h_\iota-1}T+1}-1}{r^{l^{h_\iota-1}T}(l^{h_\iota-1}T'+1)}\cdot\frac{1}{l}\sum_{x=1}^{l-1}x^{l^{h_\iota-1}T+1}, \quad \text{mod } l^{h_\iota},$$

and using the Bernoulli summation formula and the Von Staudt-Clausen theorem (cf. Vandiver,[28] 459-460), we have

$$\sum_{k=0}^{l-2} q_k r^{kTl^{h_\iota-1}}\equiv\frac{r^{l^{h_\iota-1}T+1}-1}{r^{l^{h_\iota-1}T}(l^{h_\iota-1}T+1)}\cdot(-1)^{\frac{lT+3}{2}}B_{\frac{l^{h_\iota-1}T+1}{2}}, \quad \text{mod } l^{h_\iota}$$

and therefore we have proved

Theorem VIII: A necessary condition that there exists an imaginary ideal belonging to the exponent l^{h_i} in $K(\zeta)$ is that one of the $(l-3)/2$ Bernoulli numbers

$$B_{\frac{l^{h_i-1}T+1}{2}}, \qquad T=l-4,\, l-6,\, \ldots$$

is divisible by l^{h_i}.

It is then noted that this theorem is not sufficient for the existence of an ideal of the type mentioned. However, it is shown that we have

Theorem IX: If for every odd $T<l-2$ such that

$$B_{\frac{T+1}{2}} \equiv 0, \quad \mathrm{mod}\; l,$$

$h^{(T)}$ is the greatest integer for which

$$B_{\frac{l^{h^{(T)}-1}T+1}{2}} \equiv 0, \quad \mathrm{mod}\; l^{h^{(T)}},$$

then the order of the group of classes of the imaginary ideals in $K(\zeta)$ is equal to

$$l^{\sum\limits_T h^{(T)}}.$$

For proof, the first factor A_l of the class number of $K(\zeta)$ is used in the form

$$A_l = \frac{g(Z)g(Z^3)g(Z^5)\ldots g(Z^T)\ldots g(Z^{l-2})}{(-1)^{\frac{l-1}{2}}\, 2^{\frac{l-3}{2}} \cdot \dfrac{r^{\frac{l-1}{2}}+1}{l}}, \tag{43a}$$

where Z is a primitive $(l-1)$-th root of unity and

$$g(Z) = \sum_{k=0}^{l-2} q_k Z^k.$$

Pollaczek states that the criterion follows from the reduction of A_l modulo $\mathfrak{L}^{h_i}$ where $\mathfrak{L}=(l,\, r-Z)$. (Cf. Vandiver.[illegible])

Pollaczek then considers analogous problems in $K(\epsilon)$. He refers to Hilbert's Bericht., p. 376, and remarks that a form of the class number of the field defined by m-th roots of unity reduces for $m=l^2$ to

$$A_{l^2} = \frac{G(\bar{Z})G(\bar{Z}^3)G(\bar{Z}^5)\ldots g(\bar{Z}^{l^2-l-1})}{(-1)^{\frac{l-1}{2}}\, 2^{\frac{l^2-1}{2}-1} \cdot \dfrac{r^{\frac{l^2-l}{2}}+1}{l^2}}$$

where

$$G(\bar{Z}) = \sum_{k=0}^{l^2-l-1} Q_k \bar{Z}^k$$

and $\bar{Z}$ is a primitive (l^2-l)-th root of unity,

$$Q_k=\frac{rR_k-R_{k+1}}{l^2}$$

and R_k is the least positive residue of r^k modulo l^2. Setting $Z=Z\zeta$, we find

$$G(Z^T)=g(Z^T).$$

Corresponding to each factor $G(Z^T)$ in the numerator of (43a) which is divisible by $\mathfrak{L}$, there is a factor of $G(Z^T\zeta)$, namely

$$\mathfrak{L}'=(\mathfrak{L},\ 1-\zeta), \qquad \mathfrak{L}'^{l-1}=\mathfrak{L},$$

contained to at least the first power; consequently A_{l^2}/A_l is divisible by at least l^{n_l} where n_l is the number of distinct Bernoulli numbers in the set B_1, B_2, $\ldots$, $B_{(l-3)/2}$ which are divisible by l.

The possibility of A_{l^2}/A_l being divisible by a higher power of l than l^{n_l} is then investigated. If this condition holds, then

$$G(Z^T\zeta)\equiv 0, \quad \bmod \mathfrak{L}'^2.$$

Developing the left-hand member of this congruence in ascending powers of $1-\zeta$ gives

$$\sum_{k=0}^{l^2-l-1} kQ_kZ^{kT}\equiv 0, \quad \bmod \mathfrak{L}.$$

This congruence is then transformed by methods analogous to those used in the proof of Theorem VIII and there is obtained

$$\sum_{k=0}^{l^2-l-1} kQ_kr^{kT}\equiv \frac{1}{1-r^{l-1}}\cdot\frac{r^{T+1}-1}{2Tr^T}\left(B'_{\frac{T+1}{2}+\frac{l-1}{2}}-B'_{\frac{T+1}{2}}\right), \quad \bmod l,$$

where

$$B'_n=\frac{(-1)^nB_n}{n}$$

and this yields

Theorem X: When there are exactly n_l numbers among the first $(l-3)/2$ Bernoulli numbers divisible by l, then A_{l^2}/A_l is at least divisible by l^{n_l} and in order that A_{l^2}/A_l is divisible by l^{n_l+1}, it is necessary and sufficient that

$$(44) \qquad \begin{cases} \dfrac{B_{\frac{T+1}{2}+\frac{l-1}{2}}}{\frac{T+l}{2}}\equiv\dfrac{(-1)^{\frac{l-1}{2}}B_{\frac{T+1}{2}}}{\frac{T+1}{2}}, \quad \bmod l^2, \\[2ex] B_{\frac{T+1}{2}}\equiv 0, \quad \bmod l, \end{cases}$$

for some $T<l-2$.

For fixed T, then the $\mathfrak{p}$-basis ideals in (41) in which $c \equiv r^{-T}$ (modulo l), have relative norms which give an equal number of $\mathfrak{P}$-basis classes in the irregular class group of $K(\epsilon)$ with $c \equiv r^{-T}$ and for such ideal we have

$$\mathfrak{P}^{sS-r^{-l^{\nu}T}} \sim 1,$$

as well as

$$\mathfrak{P}^{Q_0+Q_1s^{-1}+\ldots+Q_{l^2-l-1}s^{-(l^2-l-1)}} \sim 1,$$

the latter relation, which is analogous to (42), being satisfied by any ideal of $K(\epsilon)$. Using these two equivalences, the following theorem is obtained:

Theorem Xa: The existence of the congruences (44) is a necessary condition that there exists at least one $\mathfrak{P}$-class $\mathfrak{P}^{sS-r^{-l^{\nu}T}}=1$ of the type that the class $\mathfrak{P}^{1-S}$ contains no ideals in $K(\zeta)$.

The case where the class number of $K(\zeta)$ is divisible by l but not by l^2 is then treated and there is obtained

Theorem XI: If the class number of the field $K(\zeta)$ is divisible by l but not by l^2, and $B_\nu \equiv 0$, modulo l, $\nu<(l-1)/2$, and

$$B_{\nu+\frac{l-1}{2}} \not\equiv B'_\nu, \quad \text{mod } l^2,$$

then the sub-group of the irregular classes of $K(\epsilon)$ is cyclic and of order l^2, and for an ideal $\mathfrak{P}$ of a basis-class of this group we have

$$\mathfrak{P}^{s-r^{1-2\nu-(l-1)y}} \sim 1$$

and also

$$\frac{B_{\nu+y\cdot\frac{l-1}{2}}}{\nu+y\cdot\dfrac{l-1}{2}} \equiv 0, \quad \text{mod } l^2.$$

In the field $K(\epsilon)$ there exists a unit which is congruent to the l^2-th power of another unit modulo $l^2 \cdot \lambda$ which is not itself an l-th power of a unit. The proof depends on special applications of Pollaczek's methods previously described.

The above results are then applied to Fermat's last theorem and Pollaczek concludes that Bernstein's[45] results on the first case of the last theorem are equivalent to Hecke's.[46] He also points out that the criterion given by Bernstein[47] (p. 508, Assumption I), is never satisfied. (Cf. Vandiver.[48]) He shows also that the second assumption of Bernstein (p. 508) is not satisfied for $l=37$, 59, 67, hence the results of Bernstein do not include those of Kummer.[49]

VANDIVER[77] transformed the Kummer criteria (23), starting with the obvious identity

$$f_1(x)f_1(y)=\frac{(x^p-1)f_1(y)}{x/y-1}-\frac{x/y(y^p-1)f_1(x)}{x/y-1},$$

where $f_1(u)=1+u+u^2+\ldots+u^{p-1}$. Setting $x=xe^{hv}$, $y=ye^{kv}$, h and k being indeterminates not zero, multiplying through by $(x/y)^p-1$, differentiating a times ($a\leqq p-2$), with respect to v, and putting $v=0$, we have an identity in x and y which, after setting $x=xe^v$, $y=ye^v$, differentiating $p-2-a$ times with respect to v, setting $v=0$, gives another identity in x and y.

In the last identity we set $y=\rho$, where ρ is an m^{th} root of unity, $\neq 1$, and $x=t$, where $-t=\frac{u}{v}, \frac{v}{u}, \frac{u}{w}, \frac{w}{u}, \frac{v}{w}, \frac{w}{v}$; $u^p+v^p+w^p=0$. This gives

$$\begin{aligned}(t^{pm}-1)\sum{}' &\frac{A_1(t,\rho)}{((t/\rho)^p-1)(\rho^p-1)}\\ &=\sum{}'\frac{(t^{pm}-1)}{(t/\rho)^p-1)}B_2-\sum{}'\frac{(t^{pm}-1)}{(t/\rho)^p-1)}C_2+G_3p,\\ &=B_3-C_3+G_3p,\end{aligned}$$

where

$$B_3=(t^p-1)\Sigma'\sum_{n=1}^{a+1}\tbinom{a}{n-1}k^{a+1-n}(h-k)^{n-1}F_n\left(\frac{t}{\rho}\right)\frac{f_{p-n}(\rho)}{\rho^p-1},$$

$$C_3=\Sigma'\sum_{n=1}^{a+1}\tbinom{a}{n-1}h^{a+1-n}(h-k)^{n-1}F'_n\left(\frac{t}{\rho}\right)f_{p-n}(t),$$

Σ' represents a summation over all distinct values of $\rho\neq 1$,

$$F_n(z)=\sum_{r=1}^{pm-1}r^{n-1}z^r,\quad (n>1),\quad F_1(z)=\sum_{r=0}^{mp-1}z^r,$$

$$F'_1(z)=\sum_{s=1}^{pm}z^s,\quad F'_n=F_n,\quad (n>1),$$

$$f_n(z)=\sum_{r=1}^{p-1}r^{n-1}z^r,\quad n>1.$$

Using the Kummer criteria (23) in connection with this we obtain

$$k^a(t^{pm}-1)\sum\frac{f_1(t)f_{p-1}(\rho)}{\rho^p-1}\equiv B_3-C_3\pmod{p},$$

which, for $a=1$, gives

$$\begin{aligned}k(t^{pm}-1)\Sigma' f_1(t)\frac{f_{p-1}(\rho)}{\rho^p-1}&\equiv(t^p-1)k\Sigma' F_1\left(\frac{t}{\rho}\right)\frac{f_{p-1}(\rho)}{\rho^p-1}\\ &+(t^p-1)(h-k)\Sigma' F_2\left(\frac{t}{\rho}\right)\frac{f_{p-2}(\rho)}{\rho^p-1}\\ &-h\Sigma' F_1\left(\frac{t}{\rho}\right)f_{p-1}(t)-(h-k)\Sigma' F_2\left(\frac{t}{\rho}\right)f_{p-2}(t)\pmod{p}.\end{aligned}$$

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[77] **Annals of Math., (2), 26: 88-94 (1924).**

In this relation put $h \equiv 0 \pmod{p}$ and $h \not\equiv k \pmod{p}$. We have for $k=1$, $f_{p-2}(t) \equiv 0$, and hence

$$\Sigma' F_2 \left(\frac{t}{\rho}\right) \frac{f_{p-2}(\rho)}{\rho^p - 1} \equiv 0 \pmod{p}. \tag{45}$$

This congruence is reduced for $m=3$ to a relation which yields the criterion

$$1 + \frac{1}{2^2} + \ldots + \frac{1}{\left[\frac{p}{3}\right]^2} \equiv 0 \pmod{p}$$

for the solution of (1) in Case I. Cf. Pollaczek,[61] p. 59.

It is noted that this criterion is not satisfied for any $p < 32$. It is possible that analogous criteria may be obtained by setting $m=4$, 5, etc., in (45).

Pomey[78] investigated Fermat's last theorem by elementary methods. Of his three principal results (theorems IV ter, V ter, and III bis) given without reference, the first two were known previously (cf. Vandiver, Trans. Am. Math. Soc., 15: 203 (1914), relations (5) and (6), Frobenius,[56] pp. 661-3) and the third may be derived immediately from them. By means of these theorems he proved Fermat's last theorem in Case I, for various prime exponents less than 100. By employing the criterion of Sophie Germain (Legendre[1]), he proved Fermat's last theorem in Case I for eleven prime exponents between 9048 and 10000 as well as for twenty-four other exponents between 5000000 and 5003250.

Beeger[79] computed the residue r of the quotient $(2^\xi - 1)/p \pmod{p}$ where ξ is the least exponent satisfying the congruence $2^\xi \equiv 1 \pmod{p}$, for all primes $p < 14000$ and > 2000. By means of these results, which are tabulated in the paper, he found that 3511 was the only prime number > 2000 and < 14000 that satisfies the congruence $2^{p-1} \equiv 1 \pmod{p^2}$. As Dickson had proved that (1) had no solution in Case I for all exponents $p < 7000$, the present work shows that (1) has no solutions in Case I for any exponent $p < 14000$.

Vandiver[80] took the norm residue symbol

$$\left\{\frac{\nu, \mu}{\mathfrak{p}}\right\} = a^L$$

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[78] J. de Math., (9), 1925: 1-22; Abstract in Compt. Rend., 1923, 1187.

[79] Mess. of Math., 55: 17-26 (1925).

[80] Proc. Nat. Acad. Sci., 11: 292-298 (1925).

in the form given by HILBERT[81] where

$$L \equiv \sum_{s=1}^{p-1} (-1)^{s-1} l^{(s)}(\nu) l^{(p-s)}(\mu) \quad (\text{mod } p),$$

$$l^{(g)}(\omega) = \left[\frac{d^g \log (\omega(e^v))}{dv^g}\right]_{v=0}$$

$g = 1, 2, \ldots, p-1$; $\mathfrak{p} = (1-\alpha)$; $\nu \equiv 1 \pmod{\mathfrak{p}}$, $\mu \equiv 1 \pmod{\mathfrak{p}}$; ν and μ integers in the field $\Omega(\alpha)$, $\alpha = e^{2i\pi/p}$. The symbol $\omega(x)$ is defined as follows: If

$$\omega = c_0 + c_1\alpha + \ldots + c_{p-2}\alpha^{p-2},$$

then

$$\omega(x) = \sum_{r=0}^{p-2} c_r x^r - \frac{\sum_{r=0}^{p-2} c_r - 1}{p} \cdot \frac{x^p - 1}{x - 1}.$$

If ν and μ are any integers in the field prime to $\mathfrak{p}$ then

$$\left\{\frac{\nu, \mu}{\mathfrak{p}}\right\} = \left\{\frac{\nu^{p-1}, \mu^{p-1}}{\mathfrak{p}}\right\}. \tag{46}$$

Now if (γ) is a principal ideal and the p^{th} power of any ideal in $\Omega(\alpha)$ then (γ), $(x+\alpha^a y)$ being prime to each other and to $\mathfrak{p}$,

$$\left\{\frac{\gamma}{x+\alpha^a y}\right\} = 1 = \left\{\frac{x+\alpha^a y}{\gamma}\right\}, \tag{47}$$

since $(x+\alpha^a y)$ is the p^{th} power of an ideal in $\Omega(\alpha)$. Furtwängler gave a law of reciprocity between two integers in a field containing $\Omega(\alpha)$, which for the special case where the field is $\Omega(\alpha)$ itself becomes

$$\left\{\frac{\omega}{\theta}\right\} \left\{\frac{\theta}{\omega}\right\}^{-1} = \left\{\frac{\theta, \omega}{\mathfrak{p}}\right\}, \tag{48}$$

ω and θ being any two integers in $\Omega(\alpha)$ such that (ω) and (θ) are prime to each other and to $\mathfrak{p}$. Setting $\omega = \gamma^{p-1}$ and $\theta = (x+\alpha^a y)^{p-1}$ then (46), (47) and (48) give

$$\left\{\frac{x+\alpha^a y, \gamma}{\mathfrak{p}}\right\} = 1$$

or

$$L \equiv 0 \quad (\text{mod } p),$$

for

$$\nu = \theta, \quad \mu = \omega.$$

It is then shown that $\quad l^{(p-1)}(\omega) \equiv 0 \quad (\text{mod } p)$

and $\quad l^{(p-1)}(\theta) \equiv 0 \quad (\text{mod } p)$

and thence (cf. Takagi[71])

$$\sum_{s=2}^{p-2} (-1)^{s-1} a^s \left[\frac{d^s \log (x+e^v y)}{dv^s}\right]_{v=0} \left[\frac{d^{p-s} \log \gamma(e^v)}{dv^{p-s}}\right]_{v=0} \equiv 0 \quad (\text{mod } p).$$

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[81] Bericht über die Theorie der algebraischen Zahlkörper, Jahresber. deut. Math. Verein, 1894: 413.

After setting $a=1, 2, 3, \ldots, p-2$ in turn, we obtain by elimination a result which yields the theorem that if (1) is satisfied in rational integers $x, y, z,$ none zero and all prime to the odd prime p, then

$$f_{p-n}(t)\left[\frac{d^n \log \gamma(e^v)}{dv^n}\right]_{v=0} \equiv 0 \pmod{p},$$

$n=2, 3, \ldots, p-2$; $-t=x/y$. Here $\gamma(a)$ is an integer in the field $\Omega(a)$ and the principal ideal $(\gamma(a))$ is the p^{th} power of an ideal in $\Omega(a)$ which is prime to (z) and (p).

This result yields the following corollaries:

If there exists in addition to the solution (x, y, z) of (1) (xyz prime to p) another solution (x_1, y_1, z_1) z_1 prime to z and $x_1y_1z_1 \not\equiv 0 \pmod{p}$, then

$$f_{p-n}(t)f_n(t_1) \equiv 0 \pmod{p}$$

$t_1=-x_1/y_1$, $n=2, 3, \ldots, p-2$.

If (1) is satisfied for $xyz \not\equiv 0 \pmod{p}$, then

$$f_{p-n}(t)f_n(1-t) \equiv 0 \pmod{p}$$

$n=2, 3, \ldots, p-2$.

If (1) is satisfied in integers prime to the odd prime p, then

$$\left[\frac{d^{p-a} \log \gamma(e^v)}{dv^{p-a}}\right]_{v=0} \equiv 0 \pmod{p}.$$

for $a=2$ and 3.

Vandiver[82] employed the relation

$$(49) \qquad \prod_{\nu=1}^{k-1} \prod_{r=1}^{\left[\frac{\nu p}{k}\right]} (x+a^{[1:r]}y)^2 = a^{\frac{-2kyq(k)}{x+y}} \omega^{2p}$$

where k is an integer, $1<k<p$, ω is an integer in the field $\Omega(a)$,

$$q(k) = \frac{k^{p-1}-1}{p}$$

and $[1:r]$ is the least positive solution of $Xr \equiv 1 \pmod{p}$. Cf. Vandiver.[illegible] First it is proved that

$$(x+y)^{p-1} \equiv 1 \pmod{p^3}.$$

By means of this with (49) we obtain

$$(p-1)D_{np}\sum[1:r]^{np} \equiv p[D_{np}(\log \theta)]_{v=0} \pmod{p^2}$$

where

$$[D_{np}(\log \theta)]_{v=0} = \left[\frac{d^{np} \log \theta}{dv^{np}}\right]_{v=0}$$

$$D_{np} = [D_{np}(\log(x+e^v y))]_{v=0}.$$

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[82] Annals of Math., (2), 26: 217-232 (1925).

The summation extending over all values of r as given in (49), n is any integer, $\theta=\omega^{p-1}$. From this it is shown that

$$D_{p(p-2m)}\Sigma[1:r]^{p(p-2m)} \equiv -p\,\frac{2\text{ ind. } E_m(a)\,(k-k^{p-2m})}{\gamma^{2m}-1} \quad (\text{mod } p^2),$$

where m is any positive integer not greater than $p-1$ and

$$E_m(a)=e(a)e(a^{\gamma})^{\gamma^{-2m}}e(a^{\gamma^2})^{\gamma^{-4m}}\ldots(e^{\gamma^{\mu-1}})^{\gamma^{-2m(\mu-1)}}$$

γ being any primitive root of p, $\mu=(p-1)/2$. Also ind $E_m(a)$ is the value of M in the relation

$$\left\{\frac{E_m(a)}{\prod\limits_{s=1}^{i}\mathfrak{P}^{(s)}}\right\}=a^M,$$

where

$$E_m(a)^{\frac{N(\mathfrak{P})-1}{p}} \equiv \left\{\frac{E_m(a)}{\mathfrak{P}}\right\} \quad (\text{mod } \mathfrak{P})$$

$\mathfrak{P}$ being a prime ideal divisor of the principal ideal $(x+ay)$, that is,

$$x+ay=(\mathfrak{P}^{(1)}\mathfrak{P}^{(2)}\ldots\mathfrak{P}^{(i)})^p,$$

where $\mathfrak{P}^{(1)}, \mathfrak{P}^{(2)}, \ldots, \mathfrak{P}^{(i)}$ are prime ideals in $\Omega(a)$, and also

$$\left\{\frac{E_m(a)}{\prod\limits_{s=1}^{i}\mathfrak{P}^{(s)}}\right\}=\prod_{s=1}^{i}\left\{\frac{E_m(a)}{\mathfrak{P}^{(s)}}\right\}.$$

This relation yields, if $\Pi\mathfrak{P}^{(i)}=Q$

$$(50)\quad \text{ind}\left(\frac{1-a^s}{1-a}\right)$$
$$\equiv\frac{N(Q)-1}{2p}(s-1)+\sum_{m=1}^{\mu-1}\frac{(-1)^{l/2}B_{l/2}D_{p(p-2m)}(s^{2m}-1)}{2mp} \quad (\text{mod } p),$$

where s is any integer prime to p and the B's are the numbers of Bernoulli, $B_1=1/6$, $B_2=1/30$, etc. In the relation (49) put $k=2$, $a^h=a$, and take pth power characters of each side of the equation with respect to Q, and we obtain, with the use of $x+y=u^h$,

$$\prod_r\left\{\frac{\epsilon}{Q}\right\}=1$$

where

$$\epsilon=\frac{a^{\pm h[1:r]-1}-1}{a-1}$$

the sign of h being selected so that $h[1:r]\not\equiv 1$ (mod p).

Using the value of ind$((1-a^s)/(1-a))$ given in (50), and letting h range over the values 1, 2, …, $\mu-1$, we obtain $\mu-1$ congruences which yield on elimination certain congruences involving x and y, which in turn give

$$B_a D_{3p} \equiv B_{a-p} D_{5p} \equiv B_{a-2p} D_{7p} \equiv B_{a-3p} D_{9p} \equiv 0 \pmod{p^2}.$$

Finally, after some transformations, there is obtained the theorem that if (1) is satisfied in Case I, then

$$B_s \equiv 0 \pmod{p^2}$$

$s=(lp+1)/2$, $l=p-4$, $p-6$, $p-8$, $p-10$.

The above result also yields the theorem that if (1) is satisfied in Case I, then the first factor of the class number of the field $\Omega(e^{2i\pi/p})$ is divisible by p^3. It is noted that

$$B_s \not\equiv 0 \pmod{p^2}$$

$s=(lp+1)/2$, $l=p-4$, $p-6$, $p-8$, $p-10$, for any $p<167$.

Vandiver[33] attacked the last theorem by means of the theory of power characters in relative cyclic fields. Considering the relations (1) and (49), let n be a prime $\not\equiv 0$ or 1 $\pmod{p}$ and suppose that $xy \not\equiv 0 \pmod{n}$ then $x^{n-1}-y^{n-1} \equiv 0 \pmod{n}$, hence there is some integer in the set 0, 1, …, $n-2$, such that

$$x+\beta^a y \equiv 0 \pmod{\mathfrak{p}}$$

where β is a primitive $(n-1)$th root of unity, and $\mathfrak{p}$ is a particular prime ideal divisor of n in the field $\Omega(\alpha\beta)$. If θ is an integer in $\Omega(\alpha\beta)$, θ prime to (p) and $\mathfrak{p}$, then we define the p^{th} power character of θ as $(\theta/\mathfrak{p})$, which is the power of α appearing in the relation

$$\theta^{\frac{N(\mathfrak{p})-1}{p}} \equiv \left\{\frac{\theta}{\mathfrak{p}}\right\} \pmod{\mathfrak{p}},$$

where $N(\mathfrak{p})$ is the norm of $\mathfrak{p}$.

In (49) set α^m for α and in the resulting equation take the p^{th} power characters of each member with respect to $\mathfrak{p}$. Noting that

$$x+\alpha^c y = x+\beta^a y + y(\alpha^c-\beta^a),$$

and using (49), we have

$$\left.\begin{aligned} &\left\{\frac{x+\alpha^c y}{\mathfrak{p}}\right\} = \left\{\frac{\alpha^c-\beta^a}{\mathfrak{p}}\right\}, \\ &\text{and therefore} \\ &\mu(k-1)D_0 m^{p-1} + \sum(m[1:r])^{p-2} D_1 \\ &\qquad + \ldots + m\sum[1:r]D_{p-2} - \frac{mky}{x+y}\, q(k)I(\alpha) \equiv 0 \pmod{p}, \end{aligned}\right\}$$

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[33] Proc. Nat. Acad. Sci., 12: 106-109 (1926).

where $\mu=(p-1)/2$, Σ stands for

$$\sum_{\nu=1}^{k-1}\sum_{r=1}^{[\nu p/k]};\ \left\{\frac{\theta}{\mathfrak{p}}\right\}=\alpha^{I(\theta)},$$

$$D_s=\sum_{d=1}^{p-1}d^sI(\alpha^d-\beta^a).$$

Letting m range over the integers 1, 2, . . ., $p-1$, we obtain by elimination

$$(51)\qquad \left\{\begin{matrix} D_0\equiv\Sigma[1:r]^{p-s}D_{s-1}\equiv 0 \pmod p, \\ s=2,\ 3,\ \ldots,\ p-2; \\ \Sigma[1:r]D_{p-2}-\dfrac{kyq(k)}{x+y}\,I(\alpha)\equiv 0 \pmod p \end{matrix}\right\}$$

We also have, if one of the integers x, y, z, say y, is divisible by p,

$$\prod_h(z+\alpha^{[1:h]}x)^{p-1}=p^{\mu}\alpha^{\mu\Sigma[1:h]}\omega_1{}^p$$

where h ranges over the positive integers $<p$, such that $h+|rh|>p$, $|rh|$ representing the least positive residue of rh, modulo p, and ω_1 is an integer in $\Omega(\alpha)$. Taking p^{th} power characters of both sides of this with respect to $\mathfrak{p}$, and proceeding as in the preceding treatment of (49) we obtain

$$(52)\qquad \left\{\begin{matrix} D'_s\equiv\Sigma[1:h]^{p-s}D'_{s-1}\equiv 0 \pmod p, \\ s=2,\ 3,\ \ldots,\ p-2; \\ \Sigma[1:h]D'_{p-2}+\dfrac{\Sigma[1:h]I(\alpha)}{2}\equiv 0 \pmod p, \end{matrix}\right\}$$

where

$$D'_s=\sum_{d=1}^{p-1}d^s(\alpha^d-\beta^b)$$

b some integer, $0\leqq b<n-1$. A few transformations of (51) and (52) yield the theorems: (all details of proof not given).

If $x^p+y^p+z^p=0$ is satisfied in integers none zero and each prime to the odd prime p, then

$$q(n)D_0\equiv 0,\quad q(n)B_{\frac{s+1}{2}}D_s\equiv 0 \pmod p$$

$$s=1,\ 3,\ \ldots,\ p-4,$$

where

$$D_s=\sum_{d=1}^{p-1}d^sI(\alpha^d-\beta^a),\ \left\{\frac{\theta}{\mathfrak{p}}\right\}=\alpha^{I(\theta)},$$

$\mathfrak{p}$ is a prime ideal divisor in the field $\Omega(\alpha\beta)$ of a rational odd prime n which is $\not\equiv 0$ or 1 $\pmod p$; $\alpha=e^{2i\pi/p}$; $\beta=e^{2i\pi/(n-1)}$, a is some integer in the set 1, 2, . . ., $n-2$ other than $(n-1)/2$; the B's are the numbers of Bernoulli, $B_1=1/6$, $B_2=1/30$, etc.

If $x^p+y^p+z^p=0$ is satisfied in integers none zero and each prime to the odd prime p, then

$$q(n)\prod_{a=0}^{n-2}((1-v)D_{p-2}+I(\alpha))\equiv 0 \pmod{p},$$

where v has any one of the six values t, $\frac{1}{t}$, $1-t$, $\frac{1}{1-t}$, $\frac{t-1}{t}$, $\frac{t}{t-1}$, $x/y=t$ the other symbols being defined as in the above theorem.

If p is an odd prime and $x^p+y^p+z^p=0$ where $y\equiv 0 \pmod{p}$, $xz\not\equiv 0 \pmod{p}$, $x+z\not\equiv 0 \pmod{n}$, then

$$q(n)D_0\equiv 0$$

$$q(n)B_{s+1}D_s\equiv 0 \pmod{p},$$

$s=1, 3, \ldots, p-4$; and in addition one of the two relations

$$q(n)D_{p-2}\equiv 0, \qquad q(n)\left(D_{p-2}+\frac{I(\alpha)}{2}\right)\equiv 0 \pmod{p},$$

is satisfied, the other symbols being defined as above.

If $x^p+y^p+z^p=0$ is satisfied in integers none zero and all prime to the odd prime p, then

$$q(n)\prod_{a=1}^{n-2}((1-v)\operatorname{ind}(\alpha\beta^a-1)-q(n))\equiv 0 \pmod{p},$$

where
$$\left\{\frac{\alpha\beta^a-1}{\mathfrak{q}}\right\}=\alpha^i, \qquad i=\operatorname{ind}(\alpha\beta^a-1),$$

$\mathfrak{q}=(\beta-r,\ n)$; r a primitive root of n; v being defined as in the first theorem of this paper.

If $x^p+y^p+z^p=0$ is satisfied in integers none zero and all prime to the odd prime p, then

$$q(n)\prod_{a,\,b}(\operatorname{ind}(\alpha\beta^a-1)\operatorname{ind}(\alpha\beta^b-1) \\ -q(n)(\operatorname{ind}(\alpha\beta^a-1)+\operatorname{ind}(\alpha\beta^b-1)))\equiv 0 \pmod{p},$$

where a and b each range independently over the integers, 1, 2, ..., $n-2$, the other symbols being defined as in the first theorem.

If $xyz\not\equiv 0 \pmod{n}$, $z\not\equiv 0 \pmod{p}$, n a prime $\not\equiv 0 \pmod{p}$, then we have $q(n)\equiv 0 \pmod{p}$, or

$$\left.\begin{aligned}\sum_{s=2}^{p-2}(-1)^s k^s\left[\frac{d^s\log(x+e^v y)}{dv^s}\right]_{v=0}&\left[\frac{d^{p-s}\log\gamma(e^v)}{dv^{p-s}}\right]_{v=0}\\ &\equiv -\frac{kyh}{x+y}I(\alpha)+hI\left(\frac{\alpha^k-\beta^a}{1-\beta^a}\right) \pmod{p},\end{aligned}\right\}$$

where h is the integer to which the ideal $\mathfrak{Q}$ belongs, $\mathfrak{Q}$ being a prime ideal divisor of n in the field $\Omega(\alpha)$, $\mathfrak{Q}^h=(\gamma(\alpha))$, $\gamma(\alpha)$ being an integer in $\Omega(\alpha)$, k is any integer $\not\equiv 0 \pmod{p}$, a is some integer in the set 1, 2, ..., $n-2$, the other symbols being defined as in the first theorem, as criteria for the solution of (1) in rational integers, none zero.

VANDIVER[84] proved by means of Furtwängler's[80] theorem that if there exists an odd prime q such that $\xi^p+\eta^p+\zeta^p\equiv 0 \pmod{q}$ has no set of integral solutions each not divisible by q and such that $q\not\equiv 1 \pmod{p^2}$ then (1) has no solutions each prime to p. He also proved, using Sophie Germain's[1] result, a theorem due to Wendt.[26] These two theorems prove that if the above congruence has no set of integral solutions each prime to q, where $q=1+mp$ and $m<10p$, then (1) has no solutions in integers each prime to p.

VANDIVER[85] proved that if

$$u^p+v^p+w^p=0$$

is satisfied by any integers prime to the odd prime p, then

$$\sum_{l=0}^{k-1} h_n{}^{(k,\,l)}(t)\,h_{(p-n)}^{(k,\,k-l)}(t)\equiv 0 \pmod{p}, \tag{53}$$
$$n=1,2,\ldots,p-1,$$

k being any positive integer, where

$$h_n{}^{(k,\,l)}(x)=\sum_{s=0}^{p-1}(ks+l)^{n-1}x^s.$$

For $k=2$, this reduces to

$$h_n{}^{(2,\,1)}(t)\,h_{p-n}^{(2,\,1)}(t)\equiv 0 \pmod{p}.$$

The proof of this result is obtained by means of transformations of the Kummer criteria (23). We start with the relation

$$1^n+2^n+\ldots+(p-1)^n\equiv b_n p \pmod{p^2},$$
$$(n<p-1),$$

where $b_0=1$, $b_1=-1/2$, $b_{2i}=(-1)^{i-1}B_i$, $b_{2i+1}=0$, $i>0$, and $B_1=1/6$, $B_2=1/30$, etc., are the numbers of Bernoulli. From the obvious identity

$$\frac{x-y^k}{(x-1)(y-1)}=\frac{y^k}{y-1}-\frac{x(y^k-1)}{(x-1)(y-1)}$$

there is obtained the identical congruence

$$(x^p-1)\,G_n{}^{(k)}(x)-(x^m-1)\,K_m{}^{(k)}(x)\equiv H_m{}^{(k)}(x) \pmod{p}$$

where

$$K_n{}^{(k)}(x)=-\sum_{n=0}^{p-2}\left(-\frac{m}{k}\right)^n b_n f_{p-n}(x),$$
$$G_m{}^{(k)}(x)=\frac{x(x^m-1)}{x-1}\sum_{\rho}\frac{F_p(\rho)(1-\rho^k)}{mp(x-\rho^k)},$$
$$H_m{}^{(k)}(x)=x(x^m-1)\sum_{l=0}^{k-1}\sum_{\rho}\frac{\rho^l h_a{}^{(k,\,l)}(x)}{x-\rho^k}.$$

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[84] Annals of Math., (2), 27: 54-56 (1926).

[85] Annals of Math., (2), 27: 171-176 (1926).

Cf. Mirimanoff,[41] Vandiver,[58] Frobenius,[60] Pollaczek.[61] Multiplying through by (x^p-1), setting $m=1$ and $x=xe^{vk}$, differentiating $(p-1-a)$ times with respect to v, we have for $v=0$ the congruences (53).

Vandiver[66] gave complete proofs of all the theorems of Vandiver,[58] with the exception of the last.

Vandiver[67] gave among other theorems the following:

Under the assumptions: (1) The class number of the field $k(\alpha)$, $\alpha=e^{2i\pi/\lambda}$, λ, an odd prime, is divisible by λ but not by λ^2. (2) If $B_\nu\equiv 0 \pmod{\lambda}$, $\nu<(\lambda-1)/2$, and $B_{\nu\lambda}\not\equiv 0 \pmod{\lambda^3}$; the relation $x^\lambda+y^\lambda+z^\lambda=0$ is not satisfied in rational integers, none zero, where B's are the Bernoulli numbers. This theorem is equivalent to the one which Kummer attempted to prove in his 1857 memoir, and the proof is obtained here by modifications and extensions of Kummer's argument, taking into consideration also the results of Vandiver,[14] where theorems I and IV of Kummer's paper were proved. In the present article Kummer's theorem II is proved and it is noted that theorem III is unnecessary for the proof of the above result. It follows that if Kummer's computations (p. 73) are correct that the Last Theorem is established for all prime exponents $\lambda<100$.

A proof is also outlined of the following:

Under the assumptions,

1. None of the Bernoulli numbers

$$B_k,\quad k=(s\lambda+1)/2, \qquad (s=1, 3, \ldots, \lambda-4)$$

is divisible by λ^2;

2. The second factor of the class number of the field $\Omega(\alpha)$ is prime to λ; then

$$x^\lambda+y^\lambda+z^\lambda=0$$

is not satisfied in rational integers, x, y, z, none zero, if λ is an odd prime.

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OF THE
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NUMBER 63

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VIRGIL SNYDER, ARTHUR B. COBLE, ARNOLD EMCH, SOLOMON LEFSCHETZ,
F. R. SHARPE, CHARLES H. SISAM

National research council.

PUBLISHED BY THE NATIONAL RESEARCH COUNCIL
OF
THE NATIONAL ACADEMY OF SCIENCES
WASHINGTON, D. C.
1928

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BULLETIN

OF THE

NATIONAL RESEARCH COUNCIL

April, 1928 Number 63

SELECTED TOPICS IN ALGEBRAIC GEOMETRY

Report of the Committee on Rational Transformations[1]

VIRGIL SNYDER, ARTHUR B. COBLE, ARNOLD EMCH,
SOLOMON LEFSCHETZ, F. R. SHARPE,
CHARLES H. SISAM

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[1] The members of this committee of the Division of Physical Sciences are: Virgil Snyder, *Chairman*, Professor of Mathematics, Cornell University, Ithaca, New York; Arthur B. Coble, Johns Hopkins University, Baltimore, Maryland; Arnold Emch, University of Illinois, Urbana, Illinois; Solomon Lefschetz, Princeton University, Princeton, New Jersey; F. R. Sharpe, Cornell University, Ithaca, New York; C. H. Sisam, Colorado College, Colorado Springs, Colorado.

PREFACE

The purpose of this report is to give a brief, connected, and comprehensive survey of the literature on certain topics in algebraic geometry. It is designed primarily as an aid to investigators in this field, but it is hoped that it may be of service to a wider circle.

As the central theme of the report, the committee has chosen the Rational Transformation. It has been understood, however, that the authors were to feel free to depart from the theme whenever they deemed it advisable, either because a certain topic had been insufficiently covered in existing similar reports, or because they wished to present a rapid résumé of some question vital for other parts of the report.

Available reports of encyclopedic nature cover Rational Transformations only incidentally and therefore quite inadequately at best. This fact has had much to do with the choice of general subject by the committee. Although the committee has endeavored to cover the subject with a fair degree of completeness, the particular topics have been selected with due reference to the fields which the authors felt themselves most competent to discuss. It is the hope of the committee that other phases of the subject may be covered in the future.

The materials for the report have been collected, with few exceptions, directly from their original sources, then checked with current reviews. In this way, mistakes and omissions have been reduced to a minimum.

As to the arrangement of the contents of the report itself, each chapter is to be regarded as a unit. References from one chapter to another are made, but any chapter can be read almost independently of the others. Each author assumes full responsibility for the content of his chapter, but each chapter was read and criticized by all. In order to make the report as readable as possible, each chapter has a separate bibliography at its end, with articles arranged in an obvious order and with an assigned serial number. Citations in the text are made by name of author followed by serial number. Titles of papers were regretfully omitted, as their inclusion would have unduly increased the bulk of the report.

Acknowledgment is gratefully made to Colorado College, to the University of Illinois, the University of Kansas, Princeton University, and Cornell University for clerical assistance and traveling expenses. Moreover, Sharpe and Snyder were relieved from teaching for a considerable time through a grant from the Heckscher Foundation for the Advancement of Research at Cornell.

THE COMMITTEE.

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LIST OF PERIODICALS; ABBREVIATIONS

Acta Math.: Acta Mathematica.
Am. Ac. Mem.: American Academy of Arts and Sciences. Boston. Memoirs.
Am. Ac. Proc.: ———————— Proceedings.
Am. J.: American Journal of Mathematics.
Am. Math. Month.: American Mathematical Monthly.
Am. M. S. Bull.: American Mathematical Society Bulletin.
Am. M. S. Trans.: ———————— Transactions.
Amst. K. Ak. Wet. Verh.: Koninklÿke Akademie van Wetenschappen te Amsterdam Verhandelingen.
Amst. K. Ak. Wet. Versl.: ———————— Verslag.
Analyst: Analyst (1874-83).
Ann. Ac. Polyt. Porto: Annæs Scientificos da Academia Polytechnica do Porto.
Ann. di Mat.: Annali di Matematica Pura ed Applicata.
Ann. Éc. Norm.: Annales Scientifiques de l'École Normale Supérieure. Paris.
Ann. of Math.: Annals of Mathematics.
Arch. Math. Phys.: Archiv der Mathematik und Physik.
Arch. Néerl.: Archives Néerlandaises des Sciences Exactes et Naturelles.
Arxius: Arxius de l'Institut de Ciencies de Barcelona.
Assoc. Fr. Sc.: Association Française pour l'Avancement des Sciences.
Astron. Ab.: Astronomische Abhandlungen.
Berlin Ab.: Berlin. Preussische Akademie der Wissenschaften, Abhandlungen.
Berlin Monatsber.: ———————— Monatsberichte (to 1881).
Berlin Sitzungsber.: ———————— Sitzungsberichte (since 1881).
Bern. Mitt.: Berner Mitteilungen der Naturf. Gesellschaft.
Bibl. Math.: Bibliotheca Mathematica (to 1915).
Boll. Boncomp.: Boncompagni's Bollettino di Bibliografia e di Storia delle Scienze Matematiche e Fisiche (1868-87).
Boll. di Mat.: Bollettino di Matematica (Conti) (to 1911).
Boll. d. U. M. It.: Bollettino della Unione Matematica Italiana.
Boll. Loria: Bollettino di Bibliografia e Storia delle Scienze Matematiche (Loria).
Boll. Soc. It. Mat.: Bollettino della Mathesis Società Italiana di Matematica.
Bologna Mem.: Bologna. Reale Accademia delle Scienze dell' Istituto Memorie.
Bologna Rend.: ———————— Rendiconti delle Sessioni.
Bord. Mém.: Mémoires de la Société des Sciences de Bordeaux.
Bord. Proc.: Process verbaux des Séances de la Société ———————— de Bordeaux.
Brit. Ass. Rep.: British Association for the Advancement of Science. Report of meetings.
Brux. Ann. Soc. Sc.: Annales de la Société Scientifique de Bruxelles.
Brux. Bull.: Bruxelles. Académie Royale de Belgique. Bulletins.
Brux. Mém.: ———————— Mémoires. 8 vo (to 1904).
Brux. Mém. Cour.: ———————— Mémoires Couronnés et autres Mémoires. 8 vo.

Brux. Mém. Sav. Etr.: —————— Mémoires Couronnés et Mémoires des Savants Etrangers. 4 to.
Brux. Mém. Sc.: —————— Mémoires. Classe des Sciences. 8 vo (since 1904).
Bull. Soc. Sc. Hain.: Bulletin de la Société Scientifique de Hainaut. Belgium.
Bull. Sci. M.: Bulletin des Sciences Mathématiques. (Darboux.)
Calc. Bull.: Calcutta Mathematical Society Bulletin.
Cal. Ac. Proc.: Proceedings of the Academy of Science of California.
Cal. Pub.: Publications of the University of California.
Camb. Dub. Math. J.: Cambridge and Dublin Mathematical Journal (1846-54).
Camb. Math. J.: Cambridge Mathematical Journal (1837-45).
Camb. Ph. Soc. Proc.: Cambridge Philosophical Society. Proceedings.
Camb. Ph. Soc. Trans.: —————— Transactions.
Časopis: Časopis pro Pěstováni Mathematiky a Fisiky.
Cat. Acc. Gio. Atti.: Atti della Accademia Gioenia di Scienze Naturali in Catania.
Cat. Acc. Gio. Boll.: Bollettino delle Sedute della Accademia ——————
Chris. Forhl.: Christiania Videnskabs Selskabets Forhandlinger.
Colo. Stud.: University of Colorado Studies.
C. R.: Comptes Rendus. Académie des Sciences de l'Institut de France. Paris.
Crac. Bull.: Bulletin International de l'Académie des Sciences de Cracovie.
Deutsche Math.-Ver.: Deutsche Mathematiker-Vereinigung. Jahresbericht.
Deutsche Math.-Ver. Ergb.: —————— Jahresbericht Ergänzungsbände.
Dublin R. S. Proc.: Dublin. Royal Dublin Society. Scientific Proceedings.
Dublin R. S. Trans.: —————— Scientific Transactions.
Dublin Proc.: Dublin. Royal Irish Academy. Proceedings.
Dublin Trans.: —————— Transactions.
Edinb. M. S. Proc.: Edinburgh Mathematical Society. Proceedings.
Edinb. R. S. Proc.: Edinburgh Royal Society. Proceedings.
Edinb. R. S. Trans.: —————— Transactions.
Educ. Math.: L'Education Mathématique (1898-1914).
Encycl.: Encyclopédie des Sciences Mathématiques.
Encykl.: Encyklopädie der Mathematischen Wissenschaften.
Enseign. Math.: L'Enseignment Mathématique.
Erl. Ber.: Sitzungsberichte der Physischen Medizinischen Sozietät. Erlangen.
Esp. Soc. M.: Revista de la Sociedad Matematica Española.
Form. Math.: Formulaire de Mathématique (1895-1908).
Fortschritte: Jahrbuch über die Fortschritte der Mathematik.
Gaz. Mat. Buk.: Gazeta Mat. Bukarest.
Gent. D.: Gentleman's Diary or Mathematical Repository (1741-1840).
Gent. Math. Comp.: Gentleman's Mathematical Companion (1798-1827).
Gergonne's Ann.: Gergonne's Annales de Mathématiques Pures et Appliquées (1810-32).
Giorn. di Mat.: Giornale di Matematiche.
Gött. Ab.: Göttingen. Königliche Gesellschaft der Wissenschaften. Mathematisch-Physikalische Klasse. Abhandlungen.
Gött. Nachr.: —————— Nachrichten.
Gött. M. V. Ber.: Göttingen. Mathematischer Verein. Berichte.
Hamb. M. G. Mitt.: Hamburg. Mathematische Gesellschaft. Mitteilungen.
Il Pit.: Il Pitagora.
Ind. M. S.: Indian Mathematical Society. Journal.

Interméd.: L'Intermédiaire des Mathématiciens.
Jassy Ann.: Annales de l'Université de Jassy.
J. de Math.: Journal de Mathématiques Pures et Appliquées. Liouville.
J. de M. E.: Journal de Mathématiques Elémentaires.
J. Éc. Polyt.: Journal de l'École Polytechnique. Paris.
J. f. Math.: Journal für die Reine und Angewandte Mathematik.
J. Hopk. Circ.: Johns Hopkins University Circulars.
Kop. Dan. Vid. Sels. Skr.: Kongelige Danske Videnskabernes Selskabet Skrifter. Kopenhagen.
Ladies' D.: Ladies' Diary (1704-1840).
Lady's and Gents. D.: Lady's and Gentleman's Diary. (Volumes numbered in continuation of Ladies' Diary) (1841-71).
Leipz. Ab.: Leipzig. Königliche Sächsische Gesellschaft der Wissenschaften. Mathematisch-Physikalische Klasse. Abhandlungen.
Leipz. Ber.: ——————— Berichte.
Liège S. R. S. Mém.: Liège. Société Royale des Sciences. Mémoires.
Lomb. Ist. Mem.: Reale Istituto Lombardo di Scienze e Lettere. Milano. Memorie.
Lomb. Ist. Rend.: ——————— Rendiconti.
London M. S. Proc.: London Mathematical Society. Proceedings.
London M. S. J.: London Mathematical Society. Journal.
Lond. R. S. Proc.: Royal Society of London. Proceedings.
Lond. R. S. Trans.: ——————— Philosophical Transactions.
Lond. R. S. Year Book.: ——————— Year Book.
Mat. p. a.: Le Matematiche, pure ed applicate.
The Math.: The Mathematician (1843-50).
Math. Ann.: Mathematische Annalen.
Math. Gaz.: Mathematical Gazette.
Math. Mag.: Mathematical Magazine (1882-1910).
Math. Misc.: Mathematical Miscellany (1836-39).
Math. Mon.: Mathematical Monthly (1859-61).
Math. Nat. Ber. Ung.: Mathematische und Naturwissenschaftliche Berichte aus Ungarn.
Math. Qu.: Mathematical Questions and Solutions from the Educational Times.
Math. Rep.: Mathematical Repository (to 1835).
The Math. T.: The Mathematics Teacher.
Math. Zschr.: Mathematische Zeitschrift. Berlin.
Mathesis: Mathesis. Belgium.
Mem. Acc. Acireale: Memorie dell' Accademia di Scienze, Lettere ed Arti degli Zelanti.
Mess. of Math.: Messenger of Mathematics.
Modena R. Acc. Mem.: Memorie della Regia Accademia di Scienze, Lettere et Arti in Modena.
Monatsh. f. Math.: Monatshefte für Mathematik und Physik.
Münch. Ab.: München Königliche Bayerische Akademie der Wissenschaften. Mathematisch-Physikalische Klasse. Abhandlungen.
Münch. Ber.: ——————— Sitzungsberichte.
Nap. Acc. Pont. Atti.: Accademia Pontaniana di Napoli. Atti.
Nap. R. A. Atti: Napoli. Reale Accademia delle Scienze Fisiche e Matematiche. Atti.

Nap. R. A. Rend.: ———————— Rendiconti.
Nieuw. Arch.: Nieuw Archief voor Wiskunde Uitgegeven door het Wiskundig Genootschap.
Note e Mem.: Note e Memorie di Matematica (Catania).
Nouv. Ann.: Nouvelles Annales de Mathématiques.
Nouv. Corr. Math.: Nouvelle Correspondance Mathématique (1874-80).
Nova Acta: Nova Acta Academiæ Cæsareæ Leopoldinæ Carolinæ (Halle).
N. Tijds. f. Wisk.: Nieuw. Tijdschrift voor Wiskunde.
N. Y. M. S. Bull.: New York Mathematical Society Bulletin (to 1904).
Oslo Selsk. For.: Oslo Videnskabs-Selskabets. Forhandlinger.
Oslo Selsk. Skr.: ———————— Skrifter.
Pad. N. Sag.: Padova Nuovi Saggi (to 1883).
Pal. Circ. Mat.: Palermo. Circolo Matematico. Rendiconti.
Par. Ac. Mém.: Académie des Sciences de l'Institut de France. Paris Mémoires.
Par. Ac. Mém. Sav.: ———————— Mémoires Présentés par Divers Savants.
Penn. Pub.: University of Pennsylvania. Publications: Mathematics.
Per. di Mat.: Periodico di Matematiche.
Per. di Mat. Supp.: ———————— Supplemento.
Pet. Ac. Sci. Bull.: Petrograd. Académie des Sciences. Bulletin Scientifique.
Pet. Ac. Sci. Mém.: ———————— Mémoires.
Phil. Mag.: Philosophical Magazine.
Poggendorff's Ann.: Poggendorff's Annalen.
Prac. mat. fiz.: Prace matematyczne fizyczne, Warsaw.
Prag. Ab.: Prager Abhandlungen der Kgl. Böhmischen Gesellschaft der Wissenschaften.
Prag. Ber.: ———————— Sitzungsberichte.
Prag. Cesk. Ak. Bull.: (Bulletin of the Franz Joseph Academy at Prague).
P. Rep.: Pascal's Repertorium der höheren Mathematik. 2d (German) edition.
El. Prog. Mat.: El Progreso Matematico (to 1900).
Quart. J.: Quarterly Journal of Pure and Applied Mathematics.
Ras. di mat.: Rassegna di matematica e fisica, Rome.
R. de Ci. Madr. Rev.: Revista de la Real Academia de Ciencias Exactes. Madrid.
Rec. Math.: Receuil Mathématique, Moscow.
Rev. Mat. Hisp. Am.: Revista Matematica Hispano-Americana. Madrid.
Rev. de Math.: Revue de Mathématiques (1891-1906).
Rev. de Math. Spec.: Revue de Mathématiques Speciales (1890-1914).
Rev. Sem.: Revue Semestrielle des Publications Mathématiques.
Riv. di Mat.: Rivista di Matematica, Torino.
Riv. F. M.: Rivista di Fisica Matematica e Scienze Naturali (1900-12).
R. L. Ven. Ann.: Annali delle Scienze del Regno Lombardo-Veneto. Padua, Venice (1831-45).
Rom. Pont. Acc. Atti: Roma. Pontificia Accademia dei Nuovi Lincei. Atti.
Rom. Pont. Acc. Mem.: ———————— Memorie.
Rom. Acc. L. Mem.: Roma. Reale Accademia dei Lincei. Memorie.
Rom. Acc. L. Rend.: ————————Rendiconti.
Rom. Acc. L. Trans.: ———————— Transunti.
Rom. Soc. It. Sc. Mem.: Roma. Società Italiana delle Scienze, Memorie di Matematica e di Fisica.
Roum. Ac. Bull.: (Bulletin de l'Academie Roumaine).
Schl. Ges. f. v. Kultur.: Schlesische Gesellschaft für vaterländische Kultur.

Soc. Ital. Prog. Sci.: Società Italiana per i Progressi delle Scienze.
Soc. M. F. Bull.: Société Mathématique de France. Bulletin. Paris.
Soc. M. F. C. R.: ——————— Comptes Rendus des Séances.
Soc. Phil.: Société Philomathique. Bulletin. Paris (to 1912).
Sph. Oe.: Sphinx-Oedipe. Nancy.
Stockh. Ark.: Arkiv. för Matematik K. Svenska Vetenskaps-Akademien i Stockholm.
Stuttg. Math. Nat. Mitt.: Stuttgart. Mathematisch-Naturwissenschaftlicher Verein in Würtemberg. Mathematisch-Naturwissenschaftliche Mitteilungen.
Sven. Vet. Ak. Bih.: Kongliga Svenska Vetenskaps-Akademie Handlingar. Bihang (from 1872). Stockholm.
Tidskr. f. Math.: Nyt Tidsskrift for Mathematik. Kopenhagen.
Tôhoku M. J.: Tôhoku Mathematical Journal.
Tôhoku Sc. R.: Tôhoku Imperial University. Science Reports.
Torino Atti.: Torino. Reale Accademia delle Scienze. Atti.
Torino Mem.: ——————— Memorie.
Toulouse Ann.: Toulouse. Faculté des Sciences de l'Université. Annales.
Toulouse Bull.: Toulouse. Académie des Sciences, Inscriptions et Belle-Lettres. Bulletin (1897-1900).
Toulouse Mém.: ——————— Mémoires.
Ven. Ist. Atti: Venezia. Reale Istituto Veneto di Scienze, Lettere ed Arti. Atti.
Ven. Ist. Mem.: ——————— Memorie.
Wash. Nat. Ac. Sc. Proc.: Washington. National Academy of Sciences. Proceedings.
Wash. Pub.: Publications of the University of Washington.
Wektor: Wektor. Warsaw (Polish).
Wiad. Mat.: Wiadomosci Matematyczne. Warsaw.
Wien. Ber.: Wien. Kaiserliche Akademie der Wissenschaften. Mathematisch-Naturwissenschaftliche Klasse. Sitzungsberichte.
Wisk. Tijds.: Wiskundig Tijdschrift. Harlem-Rotterdam.
Zschr. Math. Phys.: Zeitschrift für Matematik und Physik (Schlömilch).
Zschr. f. Math. Unt.: Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht.
Z. Vierteljschr.: Zürich. Naturforschende Gesellschaft. Vierteljahrsschrift.

CHAPTER I

QUADRATIC CREMONA TRANSFORMATIONS

A. EARLY EXAMPLES OF QUADRATIC TRANSFORMATIONS

The purpose of this introductory note is to point out the most important phases and their originators in the early development of quadratic transformations, and to conclude with Cremona's famous memoir on the transformations which now bear his name. A report on the subsequent developments of Cremona transformations and their applications will be given in Chapter IV.

The first conspicuous instance of a quadratic transformation process appears in the discovery of the stereographic projection. According to bishop Synesius of Cyrene(231) it was Hipparchus(98) who established the law of this projection, although the first writer about the subject is Ptolemy. Only an Arabic translation of Ptolemy's work is in existence. A poor Latin translation by Jordanus was reedited and retranslated by Commandinus(37), (178).

P. G. Dandelin(43) establishes the principal properties of stereographic projection and their application to the solution of certain problems, as to Apollonius' problem of the tangent-circles of three given fixed circles. He also discovers the relation of the foci of a plane section of a right circular cone with the tangent spheres to the cone and its plane section, and comes across the relation $\rho\rho' = R^2$ between the vectors of corresponding points of lemniscates and conics.

A. Quetelet(179) refers to Dandelin's investigations which he supplements in some important particulars. Quetelet defines polar (inverse) of a curve, and establishes the well known analytic form of inversion.

In 1836 G. Bellavitis wrote a paper on inverse figures(11) in which he refers to the papers by Quetelet, Dandelin, Sturm on stereographic projection, and establishes the theory of "equipolenze." Two vectors are said to be equivalent when their projections a, b and a', b' on the $x-$ and $y-$ axes are such that $a+ib=a'+ib'$. Bellavitis gives clear-cut definitions of inversion, introduces *inverse figures, center of inversion, radius of inversion,* and establishes all principal properties of inversion, like conformity, and invariance of circularity. He also extends inversion to space. Bellavitis(9) establishes certain equalities between segments cut out on lines and curves by lines through a fixed point O (center of gravity of point-system on secant lines) and calls these "equipolenzes."

The transformation by reciprocal radii was established in special form and for special purposes by F. Vièta(236) for the plane and by P. Fermat(76) for space.

The discovery of inversion in its full and general bearing, according to information which has recently come to light, should be credited to J. Steiner, who established the laws of inversion in 1824(20), which, in his picturesque language, he called " Wiedergeburt und Auferstehung." There can exist no doubt that Steiner(214) used this principle in his solution of Malfatti's problem and a great number of other problems of inversion geometry. Pluecker was familiar with the principle of inversion and its analytic form(173).

L. J. Magnus'(144) mere inversions of given figures into new figures, according to Steiner's viewpoint, do not lead to essentially new properties.

In the course of time inversion was rediscovered by a number of investigators; for example, by W. Thomson(227), (228). See also F. Buetzberger(20).

A. F. Moebius(156), (157) made an important generalization of inversion, or transformation by reciprocal radii, in his *Theorie der Kreisverwandtschaft* by establishing in a purely geometrical manner the most general transformation in a plane, which transforms circles into circles.

The rudiments of analytic formulation(83), (84) of this transformation in the complex plane are contained in C. F. Gauss' study of the stereographic projection by means of the representation of points by complex numbers.

The first systematic treatment of the Kreisverwandtschaft, circle-transformation, in the complex plane may be found in F. H. Siebeck(208). The transformation appears in the form $Y=\dfrac{A+BX}{C+DX}$. Siebeck also studies a more general transformation (involutorial) $F(X,\ Y)=0$, where F is symmetric in X and Y.

A more detailed study of inversion and the circle transformation will be made under Section E.

The rudiments of quadratic transformations are noticeable in I. Newton's " de curvarum descriptione organica "(167). Let A and B be the vertices of two angles of given magnitudes, P the intersection of one side of the first angle with one of the second and D the intersection of the remaining two sides. Then when P describes a conic, D describes a quartic, with double points at A and B and the point E as the limiting position of D when AP and BP coincide with AB. The quartic degenerates into a cubic when the conic passes through A and B.

C. McLaurin(142) actually introduced the idea of a quadratic transformation by stating that in Newton's relations between D and P, to the lines described by D correspond conics through three fixed points. To a curve of order n corresponds a curve of order $2n$.

W. Braikenridge(18) describes a geometric mechanism which produces a quadratic transformation. If three lines A_1A_2, A_2A_3, A_3A_1 turn about three fixed points B_3, B_1, B_2, while A_3 always moves on a fixed line and A_1 describes a curve of the nth order, then A_2 describes a curve of order $2n$. He, however, did not perceive the importance of the relation between A_1 and A_2 as a quadratic transformation.

V. Poncelet(177) gave examples of involutorial quadratic transformations and studied their properties. The first instance is the transformation defined by the relation between a point P and the point of concurrence P' of its polars with respect to two circles. It is immediately recognized that this becomes identical with McLaurin's correspondence between P and D, in the case of right angles PAD and PBD, by assuming A and B as zero circles.

Poncelet(176) generalized this transformation by replacing the circles by any two conics. The polars of a point P with respect to a pencil of conics are concurrent at a point P'. Again P and P' define the involutorial transformation.

The same transformation was also defined by Steiner(215) as the correspondence between points which are simultaneously harmonic with respect to the three pairs of lines of a quadrangle. This is, of course, identical with Poncelet's definition, since the three pairs of lines belong to the pencil of conics through the quadrangle. This transformation is frequently referred to as the Steinerian transformation (Steinersche Verwandtschaft).

Another method of establishing a quadratic transformation, as pointed out by Steiner, consists in assuming a fixed triangle ABC, and a fixed triangle $A_1B_1C_1$ inscribed in and perspective to ABC. Now let $A_2B_2C_2$ be inscribed in and perspective to $A_1B_1C_1$ with center of perspective P. Then $A_2B_2C_2$ is also perspective to ABC with center of perspective P'. P and P' are thus corresponding points in a quadratic transformation. Choosing $A_1B_1C_1$ as the coordinate triangle, this latter transformation in modern form is identical with

$$x_1' = x_2x_3, \quad x_2' = x_3x_1, \quad x_3' = x_1x_2.$$

The first statement of a real general quadratic transformation for real distinct fundamental points between two planes is due to Steiner, who called it "schiefe Projektion," skew projection(212). He assumes two fixed planes e and e' and two fixed lines l and g in a general position. The common secants of l and g cut e and e' in couples of points P and P' which establish a general quadratic correspondence between e and e', with real and distinct fundamental points. Steiner discovered the fundamental points and their bearing on the transformation. The usefulness and importance of the "projection" is shown by a systematic development of a great number of new and organically connected propositions. The special transformation mentioned above as well as the general quadratic transformation were known to Steiner as early as 1828.

In the footnote on page 295 of the *Systematische Entwickelung,* immediately preceding the appendix, Steiner makes a significant statement which makes it obvious that Steiner anticipated the possibility of more general transformations than quadratic. This case was also considered by G. J. Verdam(235.2).

Magnus(143) must be credited with the first analytic treatment of the general quadratic Cremona transformation, based upon two bilinear relations, and the establishment of its essential properties by Cartesian coordinates.

J. Pluecker(174) established a quadratic transformation in a form which he considers, mistakenly, as the general expression of such a transformation.

Much later G. V. Schiaparelli(193) conceived the problem of transformations under a more general aspect. He establishes a transformation between the two planes (x, y) and (ξ, η) defined by two equations

$$f'(x, y; \xi, \eta) = 0, \qquad f''(x, y; \xi, \eta) = 0$$

so that to a "primitive point" (x, y) correspond, one, two, or even an infinite number of "transformed points" (ξ, η) according to the nature of the functions f' and f''. He discusses only the cases in which to a point (x, y) corresponds only one point (ξ, η), and conversely, and calls such transformations of the first order. By two postulates Schiaparelli further restricts the $(1, 1)$ character of this transformation to the case where f' and f'' are bilinear relations.

Discussing the "general" transformation of the "first order," Schiaparelli shows that the three quadratic expressions in ξ and η must represent three conics which all pass through three fixed points. This of course is the quadratic Cremona transformation.

Schiaparelli extends the investigation of transformations of the "first order" to space of three dimensions by defining them by a manner entirely analogous to that for the plane.

Even earlier, in 1859, a decided step in advance was made by E. De Jonquières *(44), (45) by the establishment of transformations of a certain type of higher order than the second now known as De Jonquières transformations.

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* This memoir was submitted to the Académie des Sciences in Paris, Oct. 10. 1859, and in definite form Jan. 23, 1860; but no report of it *was ever* made to the Academy.

L. Cremona(38), (39), (40) was the first (1862) who solved the problem of algebraic one-to-one correspondences between two planes in its general form: *To every point of a first figure corresponds a single point in the second, and reciprocally to every point in the latter a single point in the former.* Cremona refers to the articles by Magnus and Schiaparelli, which are particular cases of a much more general transformation which, following Schiaparelli, he calls of the first order. For this and subsequent developments see Chapter IV.

B. GENERAL QUADRATIC TRANSFORMATION

1. *Definition and General Properties of Quadratic Transformations.*—The discovery that there exist transformations between two planes $\Sigma(x)$ and $\Sigma(x')$, or simply (x) and (x') besides the earlier known collineation and inversion (transformation by reciprocal radii), such that, in general, to a point in (x) corresponds only one point in (x'), and conversely to a point in (x') only one point in (x), is of fundamental importance. The first step in this discovery was the establishment of quadratic transformations with this property as described in the preceding section, in which to a line in (x) must correspond a conic in (x'); to a line in (x') a conic in (x).

This is accomplished by a system of conics through three points, the simplest case of what Cremona, in the study of the general (1, 1) transformation, calls a homaloidal net of rational algebraic curves, that is, of rational curves through the same system of base-points consisting of singular (including ordinary) points of these curves.

Thus, a homaloidal system of conics defines a quadratic transformation between two planes.

The exceptional points of the transformation, the base points of the net, are commonly called the fundamental points of the transformation, and the lines which correspond pointwise to these points the *fundamental,* or *principal lines* of the transformation. Thus *a pencil of lines through a fundamental point A_i is transformed into a projective pencil of lines through the fundamental point A'_i.* This proposition is the basis for Seydewitz's construction of the quadratic transformation, which will be discussed later on.

The quadratic transformation, and the behavior of various singularities of algebraic curves on the fundamental system are fully discussed by K. Doehlemann(55) by a graphic method.

2. *Geometric Description of General Quadratic Transformations.*—Magnus(143), as noted before in the historical introduction, gives a discussion of the general quadratic transformation in Cartesian coordinates but did not notice the important fact that to a pencil of lines with A_i as a vertex corresponds properly a projective pencil of lines with A'_i as a vertex.

F. Seydewitz(207) was the first to establish a quadratic transformation by means of projective pencils and ranges. In a plane Σ assume two points A_1 and A_2 and the two pencils $A_1(s_1)$ and $A_2(s_2)$ with these points as vertices. Likewise in a plane Σ' establish two pencils $A'_1(s'_1)$ and $A'_2(s'_2)$. Then establish the projectivities

$$A_1(s_1) \;\overline{\wedge}\; A'_1(s'_1),$$
$$A_2(s_2) \;\overline{\wedge}\; A'_2(s'_2).$$

Any point P in Σ joined to A_1 and A_2 determines two rays s_1 and s_2 in the two pencils in Σ. To these correspond in Σ' two rays s'_1 and s'_2, by the given projectivities, which intersect in a point P'. Conversely to a point Q' in Σ' corresponds, by these projectivities, a point Q in Σ. To a line l in Σ, or a line g' in Σ', correspond evidently a conic L' in Σ', and a conic G in Σ, respectively. Thus, Seydewitz establishes a quadratic transformation between the two planes.

After Seydewitz the next important step was taken by Th. Reye(185), (187). After a short critical review of what has been done before, Reye enumerates the possible kinds of quadratic transformations between two planes Σ and Σ'.

As a first example of a quadratic transformation Reye mentions the case of a quadric F with two points S and S' on F, and two planes Σ and Σ'. Every point P of F is joined to S and S'. SP cuts Σ in P; $S'P$ cuts Σ' in P' so that P and P' form corresponding points of a quadratic transformation between Σ and Σ'.

According to Reye the simplest method to establish a general quadratic transformation between two planes is to set up two correlations between Σ and Σ', so that to every point in $\Sigma(\Sigma')$ correspond two lines (called polars by Reye) in $\Sigma'(\Sigma)$. Thus there exists a correspondence between the points of one plane and the intersections of the lines of pairs of corresponding correlative lines in the other plane, which is precisely a general quadratic correspondence.

T. A. Hirst(104), also in 1866, derives the quadratic transformation with real fundamental triangles by two couples of projective pencils without being aware of the fact that the same method had been established before by Seydewitz. He discusses fundamental points and lines, and merely indicates special cases. He finds that, in case of two superposed planes, there are, in general, four invariant points.

In his paper Hirst refers to A. Transon, who in 1865(231) establishes a quadratic transformation which is identical with Steiner's skew projection. Transon, apparently, did not know that Steiner had done this more than thirty years before. A year later(232) Transon discusses the skew projection in connection with Hirst's quadric inversion which will be explained in the next section.

E. Weyr(242) in 1869, covers analytically in the main the same ground as did Reye three years before by synthetic methods.

Weyr defines as quadratically or birationally related two plane systems when the coordinates of corresponding points are connected by bilinear relations so that to a point of one system corresponds, in general, only one point in the other.

It is a curious fact that this mistaken belief, that the quadratic transformation was the only birational transformation between two planes outside of collineation and correlation, was held by the majority of geometers up to the time of Cremona's classic memoir.

Weyr proves, what is, of course, only true for a quadratic transformation, that a (1, 1)-correspondence between two planes is determined by seven pairs of corresponding points.

H. Schubert(201) discusses the degeneration of quadratic transformations and finds that altogether, there are 18 cases of quadratic transformations, of which 17 are special.

G. Lazzeri(131) discusses birational correlation of order n in a plane. In particular, Lazzeri proves the theorem that a correlation in which every line passes through its pole (corresponding point) can only be a quadratic correlation:

$$\rho u'_1 = a_1x_2x_3,\ \rho u'_2 = a_2x_3x_1,\ \rho u'_3 = a_3x_1x_2;\ a_1+a_2+a_3=0,$$

so that evidently $\Sigma u'_i x_i \equiv 0$.

E. Czuber(41) considers the (1, 1)-point-transformations of the plane, defined by two bilinear equations

$$\begin{aligned} f(x, y;\ x', y') &= 0, \\ g(x, y;\ x', y') &= 0, \end{aligned}$$

and the special cases.

He makes the old mistake by stating that this quadratic transformation is the most general (1, 1) transformation between two planes. He also discusses cases of involutorial transformations, but misses Hirst's quadric inversion.

E. Duporcq(58) studies quadratic and rational correspondences between two plane figures.

He finds among other results that if five couples of a quadratic transformation are contained in a collineation, then they determine an infinite number of couples of conjugate points in the quadratic transformation, which are couples of corresponding points in the collineation of two conics. Duporcq extends the investigation to space. Among the results may be mentioned the notable theorem: If five points of a plane P move on five fixed spheres whose centers lie on a fixed plane P', then there exists on P a sixth point which also describes such a sphere.

M. Fréchet(81) establishes certain normal forms for birational quadratic transformations. He shows that the three cases of the fundamental points A, B, C, namely A, B, C distinct, A and B consecutive, A, B, C consecutive, may all be represented under the same form

$$\frac{X}{x \cdot F'_z} = \frac{Y}{y \cdot F'_z} = -\frac{Z}{y \cdot F'_x + y \cdot F'_y}$$

where

$$F = xy - z^2, \quad F = x^2 - z^2, \quad F = x^2 - 2yz,$$

according to the three cases. By the same method he also proves the well known fact, that every quadratic Cremona transformation may be obtained as the product of an involutorial quadratic transformation of the plane into itself, followed by a collineation.

O. Terquem(223), in a paper *Sur les méthodes métamorphiques* (de transformation) is led to the study of quadratic transformations by first attacking the problem of birational transformations between two surfaces.

Suppressing one dimension, Terquem obtains as special cases the quadratic transformation and the homographic transformation (collineation).

In a similar manner, J. G. H. Swellengrebel(222) discusses the transformations defined by two algebraic functions $f(u, v, t', w') = 0$, $F(u, v, t', w') = 0$; gives various geometric interpretations, including quadratic and collinear transformations.

C. Wiener(244) gives a short historic account of Cremona transformations, in particular quadratic transformations, and then makes a study of some multiply valued algebraic correspondences between two planes. He gives a number of examples in which is explained the derivation of properties of curves from those of lower order by the transformation.

C. Servais(205) writes on a geometric interpretation of birational quadratic transformations and has a note on the theory of transformations in (206).

P. Muth(162) in a paper on ternary bilinear forms, establishes invariant conditions on two bilinear forms $f_i(x, y) = 0$, $i = 1, 2$, that the quadratic reciprocity determined by them degenerates into a linear reciprocity.

V. Weiss(240) describes a construction of a quadratic transformation between two planes when seven pairs of corresponding points are given, by means of the pencil of correlations.

In a note on a new method of transformation(135) by a quadratic reciprocity, G. Leinekugel studies properties of rational quartics.

P. Del Pezzo(46) considers quadratic transformations with either four invariant points, or a conic of invariant points, or intermediate cases.

P. Cassani(24) constructs the quadratic transformation in a plane a by means of two skew lines r_1, r_2 and a quadric Q. Given x in a draw the transversal xr_1r_2. The polar of this transversal with respect to Q cuts a in x'. An elaborate study of the quadratic correspondence was made by E. Catalan(24.1). Accounts of the general quadratic transformation may be found in a number of text-books and treatises on algebraic geometry. As such may be mentioned: F. Aschieri, *Geometria projettiva,* 2d ed., 1888, p. 376-383. Th. Reye, *Geometrie der Lage,* vol. 2, 4th ed., 1907, p. 209-214. R. Sturm, *Die Lehre von den geometrischen Verwandtschaften,* vol. 4, 1909. K. Doehlemann, *Geometrische Transformationen,* vol. 2. Salmon-Chemin-Halphen, *Traité de Géométrie Analytic,* Paris, 1903, p. 430-463. Salmon-Fiedler, *Analytische Geometrie der Kegelschnitte,* vol. 2 (1903), 789-796. H. J. Verhagen, *Verwantschappen, etc., Quaddratische Transformatie,* Utrecht, 1910. German translation, *Verwandtschaften, vermittelt durch die Fundamentalpunkte und die Koinzidenzpunkte der quadratischen Transformation.* Diss. Groningen (1910)—Utrecht. H. Laurent, *Traité d'Analyse,* vol. 4, Paris, 1889. Clebsch-Lindemann, *Vorlesungen über Geometrie,* vol. 1 (1876), p. 474-496. H. Hilton, *Plane Algebraic Curves* (1920), p. 120-136. H. Wieleitner, *Theorie der ebenen algebraischen Kurven* (1905), p. 137-172.

H. Durège(60): *Die ebenen Curven dritter Ordnung* (1871), p. 121-127.

H. Schroeter (Steiner), *Vorlesungen über synthetische Geometrie,* vol. 2 (1876), p. 301-304.

Charlotte A. Scott, *Introduction to Modern Analytic Geometry* (1892), 218-225.

F. Klein, *Vorlesungen über höhere Geometrie,* 3d edition (1926).

G. Loria, *Il passato ed il presente delle . . .,* 2d ed. (1896).

J. L. Coolidge, *Geometry of the Circle and the Sphere,* (1916).

F. Kötter, in the *Jahresberichte d. D. Math. Ver.,* 5_2: (1901).

C. Special Quadratic Transformations

1. *Involutorial Quadratic Transformation.*—Two conics $a_x^2=0$ and $b_x^2=0$ determine a pencil of conics $a_x^2+\lambda b_x^2=0$, with the four points of intersection $B_1B_2B_3B_4$ of the two conics as base-points. Let $A_1A_2A_3$ be the diagonal triangle of the quadrangle (B). The polars of a generic point (x') with respect to the pencil of conics form a pencil $a_{x'}a_x+\lambda b_{x'}b_x=0$, whose vertex (x) is uniquely determined by (x'). Conversely, the polars of (x) all pass through (x'). In this manner is established an involutorial quadratic transformation, which has been established by Poncelet and Steiner, as stated in section A.

The diagonal triangle $A_1A_2A_3$ is the fundamental triangle of the transformation, and the vertices of the quadrangle (B) are the invariant points. It is evident that various special cases of this involutorial transformation may be obtained by choosing the two conics, or the quadrangle (B), in some special intersecting relations, which are abstractly identical with the various cases of the solutions of a biquadratic equation.

This involution also arises from Weyr's two bilinear equations between two sets of variables $x, y; x', y'$, when the equations are symmetrical with respect to the two sets; and in case of a *real quadrangle* (B) of distinct points, determined by two pairs of real lines, becomes identical with Steiner's involutorial quadratic transformation described in Section B. As this mode of establishing the transformation lends itself very easily to geometric and particularly to constructive applications, and has been used extensively for this purpose, it does not seem to be improper to use for it the name of *Steinerian transformation,* as proposed by H. Durège(60). The construction of the transformation itself is as follows: Let (a_1, b_1); (a_2, b_2); (a_3, b_3) represent the three line pairs of the pencil of conics with the vertices $A_1A_2A_3$, respectively, and P a generic point, connect P to A_1 by a line c_1, and construct the fourth harmonic line d_1 conjugate to c_1 with respect to (a_1, b_1) as the other pair of conjugate lines. Repeat the construction for A_2, (a_2, b_2); then d_2 will cut c_2 in a point P' which corresponds to P in the Steinerian transformation. As a check, d_3 must also pass through P'; i. e., $d_1d_2d_3$ are, of course, concurrent at the point P' corresponding to P in the transformation.

When the three line-pairs of the quadruple (B) are rectangular, i. e., when one of the points of (B) is the orthocenter of the remaining three, the construction becomes particularly simple. In this case c_i and d_i are symmetric with respect to both a_i and b_i. This special assumption of orthogonality does, of course, not invalidate projectively the results obtained.

The involutorial quadratic transformation may be defined somewhat differently as follows: Assume again a pencil of conics which cut every generic line of the plane in an involution, whose double points correspond to each other in the involutorial quadratic transformation determined by the pencil of conics, or by the quadrangle of base-points. M. Disteli([50]) uses this transformation in a memoir on the metrics of circular cubics in connection with geometric theorems of Jacob Steiner.

Again, the quadruple (B) and the triple (A) form the base of a net of cubics, such that any two cubics of the net intersect in a pair P, P' of corresponding points of the transformation. A. Emch, in a paper on the invariant net of cubics in the Steinerian transformation([67]), makes use of this property. See also Jacob Steiner's Vorlesungen ueber Synthetische Geometrie (by H. Schroeter)([213]), where the quadratic transformation based on a quadrangle is also called Steinerian transformation.

C. Servais([205]) gives a "geometric interpretation of the birational quadratic transformation," which practically amounts to the correspondence existing between a pole and the point of concurrence of its polars with respect to a pencil of conics.

J. Neuberg([165]) shows that the quadratic transformations determined by involutions in the pencils on B, C determine a third in the pencil on A. There exist three couples of lines, one in each pencil, which touch a conic. The involution of lines on A mark an involution on BC; Neuberg shows that three couples of points one on BC, one on CA, one on AB lie on a conic.

W. Massny([149]), in a paper of a more or less expository character, investigates a particular case of quadratic transformations.

L. Rippert([188]) treats of the dual of Neuberg's method of obtaining involutorial quadratic transformations, and extends investigation to space.

S. Loria([141]), studies pedal and antipedal transformations in the plane and in ordinary space. If (x, y) is the foot of the perpendicular from the origin upon a line whose Pluecker coordinates are u and v, Loria considers the quadratic correlations between the point (x, y) and the line (u, v) and its inverse and then extends the discussion to space.

D. N. Lehmer([133]) writes on the theory of inversion and the quadratic reciprocal transformation, and shows by a purely geometric method, how the latter gives rise to inversion.

The same author studies([134]) combinations of involutorial quadratic transformations.

Ch. Tweedie([233]) considers the general form of the involutorial (1, 1)-quadratic transformation in a plane. Tweedie points out Czuber's omission of Hirst's inversion and mistake of considering the quadratic transformation as the most general Cremona transformations. E. Beltrami(12) treats of the Steiner transformation in connection with the "conics of nine points."

A. Emch(62) in his Introduction to Projective Geometry, establishes the Cartesian form of the Steiner transformation, based upon an orthocentric quadrangle. In this case the circular points at infinity are corresponding points of the transformation, and to the line at infinity corresponds the circle through the fundamental points. On this is based an effective theory of circular cubics.

2. *Quadric Inversion.*—Although this is also an involutorial quadratic transformation, its historic and theoretical interest and usefulness for the applications are of sufficient importance to deserve a special treatment.

This transformation was discovered simultaneously by T. A. Hirst(107) and F. Geiser(85) in 1865. G. Bellavitis(10) is credited with the distinct suggestion of the quadric inversion. Two years before, in 1836, Bellavitis(11), developed very fully the principles of ordinary cyclic inversion. Hirst also refers to J. W. Stubbs' paper on the application of a new method (inversion) to the geometry of curves and curved surfaces(216).

Hirst defines as inverse points, p and p' conjugate with respect to a fixed "fundamental conic" F and collinear with a fixed point A. The several pairs of inverse points p, p' on any line R through the origin A form an involution, the double points of which are the real or imaginary intersections of R with F. For convenience Hirst assumes A in such a position that two real tangents may be drawn to F, whose points of tangency, B and C, together with A, form the fundamental points, and the polars BC, CA, AB of A, B, C the principal (fundamental) lines.

From a constructive stand-point, Hirst's transformation is extremely simple. To construct the corresponding point p' of a generic point p, connect p with A by a line R; construct the polar of p with respect to F, which will cut R in the required point p'.

Geiser(85) extends the transformation to space, by assuming a fixed point P and a fixed quadric F. The join of a generic point p to P cuts F in two points k_1 and k_2. The fourth harmonic p_1 of p with respect to k_1 and k_2 corresponds to p in an involutorial quadratic transformation in space.

Accounts of the quadric inversion by T. A. Hirst may also be found in (99), (100), and (101).

3. *Special Transformations Obtained by Various Geometric Devices.*—One finds in the literature a large number of special quadratic transformations based on special geometric constructions, of which some typical cases deserve to be mentioned in detail.

In (52), M. d'Ocagne gives an account of the following extremely simple construction: To each point P of the plane one lets correspond a point P' so that the segment PP' appears under right angles from two fixed points A and B. Thus, to obtain P', connect P with A and B, and at A and B erect perpendiculars to AP and BP, which will intersect in the required point P'. G. Battaglini(5.1) constructs T_2 by two correlations. A geometric treatment of the method of Magnus was given by P. Bretschneider(18.1). A metrical T_2 is given by G. Cordoso-Laynes(21.2). See also F. Hofmann(107.1), B. Igel(109.1), A. Jacobi(110.1), G. Kilbringer(120.1), L. Lipkin(137.1) and P. M. D'Ocagne(53.1).

D'Ocagne generalizes this transformation by collineation.

D'Ocagne does not state the fact that this involutorial quadratic transformation is a special case of that determined by a pencil of conics defined by two zero-circles, say $A[(x-1)^2+y^2=0]$ and $B[(x+1)^2+y^2=0]$. Under the latter assumption the transformation assumes the simple form,

$$x' = -x, \qquad y' = \frac{x^2-1}{y}.$$

The couple of corresponding points on a generic line l is obtained by joining the point of intersection M of l with the y-axis with A, and describing a circle K with M as center and MA as radius. K cuts l in the required couple P, P'.

D'Ocagne extends the transformation to space.

M. J. Neuberg(164) discusses the transformation established by d'Ocagne and determines the fundamental triangle of the transformation, and then makes a further generalization.

Neuberg also shows how a certain case may be reduced to the quadric inversion of Hirst and Geiser. The paper contains a number of geometrical applications of these transformations.

P. H. Schoute(195) establishes two involutorial Cremona transformations of order 2 and 5.

The first is based on a triangle ABC. Let P be a generic point in the plane of the triangle, and join P to A, B, C, so that $\sphericalangle PAB=\alpha$, $\sphericalangle PBC=\beta$, $\sphericalangle PCA=\gamma$. Construct the lines $P'A$, $P'B$, $P'C$, symmetric to PA, PB, PC with respect to the bisectors of the interior angles of the triangle. The three symmetric lines intersect in a point P' which corresponds to P in an involutorial quadratic transformation with ABC as the fundamental triangle. There is $\sphericalangle P'AB=A-\alpha$, $\sphericalangle P'BC=B-\beta$, $\sphericalangle P'CA=C-\gamma$; if A, B, C also denote the angles of the triangle; and

$$\frac{\sin\alpha}{\sin(A-\alpha)}\cdot\frac{\sin\beta}{\sin(B-\beta)}\cdot\frac{\sin\gamma}{\sin(C-\gamma)}=1.$$

In (196), Schoute applies his special quadratic transformation (by symmetric lines) to the proof of a theorem of Steiner: The series of similar conics circumscribed to a triangle *ABC* envelopes a quartic with double points, at *A, B, C,* and which has, therefore, only 4 double tangents, of which one is the line at infinity. Schoute finds several other properties of this curve and its special cases.

The same involutorial quadratic transformation is described by J. J. A. Mathieu(150), who calls it trilinear inversion. He also studies a case of quadratic correlation and applies the theory to the proof of several theorems on systems of conics (equilateral hyperbolas).

E. Vigarié(237) gives an analytic treatment of Schoute's transformation with applications to the geometry of the triangle.

A. Cayley(27) studies the transformations,

$$(1)\qquad \xi=\sqrt{x},\qquad \eta=\sqrt{y},\qquad \zeta=\sqrt{z},$$

$$(2)\qquad \xi=x^2,\qquad \eta=y^2,\qquad \zeta=z^2,$$

$$(3)\qquad \xi=\frac{1}{x},\qquad \eta=\frac{1}{y},\qquad \zeta=\frac{1}{z}.$$

The latter is an analytic interpretation of the Steiner transformation.

In an elaborate memoir(191) L. Saltel studies the following quadratic transformation: Join a generic point μ to $A, B, C,$ and these to a fixed point E, cutting the opposite sides at A', B', C'. To $A\mu$ determine the conjugate in the involution having AB, AC as a pair and AA' as a double line. Repeat the same construction on the other two vertices; then the three conjugates meet in a point M which corresponds to μ in an involutorial quadratic transformation.

P. Mansion(148) discusses Saltel's second arguesian transformation, and shows that it is identical with a general quadratic transformation with real fundamental points.

If ABC is the triangle of reference, AA', BB', CC' are three lines concurrent at $O(1, 1, 1)$, moreover AM, BM, CM three other lines movable with $M(x, y, z)$, one may determine the lines corresponding to AM, BM, CM in the three involutions determined by (AB, AC, AA'); (BC, BA, BB'); (CA, CB, CC'), with AA', BB', CC' as double lines; they intersect in a point $M'(x', y', z')$, related to $M(x, y, z)$ by a quadratic transformation.

Finally, it is proved that Saltel's triangular arguesian transformation, is reducible by a linear transformation to the one defined above.

D. Amanzio(2) discusses some special Cremona transformations and their applications to cubics and quartics, and derives properties of such curves corresponding to the theorems of Pascal and Brianchon by the process of these transformations.

A quadratic transformation based upon a cuspidal cubic is established by K. Zahradnik(245). Given a cuspidal cubic C_3. From a generic point P draw three tangents PT_1, PT_2, PT_3 to C_3. The centroid S of $T_1T_2T_3$ corresponds to P in a quadratic transformation. If C_3 is chosen as a cissoid, the transformation becomes quadratic cyclic.

G. De Longchamps(139) considers a "reciprocal transformation," defined by joining a generic point M to a fixed point P and erecting a perpendicular QI, at another fixed point Q to QM. The correspondence between M and I is an involutorial quadratic transformation.

In a "study of comparative geometry"(140), he also establishes a quadratic transformation with application to a class of sextics, and quartics of class four.

F. Ferrari notes(77) that if of two triply perspective triangles five vertices are fixed, the sixth is constructible linearly, and uses this property to construct a quadratic transformation.

E. Cavalli(25) discusses reciprocal figures and the quadratic transformation in cinematics.

R. Vercellin(235.1) writes on trilinear birational transformations, quadratic transformations with coincident fundamental triangles.

N. G. W. H. Beeger(8) studies the quadratic reciprocity between pole and polar with respect to a triangle.

J. Klima(125) considers a special quadratic reciprocal correspondence in a plane.

On this, and similar transformations also see M. D'Ocagne(51), on "polar transformations" of curves and surfaces by M. D'Ocagne(53) and H. Le Pont(136).

E. Dewulf(47) writes on a general geometric transformation of which a particular case is applicable to cinematics, by attaching movements in a plane to projective geometry.

E. Bernè(13) gives an elaborate exposition of what he calls "transformation par inversion symétrique," partly connected with inverse properties.

L. E. Dickson(48) studies a quadratic Cremona transformation defined in connection with a fixed conic. On a conic K choose four points $ABCD$. To a point P we make correspond P' as follows: Join AP and DP cutting K in F and G. Join FB and GC; they intersect in the point P' which corresponds to P in an involutorial quadratic transformation. The join PP' passes through O.

A projective treatment of this transformation is given by the same author in (48.1).

H. Wiener(243) considers the transformations which may be compounded by two reflexions: (1) projectivity on a line, (2) circular transformations of the plane, or on sphere, (3) plane (space) collineations which leave a curve (surface) of the 2. order invariant, (4) movements of rigid space-systems, and transformations of equisymmetrical space-systems.

G. Darboux(43.1) considers space constructions for T_2. M. W. Haskell(94.1) reduces the equations of T_2 to standard form.

A. J. Third(226) investigates the quadratic transformation,

$$x' : y' : z' = x(by+cz) : y(cz+ax) : z(ax+by),$$

and makes applications to the geometry of the triangle and certain conics connected with it.

In (54) D'Ocagne has a note on two transformations of plane curves. From a generic point M drop perpendiculars MP and MQ to OX and OY, then the join PQ corresponds to $M(x, y)$ in the involutorial quadratic correlation $u = -\frac{1}{x}, v = -\frac{1}{y}$. D'Ocagne studies the envelope-curve, when M describes a given curve.

K. Ono(169) studies some properties of the quadratic correlation which exists between a point P and its so-called triangular polar with respect to a fixed triangle. This polar is obtained by constructing the fourth harmonic points α', β', γ' of the intersections α, β, γ of PA, PB, PC with the opposite sides, and the corresponding pairs of vertices of the triangle. Then $\alpha'\beta'\gamma'$ are collinear on the "triangular polar."

D. INVERSION. CIRCLE TRANSFORMATION

1. Circle transformations (Kreisverwandtschaften) are of such importance in many fields of mathematical research, that, although belonging to the class of quadratic transformations, a particular section must be reserved for them.

Inversion, a special case of circle transformation, was discovered and applied to the solution of certain geometric problems long before the establishment of the general circle transformation by Moebius. As its historic aspects have been considered before in section A of this report, it is not necessary to repeat an account of them in this place. It is, however, pertinent to point out the fact that the introduction of complex quantities and their geometric representation by Wessel-Argand-Gauss gave the theory of circle transformations in its further development a tremendous impetus. Their application in the theory of functions of a complex variable as developed by Cauchy, Riemann, Weierstrass, and expounded by a large number of mathematicians, is evident to every beginner. The crowning results of the brilliant efforts in connecting the theory of circle transformations with the theory of substitution groups, or the corresponding division of the complex plain into cells bounded by circular arcs; then with the theory of automorphic functions, with non-Euclidean geometry, and with the uniformization of analytic functions are a sufficient attestation for the importance of circle transformations, especially when considered from the standpoint of function theory. An historical account of this development is outside the limits set for this report, and we must content ourselves with mentioning a few landmarks in this connection: F. Klein(122); Klein-Fricke(123), (124); Poincaré(175).

In what follows, an account will be given of the purely geometric aspects of circle transformation and its applications. Much of the work that has been done in this direction is mathematically of little consequence, while in a number of cases the results obtained deserve detailed accounting.

2. *Inversion.*—J. Liouville(137) in commenting on W. Thomson's method introduces the term "transformation par rayons vecteurs réciproques."

H. S. Hart(94) writes on transformations of curves. He refers to Salmon's mention of inversion introduced (not for the first time) by Dr. Ingram(110) and Mr. Stubbs(216). Hart finds a number of properties of inversions of conics, as the limaçon, cartesian ovals, etc., by choosing the center of inversion at a focus.

N. M. Ferrers(78) refers to Hart's method of transforming curves, consisting in writing $\frac{k}{r}$ for r in the polar equation of the curve, and discusses some metrical properties of inverse figures.

G. H. O. Boeklen(15) considers three geometric transformations: (1) Transformation of reciprocal radii-vectors. (2) Transformations by proportionally divided coordinates. (3) Transformation by bending of surfaces; triangulation of surfaces; infinitely small triangles must remain congruent by deformation.

B. Tortolini(230) makes a study of space-inversions, and some metrical relations on inverse figures.

A. Mannheim(146) shows how properties of cyclides may be obtained very simply by inversion. Mannheim solves by inversion a certain problem on circle-envelopes.

A little later Mannheim(147) investigates self-inverse curves and studies "anticaustiques."

M. Moutard(160) studies the inverse of surfaces which contain the imaginary sphero-circle as a multiple line. If m is the degree of the surface, p the order of multiplicity of the center of inversion, q the multiplicity of the sphero-circle, m', p', q' are the corresponding numbers of the transformed surface, then $m'=2m-p-2q$, $p'=m-2q$,

$q'=m-p-q$, and conversely, $m=2m'-p'-2q'$, etc. When $m'=m$, then $p'=p$, $q'=q$. Some of these surfaces are transformed into themselves. Moutard proposes to call such self-inverse figures *anallagmatic* (*α* privatif, *αλλα · τσω*, je change), a rather bombastic word now in general use. In (161) Moutard derives certain metric properties of bicircular quartic surfaces.

H. E. Habich(92) writes on the transformations of plane curves by the method of reciprocal radii-vectors and derives relations between the curvature and centers of curvature of a curve and its inverse.

A. Mannheim(145) gives a geometric demonstration of a property of the transformation by reciprocal radii. He considers two inversions in space, the first transforming F into F_1, the second F_1 into F_2, and shows geometrically that it is possible to obtain a figure F_3 symmetric to F_2 with respect to a plane, by transforming F directly.

J. C. Maxwell(151) applies inversion to the study of cyclides.

A. Cayley(29) obtains equations of cyclides by inversion of quadric surfaces.

J. Casey(23) applies inversion on circle and sphere, to establish certain properties of cyclides, for example, the fact that the bicircular quartic surface can be generated in five different ways as the envelope of a variable sphere, which cuts a given sphere orthogonally, and whose center moves on a given quadric.

In (22) Casey gives an elaborate exposition of the generation and classification of cyclides.

In a short note(171) A. Peaucellier describes the linkage which now goes by his name, a mechanism without sliding movements realizing inversion of a circle, and consequently the mechanical description of a straight line. Peaucellier had announced his discovery before(170). He calls a linkwork a "compas composé," and shows how by such mechanisms, lines, circles, conics, chonchoids, and cissoids may be described.

Lipkin, a student of Tschebichefs, rediscovered Peaucellier's inversor in 1871.

A. B. Kempe(119), (120) describes "general" methods of producing exact rectilinear motions by linkworks.

H. S. Hart(94) constructs a number of linkages and shows by their means the mechanical description of various algebraic curves.

A. Emch(65), (70) illustrates the elliptic integral of the first kind by a certain linkage consisting of a combination of Peaucellier's inversors, and is thereby able to prove a number of problems of closure on Poncelet polygons and circle-series.

H. Schroeter(200) follows, presumably, the procedure of Steiner, i. e., the method of inversion, to solve Malfatti's problem of three circles touching each other, and each two sides of a triangle, as described by Steiner(214) without proof. The solution, obtained without analysis, disproves Pluecker's insinuation, that Steiner very likely had no proof for his solution.

H. E. Grassmann(90) gives a geometric proof of a metric theorem concerning the inversion of a sphere.

G. Hessenberg(97) describes linkages for the circle transformation.

P. H. Schoute(197) shows that the six tangents from a point P to the lemniscate $(x^2+y^2)^2=2a^2(x^2+y^2)$ have their contacts on a conic. This conic cuts the lemniscate in two residual points, P_1, P_2. The line P_1P_2 cuts the lemniscate in two further points, P_3, P_4 which are residual points of a point P', a quadratic transform of P. This transformation is the inversion with respect to the circle with center at origin and radius a.

The same author(198) applies inversion to the figure of one triangle symmetric with respect to another, and to the figure of three circles symmetric with respect to a triangle, and makes extension to space.

F. Meyer(153) uses inversion and the quadratic transformation as aids in the study of nodes of algebraic curves.

M. Cochez(34) gives a general account of the geometric theory of inversion, or transformation by reciprocal radi; likewise, A. Daguillon(42).

C. Grolleau(91) solves a number of problems, as, for example, the inversion of two arbitrary circles into two equal circles.

R. Bouvaist(17) applies inversion for the derivation of certain properties of the strophoide.

F. Fouret(80) shows that there is no curve which admits ∞^1 inversions whose centers form a curve locus (though curves with ∞^1 axes of symmetry and their transforms admit (discontinuous) ∞^1 inversions).

J. Hadamard(93) finds the surfaces which admit ∞^1 inversions whose centers form a curve locus. See also F. Fouret(80.1) on the principal poles of inversion of Dupin's cyclide; and G. Humbert(109), who makes a comprehensive study of cyclides, following Moutard, Laguerre and Darboux.

R. Lachlan(127) abstract in (128), uses inversion as a working method in an extensive study of systems of circles and spheres.

S. L. Ravier(182) writes on the transformation by reciprocal radii and the mechanical description of quadrics, and J. Reveille(184) uses inversion methods to generate anallagmatic curves.

J. Kleiber(121) contributes to the cinematic theory of linkages, in which, in part, inversions play a rôle, and refers to the extensive bibliography contained in L. Burmester's Lehrbuch(21) der Kinematik, Leipzig, 1888. See also, F. J. van den Berg(235); P. H. Schoute(198). M. J. Neuberg(164), v. (1886), V. Fuhrmann(82).

A. Gob(86) establishes some properties of Neuberg circles.

In an interpretation of transformations by reciprocal radii, J. Amaldi(1) uses a mapping process of circles in a plane upon points of an S_3.

G. Lazzeri(131) gives a new geometric definition of inversion on a circle (sphere) which leads to a chain of theorems, and eventually to the usual definition.

R. Bricard(19) gives a proof that under inversion lines of curvature of a surface pass into lines of curvature of the transform.

L. Sire(209) studies some relations existing between inverse and polar reciprocal curves.

G. Valiron(234) shows how the theorem that lines of curvature are preserved by inversion, as a consequence of Dupin's theorem, can also be proved by a theorem of Joachimsthal.

Th. Leconte(132) gives an elementary proof of the theorem that inversion preserves the character of lines of curvature.

K. Ogura(168) introduces differential forms of a surface which are useful in the investigation of invariantive properties of surfaces by inversion.

R. A. Johnson(111) studies inversion with reference to directed angles and gives a proof of Schoute's theorem, and proves a number of propositions in connection with the geometry of the triangle.

D. Coelingh(35) gives an expository account of inversion and its applications. He considers first the point transformation obtained by inversion and second a line transformation which is the dual of the first, and combines these two operations to derive properties of systems of circles.

J. J. Quinn(180), (181) describes new mechanisms for inversion.

A. Larmor(130) extends the contact problem of eight circles touching three given circles to circles on a sphere, and shows how the principle of polarity is made complete and the method of inversion on the sphere is then seen in its true light. This generalization also enables Larmor to deduce the descriptive geometry of a quadric with reference to its plane sections.

J. W. Clawson(32) studies an inversion of the complete quadrilateral, and derives some of its properties by inversion.

A. Voss(238) in a memoir on the theory of reciprocal radi, investigates the metric properties of this transformation and makes application to differential geometry.

G. Bouligand(16) connects the theorem that the only real surface with all its points umbilics, is the sphere, with Liouville's theorem on the compositions of conformal space-transformations.

A few notes on inversions are given by T. Bhattacharyya(7). E. Cavalli(26) studies reciprocal figures and the application of quadratic transformations in cinematics.

3. *Circle Transformation* (*Kreisverwandtschaft*).—After the establishment of the general circle transformation by Moebius as reported in section A, A. Cayley(28) proves a theorem on circles which amounts to a description of the same correspondence. Given *ABCP* and *A'B'C'*. Through *BCP, CAP, ABP, ABC* pass circles α, β, γ, ω respectively. Construct three circles α' through *B'C'* cutting ω' (through *A'B'C'*) at same angle as α cuts ω; β' through *C'A'*, γ' through *A'B'* with similar cutting of ω'. Then α', β', γ' meet in *P'* which corresponds to *P* in a circle transformation. α', β', γ', ω' pass through *B'C'P'*, *C'A'P'*, *A'B'P'*, *A'B'C'*.

The discovery that the circle transformation in a special case of the general quadratic transformation, dates back to Magnus, Reye, Weyr, contained in papers previously mentioned in section A. When two of the fundamental points, say *B* and *C* of a quadratic transformation coincide with the circular points *I* and *J* of the plane Σ, then the homaloidal net consists of the net of circles through a fixed point *A*. The lines of the fundamental triangle consist of the isotropic lines through the point *A*, and the line at infinity. An analogous situation exists in the plane Σ. Lines and circles in Σ are transformed into lines and circles in Σ'. When *A* and *A'*, and Σ and Σ' coincide, the involutorial circular transformation arises.

In case of an involutorial circle transformation

$$z' = \frac{az+b}{cz-a},$$

and superposed planes Σ and Σ', the pencil of conics, with respect to which pole and point of concurrence of polars are corresponding points, consists of a concentric pencil of equilateral hyperbolas, through two fixed real points R_1 and R_2, whose mid-point is *A*, and two fixed imaginary points I_1 and I_2, obtained as finite imaginary intersections of zero-circles around R_1 and R_2. *AIJ* is now the diagonal triangle of the quadrangle $R_1R_2I_1I_2$ of invariant or base-points of the pencil. The involution cut out on the line at infinity by the pencil of equilateral hyperbolas has *I* and *J* as double points. In particular when the coordinates of R_1 and R_2 are (1, 0) and (−1, 0), those of I_1 and I_2 $(0, i)$ and $(0, -i)$, respectively, the transformation becomes identical with complex inversion

$$z' = \frac{1}{z}.$$

This theory and its generalization for transformations of the type $z'=f(z)/g(z)$; $z'=z+\phi(z)/\psi(z)$, and in particular,

$$z'=z-\frac{(n+1)f(z)}{f'(z)},$$

where $f(z)$ is a polynomial of degree n, and applications to projectivity between pencils of stelloids and pencils of their first polars, is given by A. Emch(68). See also C. M. Hebbert(96).

In the theory of functions of a complex variable, the circle transformations of the second kind,

$$z'=\frac{a\bar{z}+b}{c\bar{z}+d},$$

where $\bar{z}=x-iy$ are also of great importance. They differ from the direct circle transformations (of the first kind) in that they are connected with generalized reflexions, i. e., geometric inversion or transformation by reciprocal radii. Contours of closed figures in Σ' corresponding to contours of closed figures in Σ have the same—or opposite—directions to those in Σ, according as the circle transformation is of the first or second kind.

H. Durège(61) applies Siebek's representation $y=(p+qx)/(r+sx)$ of the circle transformation to the description of curves by y, and their properties, when x describes a given curve.

M. Laissant(129) expresses the desirability that the term transformation by reciprocal radii should be replaced by inversion. He studies various transformations of similar character, for example, those that preserve areas.

G. Battaglini(6) considers a non-Euclidean circle transformation established by two projective pencils of double contact conics, such that to a conic having a double contact with a fixed conic corresponds a conic with double contact with another fixed conic.

A general account of circle transformations in the complex plane is given by A. Emch(64) with various applications, for example, to the Steiner circle series.

F. N. Cole(36) discusses the linear functions of a complex variable and their geometric interpretation by points of complex planes, from a geometric point of view. In particular he studies the relation between the circle transformation of the complex plane, the stereographic projection of the sphere and the equivalent collineations of the sphere into itself. The arrangement follows more or less closely Klein's lectures on the theory of functions in 1880-81.

Ed. von Weber(239) gives a general account of and discusses circle transformations, in the main by use of the complex plane.

E. Kasner(118) develops the invariant theory of the inversion group with application to the geometry upon a quadric surface. Kasner's paper is concerned with the projective geometry on a quadric, in particular with those properties of algebraic curves on a proper quadric which are invariant under the collineation group which leaves the quadric invariant. When the quadric becomes a sphere the stereographic projection of the sphere, and on it the 6-parameter group of collineations which leaves the sphere invariant, gives in the plane the group of circular transformations; in general, inversion geometry in the plane. Kasner also studies the form problems of these geometries and their bearing upon the theory of curves on quadrics and in the plane of inversion.

In (117) E. Kasner considers a relation between the circular and the projective transformations of the plane. Kasner establishes a collineation as follows: Let τ be a circular transformation and O a finite point, not the pole of τ. Let O' correspond to O in τ. Any circle through O is transformed into a circle through O'. The centers of such circles correspond to each other in a collineation H. Conversely, to any collineation H, not an affinity, there correspond two combinations (τ, O) each of which gives rise to H.

A. Emch(63) gives simple demonstrations of some geometric properties of circular transformations which are of value in the study of ring-shaped domains as used in the definitions of certain automorphic functions of the Schottky type.

M. Pieri(172) points out the fact that up to now there are two main lines of approach to the study of inversion geometry. The first is by means of the circular transformations (6-parameter group) in the plane; the second by tetracyclic and pentaspheric coordinates.

To avoid the intermediary of a projective geometry in an S_4, in his memoir, Pieri proposes to establish inversion geometry and its principal propositions by its natural elements, the point and the circle, and a set of twenty postulates between them, without making particular use of previous modes of treatment.

A. Emch(66) studies the rectilinear congruence (2, 2) obtained by joining corresponding points of a circular transformation between two fixed planes.

W. Scheibner(192) writes on Moebius' Kreisverwandtschaft and inversion; likewise C. Klobasa(126) on the principles of inverse transformations in the plane.

A. Simionov(208.1) uses inversion and reciprocal polars for the transformation of geometric propositions.

The dissertation of A. Neumann(166) deals with quadratic transformations in the plane and in space, in particular circle- and sphere-transformation.

E. Meyer(152) investigates involutorial conformal point-transformations of space.

In the last paragraph of a paper on space-sextics of genus 2, M. Stuyvaert(220) shows that such sextics can be transformed by inversion into a quintic of genus 2, a fact already noted by D. Montesano(158). For further references on circle transformations, etc., see Holzmueller(108).

4. *Geometry of the Circle and the Sphere.*—The natural continuation of elementary inversion geometry of the circle and the sphere leads to an extensive doctrine embodied in the geometry of the 10-parameter group of conformal transformations of S_3, and the higher sphere geometry of the 15-parameter group of linear transformations of six homogeneous variables, and their extension to higher spaces. In this is included the geometry of tetracyclic and pentaspherical coordinates, and the connection with line-geometry. For all this the reader is referred to the Geometry of the Circle and the Sphere by J. L. Coolidge, and F. Klein's Vorlesungen ueber Hoehere Geometrie, 3. edition 1926. A detailed account and bibliography of this geometry should form the object of a separate report.

E. VARIOUS APPLICATIONS. GROUPS OF QUADRATIC TRANSFORMATIONS

1. *Geometric Applications. Invariant Algebraic Curves.*—G. Jung and A. Armanente(112) give a general exposition of Cremona transformations and their application to birational reduction of curves of given genus p. For example, it is shown how the general quintic may be obtained from a certain septimic with 9 double points by a quadratic transformation. See also Clebsch-Lindemann(33).

2. In some applications of involutorial quadratic transformations, E. Duporcq(59) proves that the four tangents from a point on a lemniscate have contacts on a line which envelopes the conic touching the six flex-tangents.

V. Retali(183) shows how to get cusps of the second kind by quadratic transformations.

E. Beltrami in (13) makes use of quadratic transformations to study the properties of such curves as Steiner's tricuspidal hypo-cyclide.

F. P. Ruffini(189) studies properties of conics which are polar reciprocals of each other.

K. Doehlemann(56) uses quadratic transformations to study metric properties of the net of conics on three lines.

K. Doehlemann(55), (57) investigates involutorial forms which can be contained in a plane Cremona, particularly in a quadratic, transformation. He defines first what he calls "abgeleitete," derived transformation. To (X, Y') correspond X' in $(\Sigma \to \Sigma')$, and Y in $(\Sigma' \to \Sigma)$. There exists a Cremona transformation between X' and Y, which is of degree n^2. When X' describes a line in Σ', to it corresponds in $\Sigma(\Sigma' \to \Sigma)$ a curve ϕ described by X. But, considering X in place of Y' in Σ, then Y in $(\Sigma' \to \Sigma)$ describes a curve of order n^2. This transformation between X' and Y is called the *derived* transformation, and is represented by,

$$x'_i = \phi_i[\phi_1(x),\ \phi_2(x),\ \phi_3(x)], \qquad i = 1,\ 2,\ 3.$$

A quadratic transformation may contain a general, or a rational cubic, with central or non-central involution, as an involutorial curve. If the involution on the C_3 is not central, the quadratic transformation thus defined, has no double points. If the involution is central and M its central point, then A_1B_1, $A'_1B'_1$, $A'_2B'_2$ meet in M_1, the tangential of M. Moreover the points of contact of the tangents from M are the invariant points $D_1D_2D_3D_4$ of the transformation. C_3 is the isologue of M. In this case the derived transformation is of order 4 and has C_3 as a pointwise invariant curve.

Doehlemann then considers quadratic transformations which have as an involutorial C_3 a line and a conic, and finally three lines; and some other special quadratic transformations.

He finds that a rational C_3 with non-central involution can be contained only in an involutorial quadratic transformation of the plane.

M. Disteli(49) gives a simple plane representation of the forms of plane cubics, their construction and problems of closure; utilizing the Steiner transformation. The method leads to a constructive classification of cubics.

In (50) M. Disteli applies the Steiner transformation for the proof of a large number of theorems which Steiner published mostly without proof.

Ch. Michel(154) studies transformations of cubics into themselves by a Steiner transformation. He finds two classes: first, those whose joins of corresponding points meet in a point of the cubic; second, those whose joins envelope a class cubic. See also B. Bydzovsky(31.1).

S. C. Baghi(5) proves certain properties of four pairs of inverse points on a self inverse cubic.

P. H. Schoute(194) uses quadratic transformations as an auxiliary in the investigation of quartics with three inflexional nodes.

A. Emch(70), pp. 275-284, by means of the Steiner transformation gives simple constructive methods for the Newtonian five types of cubics.

W. Binder(14) maps plane rational cubics and quartics on a conic by means of a Steiner transformation, and studies some of their properties in their relation to the conic and the point correspondences on it.

A. Emch(68) by means of the Steinerian transformation attached to an orthocentric quadrangle, develops a simple method for the study of circular cubics and its special cases.

A. Emch(67) studies the properties of the invariant net of cubics in the Steinerian transformation.

The same author(69) discusses possible classes and types of algebraic curves which are invariant under an involutorial quadratic Cremona transformation. In particular, hyperelliptic curves are considered, and a classification of hyperelliptic sextics and special quadratic transformations are studied. In the second paper the couples of corresponding points (P_1P') of the transformation, $\rho x'_1 = x_2x_3$, $\rho x'_2 = x_3x_1$, $\rho x'_3 = x_1x_2$ are mapped on a Cayley cubic surface by means of the invariant functions, $\rho y_1 = x_1(x_2^2 + x_3^2)$, $\rho y_2 = x_2(x_3^2 + x_1^2)$, $\rho y_3 = x_3(x_1^2 + x_2^2)$, $\rho y_4 = 2x_1x_2x_3$. The Cayley cubic thus obtained is $2y_1y_2y_3 - (y_1^2 + y_2^2 + y_3^2)y_4 + y_4^3 = 0$, and the mapping process between its points and the couples $(x)(x')$ or lines (n) in the plane of transformation is studied in detail.

The characteristic difference between two classes of invariant curves is exemplified by the two classes of invariant cubics,

$$a_1x_1(x_2^2 - x_3^2) + a_2x_2(x_3^2 - x_1^2) + a_3x_3(x_1^2 - x_2^2) = 0,$$

and

$$a_1x_1(x_2^2 + x_3^2) + a_2x_2(x_3^2 + y_1^2) + a_3x_3(x_1^2 + y_2^2) + 2a_4x_1x_2x_3 = 0.$$

Only the square of the first may be rationally represented on the Cayley cubic.

The classification of invariant curves by means of this mapping process, which is, in general, possible for any involution in a plane is extended to quartics, quintics, and sextics.

The investigation of invariant algebraic curves and surfaces is extended by A. Emch to general involutorial Cremona transformation in the plane and higher spaces(69, 71). It is found that in any involutorial Cremona transformation of any dimension, there are, in general, two classes of invariant forms (A) and (B), so that the square of (A) belongs to class (B).

The problem of invariant curves in a Cremona transformation is also considered by A. Myller(163). See also F. G. Taylor(222.1) on birationally related cubics.

E. Ciani(30) studies some constructions of the plane quartic with a double point, which he generates by the use of quadratic transformations.

The problem of mapping couples of corresponding points of an involutorial quadratic transformation upon certain cubics has also been considered by L. Godeaux(88). See also (87).

O. M. Thalberg(224) investigates the geometric relation between the tangents at multiple points of certain algebraic curves by means of involutorial quadratic transformations.

In a previous paper(225) the same author considers problems of this sort of a more general type by means of plane involutorial Cremona transformations, in particular properties of curves which are invariant in such transformations.

Concerning birational transformations of cubics into themselves see also E. Weyr(241), and C. Segre(204).

K. Ogura(168) applies involutorial quadratic transformation to cubics which are invariant, and studies their relations with the geometry of the triangle.

E. Ouivet(169.1) applies Cremona transformations to certain types of differential equations.

H. E. Timerding(229) investigates quadratic Cremona transformations also in connection with certain important classes of curves. Reference is made to G. B. Guccia's investigations(91.1) and first an account is given of curves and their properties connected with the general Cremona transformation. Timerding applies the results to the quadratic transformation, thus leading to a treatment which already W. Godt explained in (89). Timerding's object is to return once more to this analytic treatment and to add some new results.

3. *Repeated Quadratic Transformations.*—S. Kantor(113) determines the number of cyclic groups in a plane quadratic Cremona transformation. He first establishes the fact that in every quadratic transformation there is, in general, one pair of points which are in involutorial correspondence. The four invariant points of the transformation form the base points of a pencil of conics in the net of conics determined by three pairs of corresponding sides of the fundamental triangles. There are three cyclic groups of three points each.

Kantor finds the following formula for the number of cyclic groups of N points. If we put $N=a_1^{m_1}a_2^{m_2}\ldots a_\nu^{m_\nu}$, where $a_1a_2\ldots a_\nu$ are prime factors, then the number of groups, in any quadratic Cremona transformation, for which every point of a group returns to its original position after applying the transformation N times is,

$$Z_n=\frac{2-\Sigma 2^{\frac{N}{a_r}}+\Sigma 2^{\frac{N}{a_ra_s}}+\ldots+(-1)^\nu 2^{\frac{N}{a_1a_2\ldots a_\nu}}}{N}$$

$$=\frac{1}{N}\cdot\sum_{\mu=0}^{\mu=\nu}\sum_{r=1}^{r=\nu}(-1)^\mu 2^{\frac{N}{a_{r_1}a_{r_2}\ldots a_{r_\mu}}},$$

in which for $r_1r_2\ldots r_\mu$ any permutation of the numbers 1, 2, ..., ν must be set.

Thus there is one involutorial couple, there are two periodic triples, three periodic quadruples, 6, 9, 18 periodic quintuples, sextuples, septuples, etc.

For $N=a$, a part of Fermat's theorem results. Kantor's paper is dated Rome, February 13, 1880, and its results are generalized for a Cremona transformation of any order, in the same volume(114).

The same problem and some additional questions are considered by S. Kantor(115).

T. A. Hirst(105) studied the same problem of periodic quadratic transformations in 1881. The formula for the number of periodic transformations of cycles of m points, which is identical with that given by Kantor, was derived for Hirst by Professor H. J. Smith.

V. Snyder(211) refers to Kantor's and Wiman's results on periodic transformations and points out the omission by Kantor of an important category of such transformations. Snyder's purpose is to obtain the equation, define the system of fundamental elements, and discuss some of the properties of the transformation, and to show the existence of new ones not heretofore considered.

4. *Groups of Quadratic Transformations.*—L. Autonne(4) was one of the first to investigate quadratic Cremona substitution groups of three homogeneous variables.

H. E. Slaught(210) investigates the "cross-ratio group" of 120 quadratic Cremona transformations of the plane. This is a particular case of E. H. Moore's cross-ratio group(159). In the second part of the paper Slaught studies the complete form system of invariants of the cross-ratio group of order 120.

P. Field(79), in a note on certain groups of transformations of the plane into itself, considers the case of Slaught's diagram of five points and the 120 regions of the cross-ration group of the 120 quadratic Cremona transforms, when 4 of the 5 points are imaginary.

E. Ciani(31) investigates some notable groups of quadratic transformations in a plane. They have the property that they transform into themselves two groups of four lines each, which correspond to each other in two non-concentric projective pencils in the same plane. Ciani studies and classifies the groups formed by such transformations. The group is of order 8 when the two quadruples of lines are generic; of order 48 when they are harmonic; and of order 96 when they are equianharmonic. The most notable of these transformations are the quadric inversions, others are involutorial with four invariant points, and the remaining ones are of period 3 or 4.

Ciani applies the results to the elliptic quartic, which has the property that the quadruples of tangents drawn from the two double points are projective.

Earlier, G. Fano(74) investigates continuous groups of Cremona transformations in a plane, and certain groups of projective transformations.

The determination of Cremona transformations which form continuous groups has been carried out by F. Enriques(72).

The same problem for space was solved by Enriques-Fano(73).

A good account of continuous Cremona transformations may be found in G. Fano(75).

F. SYSTEMS OF CORRELATIONS AND COLLINEATIONS IN CONNECTION WITH QUADRATIC CREMONA TRANSFORMATIONS

1. *Hirst's Investigations on the Correlations of Two Planes.*—T. A. Hirst(106) takes as a starting-point the correlations determined by four points, $A_1B_1C_1D_1$ in Σ and four corresponding lines $a_2b_2c_2d_2$ in Σ'. If a line corresponding to A_1 passes through A_2, then the line corresponding to A_2 passes through A_1. Two points like A_1 and A_2 are called conjugate; likewise, a_1 and a_2 are conjugate lines if the point corresponding to a_1 lies on a_2, and conversely. There is *one* condition that in a correlation two points shall be conjugate, or two lines shall be conjugate. The problem to determine a correlation which shall satisfy *any* eight given conditions admits, in general, a limited number of solutions. Hirst finds six types.

The systems determined by α points in Σ with α given polars in Σ; β lines in Σ with β polar in Σ'; γ points and δ lines in each plane with given conjugates in the other, are as follows: if $(\alpha\beta\gamma\delta)$ denotes signature of system of correlations, and noting that $(\alpha\beta\gamma\delta) \equiv (\beta\alpha\gamma\delta)$; $2\alpha+2\beta+\gamma+\delta=7$,

I	(3010)	(0301)
II	(2110)	(1201)
III	(2030)	(0203)
	(2021)	(0212)
IV	(1130)	(1103)
	(1121)	(1112)
V	(1050)	(0105)
	(1041)	(0114)
	(1032)	(0123)
VI	(0070)	(0007)
	(0061)	(0016)
	(0052)	(0025)
	(0043)	(0034)

altogether 26 distinct fundamental systems.

Of particular importance are the cases (3010), (2030), (1130), (1050), (0070), each of which leads to a pair of conjugate connexes of the first class and second order. In each of these five systems, a point in either plane has a conjugate in the other plane, while every line is represented by a conic passing through three fixed singular points of the three exceptional correlations which each system includes.

We have here the quadratic transformation in its most general form; the three pairs of singular points being identical with the three pairs of fundamental points.

Hirst in (103) derives new numerical relations connecting exceptional correlations and shows how by means of these the above classification may be obtained in a much simpler manner.

Before this simplification Hirst extended the investigation, contained in the first paper, to correlations in space(102), which led to much more complicated questions.

2. *Further Investigations on Correlations and Collineations.*—Th. Reye(186) studies reciprocal transformations (correlations), polar, and null-systems; F^3— and ϕ^2— systems; and apolar relations.

R. Sturm(218) investigates correlative pencils, and in (217) correlative bundles, where he extends the idea of Hirst's signature to space-relations.

R. Sturm(219) considers correlations and correspondences connected with them, and first remarks that Hirst showed that in every ∞^1 system of correlations there are two kinds of singularities according as there is a singular point in each plane or a singular line in each plane.

In the system (0070) of which the remaining four mentioned by Reye and Hirst are special cases, the three pairs of singular points are those from which the seven points in one field and the seven corresponding points in the other field are projected by projective pencils.

Sturm also discusses the quadric inversion by Hirst-Geiser, as generated by ∞^1 correlations (1130). W and w as pole and polar in both senses (Σ and Σ'), and 3 points of K^2 as self conjugate. K^2 is common to all correlations of the system.

Sturm also points out an error made by A. Millinowski(247) in a paper on correlations of the second order, obtained by two planes collinear in two ways. The latter thought, mistakenly, that the correspondence established by lines between Σ and Σ' was (1, 1).

After extending the investigation to correlative bundles, Sturm studies infinite numbers of quadratic transformations between Σ and Σ' with the same seven pairs of corresponding points. In order that this may be possible, the fundamental points in $\Sigma(\Sigma')$ must lie on a cubic $K^3(K'^3)$, as pointed out in connection with Millinowski's paper. In this case there are ∞^1 singular correlations among the ∞^2.

Also, the null-system of the second order, formed by A. Ameseder(3), is discussed. This may be put in simple form,

$$\begin{aligned}\rho u_1 &= a_1 x_2 x_3,\\ \rho u_2 &= a_2 x_3 x_1,\\ \rho u_3 &= -(a_1 + a_2) x_1 x_2,\end{aligned}$$

from which there is obviously $u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$.

R. Sturm(219.1) proves a theorem due to Clebsch, published without proof, which, in terms of correlation, may be stated as follows: All ∞^3 correlations of two fields A, B which have five pairs of conjugate pairs of conjugate points A_1, B_1; ...; A_5, B_5 in common, have a sixth pair A_0, B_0 in common. Two points in two correlated fields (planes) are said to be conjugate, when each lies in the corresponding line of the other. Sturm discusses investigations by Rosanes and Voss on collinear and correlated systems and their connections with quadratic transformations.

S. Kantor(116) discusses the relation between nets of collineation and quadratic transformations.

F. London(138) discusses important constructive problems connected with the theory of correlation and the theory of quadrics and quadric systems. He succeeds in finding a simple construction of a quadratic transformation when seven corresponding pairs are given. Then the problem is solved to construct the common null-pair of three correlations when six of them are given. This is important because the construction of space quartics of the first kind, when eight of its points are given, may be reduced to a special case of that problem. Similarly, the construction of a quadric, when nine points are given, becomes identical with the construction of a quadratic transformation when seven pairs of corresponding points are given.

Other solutions of the latter problem are given by H. Schroeter(199), and by Th. Reye(187).

3. *Trilinear Correspondence.*—H. Schubert(202), (203) establishes trilinear relations between three one-dimensional forms g, g', g'', whose parameters are connected by an equation,

$$xx'x'' + ax'x'' + a'x''x + a''xx' + bx + b'x' + b''x'' + c = 0.$$

It has six singular pairs. Such trilinear relations lead to forms of order 3; for example, 3 trilinear pencils of planes generate a cubic surface. By the properties of this relation Schubert is able to construct the cubic surface linearly.

G. Hauck(95) develops a theory of trilinear correspondence between plane systems Σ_1, Σ_2, Σ_3, and threefold (dreibuendig-eindeutig) quadratic Cremona transformations with three pairs of fundamental conics.

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CHAPTER II

ANALYSIS OF SINGULARITIES OF PLANE ALGEBRAIC CURVES

1. *Account of the Development of the Theory of Singularities of Plane Algebraic Curves.*—In the Lezioni Sulla Teoria Geometrica delle Equazioni e delle Funzioni Algebriche, by F. Enriques, edited by O. Chisini, Vol. 2, the whole theory of singularities is divided into three chapters: (1) Singularities and expansions in Puisseux series, (2) singularities with respect to quadratic transformations, (3) singularities with respect to differential calculus. The three aspects and their connections are critically illuminated. The treatment concludes with a historic survey of the whole problem.

As there are important connecting links between the various points of view, the historic part of the report will not be limited to the treatment of singularities by quadratic transformations, and will follow in the beginning mainly Enriques' Lezioni(^{50}).

The theory of algebraic curves or functions has its origin in the study of successive approximations, and as such is connected with the work of Newton. In reality Newton did not occupy himself particularly with singular points, but in(^{99}) giving examples, studied the problem of successive approximations of the implicit function defined by the equation of $f(x, y)=0$, from which results the expansion of y in powers of x. Newton considered this problem as an extension of the determination of the successive decimal places of a root of an equation $f(x)=0$. Considering $f(x, y)$ in the neighborhood of a point (o, a), he puts $y=a+z$, and obtains the resulting equation in z, which is supposed to be satisfied by putting $z=bx+z_1$, thus giving rise to a new equation in z_1, which is then solved by putting $z_1=cx^2+z_2$, and so forth.

To write effectively the equation which determines z, and analogously the succeeding equations, Newton arranges the terms of $f(x, a+z)$ in rows and columns, such that the terms of each row or column have the same degree with respect to one of the variables. Thus *Newton's parallelogram* arises(^{100}).

De-Gua(^{43}) applied this diagram to the problem of the separation of the branches of the algebraic function $y(x)$ at a singular point. For this purpose he assumes the singular point at the origin, and thus determines the groups of characteristic terms to which the equation reduces, by means of infinitesimals of higher order. In De-Gua's treatment, Newton's parallelogram assumes the form of "algebraic triangle," which Cramer later called "analytic triangle." De-Gua thought that singularities at the origin could be approximated by $y^{\nu}=ax^{\mu}$. This error was corrected by Euler, 1749, who observed that a branch may be imaginary, when x is negative, but still admits of a real approximation, and illustrated this by the cusp of the second kind $y=ax^2+bx^{\frac{5}{2}}$, already discovered by De L'Hopitale.

The complete separation of the branches at a singular point by means of the calculus of successive approximations is found in Cramer(38). The same method is used in Chapter 7 for the asymptotic approximation of infinite branches. According to Enriques, Cramer's method, for practical purposes, has not been surpassed even today, although other methods have been added since that time.

For Cramer the purpose of the investigation of a singularity is attained, when it is possible to separate the branches forming the singularity, and the approximation may be carried as far as desired. But Cramer's method does not examine the infinite power series to which the branches give rise and the possibility of an analytic representation of the branches of an implicit function $y(x)$. This problem was considered by Newton's school, and Newton himself stated how one may obtain an "exact" value of a root of an equation by an infinite expansion.

B. Taylor(145) followed the path of Newton and found as his main contribution to the theory of implicit functions, the series bearing his name. C. MacLaurin(94), based on the method of undetermined coefficients, found the particular case of Taylor's series, known as MacLaurin's series.

The question concerning the validity of Taylor's series goes back to Lagrange(85), who gives as a definition of an arbitrary function $f(x)$ of one variable an expression in which the variable enters in any manner. According to this a function $f(x+i)$ may be expanded into a power series of integral powers of i, with the exception of particular values of x. This expansion becomes illusory when the consecutive derivatives of f, for the point x, up to a certain order become infinite. Then $f(x+i)$ may be expanded in power series with fractional and also negative exponents when $f(x)=\infty$. These conclusions suppose a particular form of the function, constructed with a small number of symbols of calculation. Lagrange was the first to give an expression for the remainder of Taylor's series.

A complete justification of this expansion was accomplished only by the extension to functions of a complex variable by the efforts of Cauchy, Laurent, and Puisseux. Cauchy(28) as early as 1826, gave a precise answer to the problem of the convergence of Taylor's series and the possibility of expanding a function by such series. In particular he determined rigorously the expansion of functions into entire power series, under the conditions of regularity which exclude singular points.

Laurent showed, in 1843, how in the algebraic domain, the case of poles gives rise to negative exponents. Puisseux(123) admitted the case of branch points, and shows how the branches of an algebraic function in the neighborhood of a singular point may be represented by power series with fractional powers of the independent variable, and by using Newton-Cramer's method also gives effective calculations. Puisseux' method has been treated and in some points perfected by Fischer(55). See also the treatises by Clebsch-Lindemann(33) ; Baltzer(5) ; Netto(98) ; Hensel-Landsberg(70) ; E. Picard(113) ; Jordan(73) ; and other references in the Encyklopädie.

With this ends the first period of the theory of singularities of which the "Théorie des fonctiones elliptiques" by Briot et Bouquet(26) may be mentioned as a representative treatise.

The second, or modern period of the theory of singularities of algebraic curves, may be characterized by the problem of intersections, to which one is led also by those researches whose object is the absolutely general formulation of Pluecker's relations. Cayley(31) in 1865 found the multiplicity of the intersections of two curves by taking the definition of the resultant as the product of the differences of the roots as a starting point. This multiplicity is the sum of those of the couples of branches, and, when considering the singularity at the origin and the expansions of x and y by entire powers of a parameter t, its determination is reduced to the evaluation of the minimal degree in which t figures in the resultant of Weierstrass(151).

The problem of evaluating the intersections of two tangent branches when one knows the respective Puisseux expansions, is solved by H. J. Smith, in (139) and by G. Halphen(65).

These investigations put in evidence the purpose which certain "characteristic terms" and the exponents of the variable have in the Puisseux series.

A more original and very fruitful point of view is taken by Noether(101), by introducing quadratic transformations. This method is able to serve equally well for the analytic calculation of the Puisseux expansions and for the geometric study of the nature of the singularity.

As for the first task, the idea of using quadratic transformations was conceived by Hamburger(67) and afterwards by Weierstrass(151). See also Koenigsberger(81).

An extensive development with respect to the geometric aspect is given by Noether in a fundamental memoir(103).

The possibility of transforming a curve with any singularities into a curve with multiple points with distinct tangents goes back to Kronecker, who communicated the theorem verbally to Riemann and Weierstrass in 1858. Kronecker(84) based his argument on Puisseux's expansions and imposed a rational substitution on a single variable x, which is rationally invertible for the points of the curve $f(x, y)=0$, as may also be seen in his researches on discriminants of algebraic functions of one variable, in which he distinguishes between an essential and non-essential factor. The first factor which corresponds to the branch points of the function, turns out to be invariant in rational transformations of the curve.

It is in the introduction of this paper in which Kronecker writes of his contact with Riemann and Weierstrass on this matter.

Kronecker's method is based upon the fact that the discriminant of y, i. e., the resultant $R(x)$ of the elimination of y between $F(x, y)=0$ and $\frac{\partial F}{\partial y}=0$ is composed of two factors D and D_1, of which one is an essential divisor, the other an unessential divisor which enters as a square, so that $R=D \cdot D_1^2$. The first factor depends on the branch points x of y and their multiplicities only. The second factor depends on the multiple points of the curve, and the transformation permits the separation of its quadratic factors from each other and from the factors of D. Riemann(125) accomplished the same result by transcendental means. In this connection see further G. Landsberg(86); E. Lasker(87); L. W. Thomé(146); S. Kantor(76). G. A. Bliss(18.1) gives a valuable critical review of what has been done concerning the reduction of singularities of plane curves by birational transformations, and makes a study of birational transformations simplifying singularities of algebraic curves. The method followed consists in the construction of a pair of rational functions $\xi(x, y)$, $\eta(x, y)$ satisfying an irreducible equation $\phi(x, y)=0$ birationally related to $f(x, y)=0$, having certain properties described subsequently. Also, the distinction between the theorems for the projective and the function theoretic planes is explained.

V. Snyder(139.1) shows that curves of given order and genus with ordinary double points exist for every value of the genus p not exceeding $\frac{1}{2}(n-1)(n-2)$, n being the order of the curve, and determines the equation in each case.

S. Lefschetz(88.1) considers the loci with given singularities and shows the existence of a curve having assigned Plueckerian characters. The question is equivalent to the determination of the maximum number of cusps that a curve of order *n* and genus *p* may have. He also points out the extension of the theory to surfaces and higher varieties.

J. L. Coolidge(36.1) discusses the characteristic numbers of real algebraic plane curves and proves the theorem that the only algebraic identities involving any combination of real or total singularities, which are valid for all real algebraic plane curves, are those which are deducible from the known equations of Pluecker and Klein(79).

T. R. Hollcroft(70.1) considers problems such as to find the greatest number of cusps that may be added to the singularities of a plane curve of given order and number of cusps and nodes.

In (70.2) Hollcroft studies maximal cuspidal curves, i. e., curves which for a given order and genus have the maximum number of cusps, and as such are unique.

Operating with birational transformations on a curve one arrives at a much more expressive result, namely, the possibility of reducing an arbitrary curve to one possessing only double points. See Hensel-Landsberg(70).

Klein seems to have been the first to point out this consequence of Kronecker's theorem, by recalling an observation made by Clebsch that the curves with multiple points with distinct tangents can be reduced to curves with double points only, by the medium of space curves and the plane mapping of the cubic surface. More precisely, Halphen, in Étude referred to above(58), points out, that any algebraic singular curve can be transformed into a space curve without singular points, from which by projections one obtains a plane curve with nodes only. A simple geometric proof of the theorem is given by Bertini(13). Other proofs are given by Picard-Simart(116); Halphen(58); Appell-Goursat(2); Simart(138); A. R. Forsyth(56); E. Vessiot(150); A. Del Re(47).

Noether(101) used Cremona transformations, in particular quadratic Cremona transformations. By passing from a given curve to a series of transforms by quadratic transformations, one obtains a concept of the composition of the singularity by means of consecutive multiple points, which may be considered as the most important acquisition in the theory of singularities. Noether is thus led to conceive any extraordinary singularity as a limit-singularity due to the reunion of several multiple points which approach each other indefinitely, and of simple branch points.

Noether's theory has been given a simpler geometric treatment by Bertini in several papers(12), and in the systematic expositions in the appendix of his Introduzione alla geometria projettiva degli iperspazi(9); German translation by A. Duschek(48).

The reduction of singularities by transformations also permits one to extend algebraically to algebraic curves with any sort of singularities the concept of that invariant characteristic which constitutes the genus, which on the other hand may be defined with equal generality by the order of connectivity of the corresponding Riemann surface.

From the relation of invariance of the genus applied to the correspondence between point curves and envelope curves, Noether(102) deduces the extension in case of any singularity of one of Pluecker's formulas; on the other hand, by calculating the intersections of the curve with a polar, he determines the reduction of the class due to any singularity, which by another method was also evaluated by Halphen.

The complete extension of Pluecker's formulas requires us, moreover, to pass from the characteristics of singularities of a point-curve to those of the dual curve, which is accomplished by making use of the observation of Cayley(31) and Halphen(66) that in a dual transformation order and class of a branch are permuted.

Other relations between the characteristics of a curve capable of valuable applications were established by H. J. Smith(139) and Zeuthen(159), (161).

With the extension of Pluecker's formulas is also associated the question of Plueckerian equivalents. Cayley(31) in 1866, is led inductively to the concept that every singularity in its effects on Pluecker's formulas may be replaced by a certain number of double points and cusps, double tangents and flexes.

A purely arithmetical justification for this was given by H. J. Smith(139). There remains the essential problem to recognize in the Plueckerian equivalents the trace of a passage to the limit, by which every singularity results from the superposition of elementary singularities. This question may be illuminated as well by the expansions into series, as by the method of quadratic transformations.

The first method is adopted by A. Brill(23) who substitutes for a branch of the curve an "osculating hyperparabola" which he lets vary conserving rationality. The second method is implicitly contained in Noether's definition of a singularity as a limit of ordinary multiple points which approach each other. This was done by Miss Scott(129).

The relation between the theory of singularities by Noether and that of Smith-Halphen, based on Puisseux expansions comes to light when one proposes the effective calculation of those expansions by quadratic transformations with Hamburger's method mentioned above. The explicit determination of the successive multiple points of a branch in terms of the exponents of the development into series, following from the inversion of the preceding procedure, is contained in a memoir by

Noether(105), and is obtained by the use of quadratic transformations, substantially as applied by Noether in his preceding investigations.

The various methods are coordinated and supplemented by F. Enriques(52), and again systematically presented in his Lezioni(51). The importance of the new concept, in which the "satellites" and "free" points connected with a singularity are arithmetically illuminated, consists in the completion of Noether's analysis. For Noether the singularity is simply defined as reunion of consecutive multiple points; for Enriques the idea of the position of these points must be added, which is expressed by the concept of "satellitisme," on which depends the existence of superlinear branches.

The notion of "satellite points" is based upon the following considerations on the intersection of a branch of the second order

$$(1) \qquad y=a_1x+a_2x^2+\ldots+a_sx^s+a_{s+1}x^{s+\frac{1}{2}}+a_{s+2}x^{s+1}+\ldots,$$

$$s\geqq 1, \qquad a_1\neq 0, \qquad a_{s+1}\neq 0,$$

and the linear branch

$$(2) \qquad y=a_1x+a_2x^2+\ldots+a_ix^i+b_{i+1}x^{i+1}+\ldots.$$

When $i=s-1$, the branch (1) has $s-1$ double points $O_1\ldots O_{s-1}$ consecutive to O, and succeeding each other on the linear branch

$$y=a_1x+a_2x^2+\ldots+a_{s-1}x^{s-1}.$$

When $i\geqq s$, to the $(s-1)^{\text{st}}$ double point O_{s-1} consecutive to O on the branch (1) follows an s^{th} point O_s which is simple on (1). In this case (1) is defined as a branch of *species* $s-1$.

Now on a branch of the second order and of species $s-1$, there exist $s-1$ double points consecutive to O, and then the simple points O_s, O_{s+1}, O_{s+2}, ..., then the second simple point O_{s+1} is defined as the *satellite* of O_s, so that every branch of the second order passing properly through O_s (i. e., through $O\ O_1\ldots O_s$) passes in consequence also through O_{s+1}.

Another new feature of Enriques' treatment is the study of the differential conditions which characterize the passage of a curve through consecutive points and the relative multiplicities. In this manner is obtained a more precise determination of the concept of extraordinary singularities as limit-singularities, from which follow interesting applications, in fact a complete theory of algebraic singularities from the point of view of infinitesimal analysis. Enriques gives some historic data leading to this development.

Hamburger(67), in 1871, calculated the Puisseux expansions for linear branches, by determining the coefficients as successive derivatives of the algebraic function $y(x)$, and Stolz(142) in 1874 reconsidered and continued Pluecker's analysis in the form later generalized by Enriques, and, like Hamburger, reduced by proper transformations the case of superlinear branches to that of linear branches. See also Koenigsberger(81); De La Gournerie(45); Darboux(40).

The Enriques investigation begins with the demonstration and extension of Stolz's formulas for the successive derivatives of implicit functions and of composite functions. The symbolism introduced permits of a treatment in terms of the theory of operations together with the expansion of the power of a polynomial. The application of the formulas of the calculus thus obtained leads in a simple manner to the required result to characterize with differential conditions the composition of the points which make up the elements of a singularity. The result seems particularly important when taking into account the case of satellite points and hence of the superlinear branches, whose origin appears thus in its true light; likewise the classical method of Newton-Cramer, which thus attains a more profound significance. Also, the relation with the arithmetical point of view of the theory becomes apparent in the diagram by which Enriques has illustrated his differential conditions, thus moving in an order of ideas which have contact with some recent arithmetical theories, as, for example, contained in Minkowsky's "Geometrie der Zahlen"(97).

On the theory of singularities of an algebraic curve and the neighboring points by means of differentials of various orders of the coordinates, or the "elements of first and higher order," which is of importance in the treatment of differential equations of first and higher order, one may find comment by F. Engel(49); E. Study(144); M. Noether(106), (107); Zeuthen(161).

The double points with a single tangent may be put in two classes, into proper nodes (tacnodes, oscnodes) and cusps (of the first, second, ... kind), according as they may be resolved into two simple points or a single point after a finite number of quadratic transformations. The characteristic form of the equation of a curve with such a singularity at the origin is given by B. Levi(90). Let Δ be the order of the branch at A, i. e., the number of intersections at A of a generic line r, sufficiently close to A, so that the sum of the orders of all branches through A is equal to the order s of the multiplicity. There exists a line a, however, one of the tangents to f at A, such that when r forms a sufficiently small angle with a, r will cut the branch in more than Δ points. a is then called the singular tangent of said branch through A. The class Δ' of the branch is the number of tangents to the branch from a point different from A whose distance from a remains below a certain limit. Thus for an ordinary point, a cusp of the first, and a cusp of the second kind, Δ and Δ' have the values 1, 1; 2, 1; 2, 2. The idea of order and class of a branch (in case of reality) goes back to Pluecker(119). Further references on branches and corresponding expansions may be found in E. Picard(114); Appell-Goursat(2); C. Jordan(78); Halphen(58), (63) with abstract of this in (64). For real branches see E. Cosserat(37); E. Vessiot(149). An analytic treatment of a curve in the neighborhood of a finite or infinite point is given by Ch. Bieler(17). M. de Franchis gives a synthetic treatment of higher singularities in (42). A. Cayley(31), and in particular G. Halphen(63), (64) have investigated the mapping of singularities between correlated curves. To a branch (Δ, Δ') of one curve corresponds a branch (Δ', Δ) of the other. The number of intersections of the branch with a is equal to the number of tangents from A coincident with a, i. e., equal to $\alpha = \Delta + \Delta'$.

When $x_1 y_1$ are the coordinates of A, and assuming the singular tangent different from $x - x_1 = 0$, which is always possible, from the series of quadratic transformations which resolves the singularity, and the possibility of expansion of the coordinates x, y of $f(x, y) = 0$ in series of integral exponents of a parameter t in one-to-one correspondence with the points of the branch, i. e., t a rational function of x, y, follows the representation of the branch

$$y - y_1 = (x - x_1)^{\frac{\alpha}{\Delta}} \cdot \phi\left[(x - x_1)^{\frac{1}{\Delta}}\right],$$

in which, according to Halphen ϕ is a series of ascending positive powers of $(x - x_1)^{\frac{1}{\Delta}}$. In this expansion $\alpha \geqq \Delta$, for $y = y_1$ is a singular tangent $\alpha > \Delta$, in fact according to Cayley-Halphen $\alpha = \Delta + \Delta'$. Compare O. Stolz(142); Weierstrass(153); L. Koenigsberger(81); O. Biermann(18); Pluecker(119).

A. Brill in (25) and (19) gives a rigorous function-theoretic treatment of algebraic functions and its geometric interpretation for algebraic curves. Much use is made of the Weierstrassian "Vorbereitungssatz" which in the hands of Brill becomes a powerful instrument of analysis and by which he proves Noether's fundamental $A\phi + B\psi$-theorem. By these methods is also shown how a reduced polynomial may be resolved explicitly into linear factors concerning y; thus,

$$f(x, y) = c(x - a) \prod_{i=1}^{n} \left\{ y - \frac{1}{(x - a)^{\nu i}} - Q^{(i)}\left[(x - a)^{\frac{1}{\Delta i}}\right] \right\}$$

in which the Q's are power series. Making use of these developments, the discussion of the branches of the curve offers no particular difficulty.

Weierstrass' investigations concerning the aforesaid theorem are contained in (152). See also W. F. Osgood(108).

2. Consecutive Points of a Singularity; Characteristics. Birational Transformation of Plane Algebraic Curves into Non-Singular Space Curves.—On account of the importance of certain terms used in the theory of singularities, a more detailed presentation of their definitions will perhaps be helpful in reading the report.

In the first place Noether's definition of consecutive singular points must be mentioned. The behavior of curves and its ordinary singularities in a quadratic transformation has been discussed previously in Chapter 1. Now consider an r-fold point O of a curve f. Assuming O as a fundamental point of a quadratic transformation in Σ, the r-fold point is transformed into the points O'_1, O'_2, ... of multiplicities r_1, r_2, ..., ($r_k \geqq 1$) in Σ'. Every one of these points whose order is >1, may be treated similarly by new quadratic transformations, and so forth; so that finally after a finite number of quadratic transformations, a transformed curve with only ordinary singularities (distinct tangents) is obtained. This constitutes Noether's famous theorem. That by birational transformation the reduction may be made to a curve with ordinary double points only, was first proved implicitly by Kronecker(84) as pointed out before; then in a purely geometric manner by Bertini(18) and reproduced by Picard in his Traité(115). For other proofs see Halphen(60), Appell-Goursat(8), Simart(138). See also Hensel-Landsberg(69).

That the series of quadratic transformations is limited is proved by Hamburger(67). See also Bertini(18); Severi(137); Picard(115). Reversing the order of transformation by their inverses, the original curve f is obtained with an r-fold point at O, which now appears as surrounded by consecutive points $O_1 O_2 \ldots O_i$ in the first neighborhood, O_{11}, O_{21}, ... in the second neighborhood, O_{111}, O_{211}, ... in the third neighborhood, etc. whose multiplicities are $r_1, r_2, \ldots, r_i$; $r_{11}, r_{21} \ldots$; $r_{111}, r_{211}, \ldots$ respectively, and which satisfy the fundamental neighborhood relation

$$r = r_1 + r_{11} + r_{111} + r_2 + r_{21} + r_{211} + \ldots.$$

This is the fundamental idea used by Noether in his classic memoir of 1871.

The multiplicities of the points of a branch whose Puisseux expansion is $y = ax + a_1 x^{\frac{\nu+\nu'}{\nu}} + \ldots + a_{m+1} x^{\frac{\nu+\nu'+\ldots+\nu^{(m+1)}}{\nu}} + \ldots$, depend essentially on those terms in which $\nu^{(i)}$ is not divisible by the highest common divisor of the preceding ν's. The positions of the successive multiple points of the branch depend on the coefficients a, a_1, ..., a_m, and on the positive integers $\nu (>1)$, ν', ν'', ..., ν^{m+1} in which m denotes the smallest index for which $\nu = \rho\rho_1\ldots\rho_m (\rho_m > 1)$. It is assumed that $a_1 \neq 0, \ldots a_{m+1} \neq 0$, and $\frac{\nu'}{\nu} = \frac{\rho'}{\rho}, \ldots, \frac{\nu^{(m+1)}}{\nu} = \frac{\rho^{(m+1)}}{\rho\rho_1\ldots\rho_m}$; $\rho \geqq 1, \rho_1 \geqq 1 \ldots$, and ρ^{i+1}/ρ_i are irreducible fractions. The orders of multiplicity of said points are the remainders of the successive divisions: $\nu' : \nu; \nu : \nu_1; \ldots, \nu_{\sigma-2} : \nu_{\sigma-1}; \nu'' : \nu_\sigma; \ldots; \nu''' : \nu_{\sigma+\sigma'}, \ldots$, which lead to the determination of the highest common divisor of $\nu\nu', \nu\nu'\nu'', \ldots, \nu\nu'\nu''\ldots\nu^{(m+1)}$. Under the hypotheses made

$$\text{h.c.d. } (\nu\nu') = \nu_\sigma = \frac{\nu}{\rho} = \rho_1\rho_2\ldots\rho_m,$$

$$\text{h.c.d. } (\nu\nu'\nu'') = \nu_{\sigma+\sigma'} = \frac{\nu}{\rho\rho_1} = \rho_2\ldots\rho_m,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\text{h.c.d. } (\nu\nu'\nu''\ldots\nu^{(m)}) = \rho_m,$$

$$\text{h.c.d. } (\nu\nu'\nu''\ldots\nu^{m+1}) = 1.$$

The characteristic terms introduced by Smith and Halphen mentioned in Section 1 are substantially $a_i x^{\frac{\nu+\nu'+\ldots+\nu^{(i)}}{\nu}}$ and the characteristic couples of numbers

$$\text{h.c.d. } (\nu\nu'\ldots\nu^{(i-1)}) = \rho_{i-1}\ldots\rho_m,$$

$$\nu^{(i)} - \rho_{i-1}\ldots\rho_m \left| \frac{\nu_i}{\rho_i\ldots\rho_m} \right|.$$

By another notation, to the branch, as represented above, correspond the characteristic couples $\nu, \nu_1; \nu_\sigma, \nu_{\sigma+1}; \ldots$ where

$$\nu_1 = \nu' - \nu \left| \frac{\nu'}{\nu} \right|, \quad \nu_\sigma = \text{h.c.d. } (\nu\nu') = \rho_1\ldots\rho_m; \quad \nu_{\sigma+1} = \nu'' - \nu_\sigma \left| \frac{\nu''}{\nu_\sigma} \right|, \text{ etc.}$$

The number of characteristic couples is called genus of the branch by Enriques.

When a singular point is at the origin O, then a generic line a through O cuts f in an intersection of multiplicity Δ. The branch is linear or superlinear according as $\Delta = 1$, or > 1. When the multiplicity of the intersection of a is $\Delta + \Delta'$, $\Delta' > 0$, then a is a singular tangent to the branch at O, and f may be represented parametrically as $x = ct^{\Delta} + \ldots$, $y = c't^{\Delta+\Delta'} + \ldots$, where $c, c' \neq 0$ are constants. Δ' is called the class of the branch. The tangents at the points of a branch are dual to the branch (described by points) and Δ and Δ' are dually corresponding. A. Cayley(30) conjectures the theorem that Δ' is the number of tangents coinciding with a which pass from a point of a (except O) to the branch and $\Delta + \Delta'$ the number of tangents from O coinciding with a. This was proved by Halphen(63), (64). See also Zeuthen(160); Halphen(59); Bertini(8); Segre(133).

As already indicated in Section 1, Noether(105) reconsiders the question of singularities in order to establish the connection between his former results obtained by quadratic transformations and the characteristics of Halphen and Smith. It is known that analytically these developments must be derived from a succession of quadratic transformations of the curve which resolves the singularity step by step. But this analytic process leads to a series of continued fractions by which numbers defining the singularity result directly. Noether asks and answers the question, which of the combinations of these numbers of transformations are the characteristic numbers of Smith and Halphen.

A very important aspect of the problem of singularities is gained by the method of projection between higher spaces. Veronese(147) proved the theorem that any algebraic curve whatsoever may be obtained as a projection of a hyperspace curve without singularities. This is implicitly contained in Brill-Noether's theory of linear point-series on an algebraic curve. The theorem concerning the birational transformation of an ordinary space curve into a non-singular curve of the same space was proved by Poincaré(122).

See in this connection also Del Pezzo(46); Vessiot(148); Segre(134); Levi(89); Hensel-Landsberg(70).

As a matter of fact, by the theory of linear point series it is easily proved, (Severi-Loeffler(137)), that every plane algebraic curve may be obtained by a birational transformation from a space curve without singularities in a space of $r \geqq 3$ dimensions. Moreover, every plane algebraic curve, with any singularities whatsoever, may be obtained as projection of a space curve without singularities in a space of four or more dimensions. Again, as stated before, an algebraic curve with any singularities, may always be transformed birationally into a plane curve with only ordinary double points.

M. De Franchis(42) develops a geometric theory of the singularities of a plane algebraic curve by taking the definition of a cycle as a starting point. If C is a curve and P a point on it, which by a certain Cremona transformation is transformed into a curve C_1, so that to a point P correspond simple points $P'_1 P''_1 \ldots P_1^{(s)}$ of C_1, then de Franchis calls cycle of C the totality of points of C consecutive to the same point P_1.

The graphical features of higher singularities are investigated by Ch. A. Scott(129) who gives a graphical analysis of the singularities of algebraic curves, by means of the quadric inversion, and is thus enabled to ascribe a definite geometric meaning to the numbers δ and κ in Pluecker's formulas, also in cases of higher singularities.

A. B. Basset(6) by means of birational transformations discusses compound singularities of quintic curves, and in the second paper uses special quadratic transformations to resolve certain types of singularities of plane algebraic curves.

Ch. A. Scott(130) gives an account of quadric inversion in an introductory study of certain modern ideas and methods in plane analytic geometry.

H. W. Jung(74), (75) treats singular points of plane algebraic curves by the arithmetical theory of algebraic functions, and establishes and clarifies the connection of his theory with that of the geometric-algebraic method.

3. *Multiplicities of Intersection; Additional References on Singularities.*—The question of the multiplicities of intersections of two curves with given singularities at a given point is closely connected with that of singularities in general, and deserves particular discussion. The importance of this will not be diminished by some repetitions of references given in the preceding sections.

Weierstrass(154) investigates the multiplicities I of the intersection of two curves f, f' at a common point, which may be determined by adding the multiplicities of the intersections of one of the curves with the branches of the other. I may also be determined by the formation of the resultant of f, f' as a product of differences, as was done by Cayley(31). Compare also Smith(139); Halphen(61), (66); De La Gournerie(44), (45); Painvain(109).

Segre(136) (see also Bertini(10)) points out that it is not sufficient for the characterization of a singular point merely to establish the orders of the branches and the multiplicities of the consecutive infinitely close branches. The nature of the branches passing through the multiple point must be ascertained also.

Another method for the determination of the multiplicities of intersections was established by Brill(24). If f, and f_1 are curves of order n and n_1 with arbitrary singularities, but no common parts, the common intersections are obtained by the linear factors of the resultant R. Cayley(31) and H. J. Smith(139) give as definitions of the resultant the product of the differences of the values of y which from $f=0$, $f_1=0$ correspond to one and the same value of x. For an intersection $A(x_1, y_1)$ it is sufficient to use the expressions for y which assume the value y_0 for $x=x_1$. Substituting in R for the y's their expansions in terms of $x-x_1$, then the least exponent of the result will give the required multiplicity. Concerning this see also O. Stolz(143), Noether(104), Segre(135). Halphen(66) gives a geometric form to the definition of multiplicities of intersections, as the sum of the orders of the infinitely small segments which are cut out between the two curves by a transversal whose distance from A is infinitely small of the first order. For this see also Zeuthen(157), (158).

Noether studies multiplicities of intersections by means of quadratic transformations which resolve the singularities of f and f_1 at A. If $r_i, r_{ik}, \ldots$ and $s_i, s_{ik}, \ldots$ denote the multiplicities appearing in the succession of quadratic transformations of f and f_1 with multiple intersections at A, then the multiplicity of intersection of f and f_1 at A is expressed by

$$rs+\sum_i r_i s_i+\sum_{i,k} r_{ik}s_{ik}+\ldots.$$

Macaulay(92) gives a method for determining the multiplicities of intersection in connection with Noether's fundamental theorem. See also Ch. A. Scott(131), and Macaulay(93).

Bacharach(4) gives a geometric form to the fundamental theorem of intersections in the form: If mp of the intersections of a C_m with a C_n lie on a C_p, then the remaining $m(n-p)$ lie on a C_{n-p}. Zeuthen(162) proves this theorem directly and derives from it the analytic representation of such related curves. Bacharach also shows that when a C_p is made to pass through the intersections of C_m and a $C_n (p \geqq m, p \geqq n)$, then for $p > m+n-3$ the mn conditions to be imposed on C_p are independent; however, for $p \leqq m+n-3$, of these conditions exactly

$$d=\tfrac{1}{2}(m+n-p-1)(m+n-p-2)$$

are a consequence of the remaining ones. In other words, a C_p through $mn-d$ of those intersections also passes through the remaining d, except in the case that these lie on a $C_{m+n-p-3}$.

See also Jacobi(71); Pluecker(121); Cayley(30), (32); Clebsch(35); Meray(96).

B. M. Walker(150.1) gives a detailed account of the proof suggested by Bertini by mapping the curve by cubics on six points upon a space-curve and projecting the space-curve back into the plane.

In (156.1), O. Zariski extends Puisseux's theorem on the expansion in series of analytic functions in the neighborhood of a critical algebraic point to the case of algebraic functions $y(x)$ in a field which contains several critical points. In particular he asks the question whether the relation between x and y may be uniformized by the introduction of a parameter z which is an algebraic function of x, and succeeds in establishing the condition for such a possibility, i. e., the substitutions of $y(x)$ in the critical points of C must belong to an algebraic function of genus $p=0$.

Additional references concerning algebraic curves, their singularities, correspondences, and their properties in general will be found in the following alphabetical list.

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CHAPTER III

LINEAR SYSTEMS OF PLANE CURVES

1. *Definitions.*—The system of curves represented by an equation of the form

$$(1) \qquad \sum_{i=1}^{k+1} \lambda_i f_i(x_1, x_2, x_3) = 0$$

where the f_i are of the same order n and linearly independent, is a linear system of curves of dimension k. A point which is at least r fold for each curve of the system is called an r fold basis point. It was shown by E. Bertini(1) that if a linear system of curves has a variable r fold point, the system consists of a fixed curve counted $r-1$ times and a variable curve. He also showed that if every curve of a linear system is composite, either there is a fixed component or else each curve consists of the same number of curves belonging to the same pencil. For another proof see J. Lüroth(14) and (15). In studying linear systems of curves it is convenient to consider systems of maximum dimension with a given set of basis points. An r fold point imposes $\frac{r(r+1)}{2}$ conditions on a curve so that counting the conditions separately we have the virtual dimension

$$(2) \qquad K = \frac{n(n+3)}{2} - \sum \frac{r(r+1)}{2}.$$

In certain cases the conditions are not independent so that the effective dimension is

$$(3) \qquad k = K + s,$$

where s is the superabundance.

A system for which $s=0$ is said to be *regular.*

A curve of the system is determined by k arbitrary points. The virtual genus of a general curve of the system as calculated from the basis points is

$$(4) \qquad P = \frac{(n-1)(n-2)}{2} - \sum \frac{r(r-1)}{2}.$$

The effective genus p is the number of linearly independent adjoints of order $n-3$ having an $r-1$ fold point at each r fold point of the system. For an irreducible system $p=P$; G. Castelnuovo(5). For reducible systems $p=P+c-1$ where c is the number of component curves which make up a curve of the system, G. Biermann(3). The number of variable intersections of 2 curves of the system is the grade

$$D=n^2-\Sigma r^2. \tag{5}$$

The numbers K, D, and P satisfy the relation

$$K=D-P+1 \tag{6}$$

so that

$$k=D-p+s+1 \tag{7}$$

for an irreducible system, where s is $\leqq p$ because $D \geqq k-1$, V. Martinetti(20).

Since $D \geqq 2(k-1)$, s is $\leqq p-k+1$; G. Castelnuovo(5).

The numbers D, p, and from (1) k are invariant under birational transformation, see G. Jung(11). See also G. B. Guccia(9).

2. *Superabundance.*—C. Küpper(13) determined the maximum number of simple basis points such that through them pass a system of curves of dimension K and order N.

F. S. Macaulay(16) and (17) showed that the superabundance of the complete intersection of C_l, C_m for C_n is $(l+m-n-1)(l+m-n-2)$ for $n \geqq l \geqq m < l+m$ and is zero for $n \geqq l+m-2$. The special case $n=l+m-2$ is given by C. Küpper(13). Macaulay also found an expression for the superabundance of the residual intersection of C_l and C_m through $N(<lm)$ points for $C_{l+m-n-3}$, and applied the theorem to construct point groups with required superabundance for curves of given orders. V. Martinetti(18) determined the superabundant systems for $p=2$, $k>1$, and M. de Franchis(7) and (8) found those with $p=2$, $k=1$ and also for $p=3$. H. Krey(12) showed that through $\frac{n(n+3)}{2}-4$ arbitrary points and 2 points properly chosen pass ∞^3 curves of order n for $n>4$.

3. *Characteristic Series.*—C. Segre(21), by considering the G_D^{k-1} cut on a curve of the system by the other curves of the system, showed that if $D>2p-2$ or $k>p$, $k=D-p+1$ so that $s=0$ and if D is $>2p$, or $k>p+1$ the system is *simple,* that is, if a curve of the system passes through a point it does not therefore pass through one or more associated points.

4. *Maximum Dimension.*—G. B. Guccia(10) and V. Martinetti(19) had determined the linear systems of least order for $p=1$, namely cubics, and C_4 with $2P_2$, and C. Segre(21) had found for $p=2$, the C_4 with P_2, the C_5 with P_3 and P_2, and the C_6 with $8P_2$. Hence if $p=1$, the maximum $K=D=9$, and for $p=2$, the maximum $K=11$, and the maximum $D=12$.

G. Castelnuovo(4), by adding additional double points, each of which reduces p by 1, k by 3, and D by 4, found the maximum K to be $3p+5$ and the maximum D to be $4p+4$, limits actually attained by hyperelliptic curves.

5. A *fundamental curve* is one which does not meet a general curve of the system except at basis points. If we impose the condition that a point on a fundamental curve F lies on a curve of the system we have a system of dimensions $k-1$ consisting of F and a residual curve. E. Bertini(2) proved that if $s=0$, then the genus of F is 0, and G. Castelnuovo(5) showed that the genus of F is $\leq s$. He also proved that a superabundant system has at least 9 basis points. E. Ciani(6), continuing Castelnuovo's work, distinguished between a proper fundamental curve and an improper one which can be transformed into a point which is not a basis point of the system. G. Jung(11) showed that the difference between the number of fundamental points and curves is invariant under a birational transformation. C. A. Scott(20) considered cases in which a fundamental curve is a multiple factor of partial systems of a linear system.

6. *Adjoints.*—G. Castelnuovo(5) studied linear irreducible systems $|C|$ by the aid of the adjoints of order $n-3$ which contain as a fixed component any fundamental curve for which $s=0$. The residual system $|C'|$ is termed the pure system of adjoints and cuts on a curve of $|C|$ the series G^{p-1}_{2p-2}.

7. *Virtual Characteristics of a Composite System.*—G. Castelnuovo(5). If the system $|C|$ has a residual $|C''|$ with respect to $|C'|$, with $|C'|$ and $|C''|$ meeting $|C|$ in Δ' and Δ'' points and $|C'|$ meeting $|C''|$ in I points then

$$\begin{aligned}
K &= K'+K''+I\\
p &= p'+p''+I-1\\
D &= D'+D''+2I\\
K'' &= K-\Delta'+p'-1\\
\Delta'-D' &= \Delta''-D''=I\\
\Delta'+\Delta'' &= D\\
p-p' &= \Delta''-K''.
\end{aligned}$$

8. *Adjoints and Dimension.*—G. Castelnuovo(5). If $|C'|$ is the pure adjoint system of $|C|$ then $\Delta'=2p-2$ and $K''=K-2p+p'+1$. If now k is $>p+1$, $K=k$ and $D>2p$.

Hence Δ'' is >2 and $\Delta''-K''$ is ≤ 2.

Hence $p'<p-1$ so that s' which is $\leq p'-k'+1=0$ and therefore if k is $>p+1$, $|C'|$ is regular and p' is $<p-1$.

Since $p-p''=\Delta'-K'$ therefore $p''=1$ so that K'' is ≤ 9. Hence the maximum value of k is $2p-p'+8$ when $|C|$ can be transformed into all the curves of a certain order. With this exception we have $k \leq 2p-p'+7$. The minimum value of p' is $2-p$ so that the maximum value of K is $3p+5$.

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CHAPTER IV

PLANAR CREMONA TRANSFORMATIONS

The more important general properties of these transformations have been developed for the plane by Luigi Cremona in the two memoirs (86) (1863); (88) (1865). He remarks(86) that earlier writers had regarded the quadratic transformation as the most general birational transformation from plane to plane and had himself apparently accepted this view two years earlier (85) (1861-62). He notes, however, that the sequence or product of two or more quadratic transformations is usually of higher order. In the present account sections 1, 2, 3, 4 are devoted to those properties of the transformations which appear when the two planes in correspondence are not necessarily superposed; sections 5, 6, 7, 8, 9, 10 to properties for superposed planes; and sections 11, 12 to applications of the transformations in geometry and algebra. General accounts of this subject have been given by Clebsch-Lindemann(67), Doehlemann(97), Sturm(272), Malet (192), Coble(77), and Mlodziejowski(201). Historical résumés are given by Pascal[(242.1), pp. 356-365] and by Loria[(188.1), pp. 253-259]. A comprehensive exposition of the theory is given by Hudson[(144.1), Part I].

1. *Definitions and General Properties.*—We use the terms *pencil, net,* and *web* for linear systems of dimension one, two, and three respectively. Sylvester(276) calls a flat space *homaloidal* or a *homaloid* (Gr. ομαλοσ = level); Cremona(89) extends the term to cover a rational surface, and calls a web of surfaces, such that the variable net in the web has a single base point variable throughout the space, a *homaloidal* web. In (90) Cremona defines a *homaloidal net* of plane curves to be a net whose variable pencil has a single base point variable throughout the plane.

Cremona(88) establishes the general transformation of *order* n by setting up a projective correspondence between the net of lines on a plane E_x and a homaloidal net H_y of order n on a plane E_y by making four lines (no three on a point) of E_x correspond respectively to four curves (no three in a pencil) of H_y. If the pencil of lines on x in E_x corresponds to a pencil of H_y with variable base point at y then y is the image of x in the Cremona transformation T from E_x to E_y. The net is determined by fixed points of given multiplicity called *fundamental points* or *F-points* (also base-points, principal points). If these consist of β_k k-fold points ($k=1, \ldots, n-1$) then the linear conditions they impose on the net and the intersections they absorb lead to(86) the relations

$$(1) \qquad \sum_{k=1}^{n-1} \frac{k(k+1)}{2}\beta_k = \frac{(n-1)(n+4)}{2}, \quad \sum_{k=1}^{n-1} k^2\beta_k = n^2-1.$$

Since the genus of a curve of H_y must be zero the independence of the linear conditions is confirmed. Lines on E_y correspond under the inverse transformation T^{-1} to curves also of order n of the *inverse* (or conjugate) homaloidal net H_x on E_x which is determined by α_k k-fold ($k=1, \ldots, n-1$) F-points on E_x where the numbers α_k satisfy the relations (1) (88). We assume unless expressly stated otherwise that the F-points in E_x, E_y are all distinct, i. e., the homaloidal nets have ordinary multiple points all of whose tangents are variable in the net.

Cremona(88) shows that the number of F-points in E_x is the same as the number ρ of F-points in E_y. Clebsch(68) arranges the F-points in E_x, F_i, in order of decreasing multiplicity r_i, i. e., $r_i \geqq r_{i+1}$; and the F-points in E_y, F_j of multiplicity s_j in similar order, i. e., $s_j \geqq s_{j+1}$ ($i, j=1, \ldots, \rho$). Cremona(88) shows that the pencil of curves in E_y which is the transform of a pencil of lines on F_i in E_x has for common factor a curve P_i of order r_i—a curve which we call a *principal* (or fundamental) *curve* or *P-curve.* He proves that the various points of P_i on E_y correspond to the points on E_x consecutive to F_i in various directions or more briefly to directions at F_i. An algebraic proof by r-th polars of this theorem is given by Cayley(58); see also Rosanes(251). If the singularities of the net at the F-points are compound, or if the F-points are in special position, the P-curves may break up; cf. Autonne(23.1), Biasi(38.1) and Hudson[(144.1), Chap. II].

With respect to the relative behavior of the P-curves and F-points, Cremona(88) proves the following: Every curve which is a fixed part of a composite pencil of the homaloidal net is a P-curve and conversely; every composite curve of the homaloidal net has at least one P-curve as a part; a curve which cuts the curves of the net only at the F-points is a P-curve and conversely; a P-curve is of genus zero, has no multiple points except at the F-points, and is completely determined by its multiplicities at the F-points; the number, α_{ij}, of times a P-curve P_i passes through the F-point, F_j, on E_y is the same as the number of times the P-curve P_j passes through the F-point, F_i, on E_x; the aggregate of the P-curves of a homaloidal net is the Jacobian of the net [proved earlier by T. A. Hirst in "The Reader," Oct. 1 (1864)]. The class, the number of flexes, and the number of double tangents of P_i is transferred to the number of curves of the net with a given behavior at F_i. Cremona(87) also notes that the sum of the orders of the P-curves of a homaloidal net is equal to the sum of the orders of the F-points. For the particular case of a Jonquières transformation for which $\alpha_{n-1}=\beta_{n-1}=1$ and $\alpha_1=\beta_1=2(n-1)$ de Jonquières [(145) (1859); (146) (1860); (147) (1864); in full (148) (1885)] had noted the F-points and P-curves. The first algebraic treatment is due to Rosanes(251) (1871). For such treatment the following theorems of Bertini(32) concerning linear systems in any space are important. If every member of a linear system has an r-fold point outside the base of the system, the locus of these r-fold points is an $(r-1)$-fold factor of the system; if every member of the system is composite, then either the members have a common factor or the members are composed of j members of a fixed pencil [cf. Lüroth(190)]. Pasch(243) proves that between three ternary forms of order n and no common factor with vanishing jacobian there exists an irreducible equation of genus zero and degree $\nu(n=\nu\mu)$; and the three given forms can be expressed as binary forms of order ν whose arguments are ternary forms of order μ. Darboux(91) (1872) gives a review of the above and Dewulf(94) an excellent résumé of progress up to 1873.

The theorems stated above lead to the following relations on the positive integers n, r_i, S_j, a_{ij}:

$$
(2)\qquad
\begin{array}{ll}
\Sigma_i\ r_i^2=n^2-1 & \Sigma_j\ s_j^2=n^2-1\\
\Sigma_i\ r_i=3(n-1) & \Sigma_j\ s_j=3(n-1)\\
\Sigma_i\ a_{ij}^2=s_j^2+1 & \Sigma_j\ a_{ij}^2=r_i^2+1\\
\Sigma_i\ a_{ij}=3s_j-1 & \Sigma_j\ a_{ij}=3r_i-1\\
\Sigma_i\ a_{ij}r_i=ns_j & \Sigma_j\ a_{ij}s_j=nr_i\\
\Sigma_i\ a_{ij}a_{ik}=s_js_k & \Sigma_j\ a_{ij}a_{kj}=r_ir_k
\end{array}
\qquad (i, j=1, \ldots, \rho).
$$

Clebsch(68) (1871) notes that the absolute constants of T are the $2\rho-8$ absolute projective constants of the ρ F-points in either plane. He finds that

$$
\Delta=|a_{ij}|=\pm n, \qquad \pm r_i=\Sigma_j s_j\frac{\partial\Delta}{\partial a_{ij}}, \qquad \pm s_j=\Sigma_i r_i\,\frac{\partial\Delta}{\partial a_{ij}},
$$
$$
r_is_j=na_{ij}\mp\frac{\partial\Delta}{\partial a_{ij}}.
$$

He proves that the matrix $||a_{ij}||$ for a group of α_l equal r_i's and a group of β_m equal s_j's has elements a_{ij} which are all equal except that for the group of $\alpha_l>1$ points there is one and only one square matrix $(\beta_m=\alpha_l)$ in which the elements a_{ij} are all equal to a except for α_l elements which by proper arrangement of the β_m points can be brought into the principal diagonal and these α_l elements are all equal to $b\neq a$. Thus *the numbers* α_l, β_m *coincide except as to order.* By subtracting one column of Δ from the others he finds that $(a-b)^{a_i-1}$ is a factor of $\Delta=\pm n$. Bertini(30) shows, however, that $a-b=\pm 1$; and that $a_{ij}>a_{ij'}$ if $s_j>s_{j'}$.

The P-curves of a Cremona transformation are rational, and are completely determined by their multiplicities at the F-points which themselves may be in general position. Thus not every rational curve can be a P-curve. Bertini(27) calls a rational curve of order t for which the positions of all its multiple points of multiplicities

$$t_1, t_2, \dots \; (t_1 \geqq t_2 \geqq \dots)$$

may be arbitrarily assigned an L-curve. He proves that $t_1+t_2+t_3>t$ and that an L-curve can be transformed by quadratic transformation into an L-curve of lower order and eventually into the directions about a point. Since any Cremona transformation is a product of quadratic transformations (cf. §7) Bertini's L-curves are the P-curves of Cremona transformations. The L-curves for which $t_1+t_2+t_3=t+1$ belong to seven types: $t=5$, $a_2=6$; $t=6$, $a_3=1$, $a_2=7$; $t=8$, $a_3=7$; $t=11$, $a_4=7$, $a_3=1$; $t=17$, $a_6=8$; $t=20$, $a_7=8$, $a_3=1$; $t=t$, $a_{t-1}=1$. Brahana and Coble(42) note that all the P-curves, including the directions at a point, completely defined by a set of ρ F-points (which are infinite in number for $\rho \geqq 9$) give rise to a discontinuous division of the plane and study the simple finite map for $\rho=4$.

H. W. E. Jung(159) treats the general Cremona transformation with singularities of any kind whatever at the F-points. The theorems obtained differ in some respects from those which are valid for the transformations with ordinary discrete F-points. Both the exposition and the results are given in the notation of the Hensel-Landsberg theory of algebraic functions.

2. *Determination of the Types.*—The type of a Cremona transformation is determined when one of the two homaloidal nets is given. The net is determined when the position of the F-points is given as well as the order n and multiplicity r_i of the curves of the net. We consider here the determination of the positive integers n, r_i which satisfy the relations

$$(3) \qquad r_1+r_2+\dots+r_\rho=3(n-1), \qquad r_1^2+r_2^2+\dots+r_\rho^2=n^2-1.$$

This is not merely a diophantine problem. The solutions are subject to the inequalities which express the fact that a curve of order m cannot cut all the curves of the net in more than mn points. Thus de Jonquières(150) distinguishes between the algebraic solutions for which some of the integers are negative; the arithmetic solutions for which the integers are positive but the net does not exist; and the geometric solutions for which the integers are positive and the net exists [cf. also Palatini(237)].

The types may be listed either according to the order n or according to the number ρ of F-points, the latter classification being perhaps more fundamental. Cremona(86) gives the first list: $n=2$, $\alpha_1=3$; $n=3$, $\alpha_2=1$, $\alpha_1=4$; $n=4$, $\alpha_2=3$, $\alpha_1=3$; $n=4$, $\alpha_3=1$, $\alpha_1=6$; and the de Jonquières type, $n=n$, $\alpha_{n-1}=1$, $\alpha_1=2(n-1)$. He adds(88) the inverse nets and P-curves for all solutions from $n=2$ to $n=10$ inclusive; for $n\equiv 0$, or 1 (mod. 2) one pair of inverse nets; and for $n\equiv 0$, 1, 2, or 3 (mod. 4) four pairs of inverse nets. Roberts(250) determines the numbers α_{ij} for the types given by Cremona as reproduced by Cayley(58). Guccia(128) gives algebraic expressions for the homaloidal nets and their inverses of four of the general types given by Cremona. Bianchi(38) first derives new types of nets by transforming others. By transforming a Jonquières type of order m by one of order n he finds new self-inverse types for any order. Ruffini(252) made little effective progress by arithmetic methods. Guccia(135) derives the type of a product from that of the factors. Thus if T_k (order k) has inverse F-points of orders $s_1, \ldots, s_\rho$ and T_l of orders $s'_1, \ldots, s'_{\rho'}$ then $T_kT_l=T_{kl}$ has F-points of orders

$$s_1, \ldots, s_\rho, ks'_1, \ldots, ks'_\rho.$$

He notes that a type given by de Jonquières(149) is the product of two Jonquières types. G. Jung(158) determines the jacobian of the product of two T's and the relations on the integers r_i, α_{ij} of the product. Palatini(238) finds a variety of new types by combining in various ways the known types with some general values in their integers α_i, β_k. De Jonquières(150), (149.2) devises a rather complicated transition from solutions for one order n to another $n+k$, gives the geometric solutions for $n=12$, and for $k=1$ deduces those for $n=13$. It is a method of trial with high orders used as illustrations. Earlier(149.1) he discussed composition as a method for finding types. Guccia(129.1) gave the type which arises from a product of quadratic transformations which are not particularly related. Ruffini(252.1) gives types with simple and double F-points.

Clebsch(67.1) and Diekmann(96) recognize that two different mappings of a rational surface on a plane lead to a Cremona transformation of the plane—the extension of the construction of a quadratic transformation by two projections of a quadric surface. The latter applies this to two mappings of a cubic surface by cubic curves on six points. Sturm(275) maps the cubic surface on a plane first by transversals of two skew lines u, v and second by bisecants of a twisted cubic K on the surface. According as (σ, τ), the respective numbers of points at which u, v meet K, are (0, 0), (1, 0), (1, 1), (2, 0), (2, 1) and (2, 2), T's of orders 4, 6, 8, 8, 10, 12 are obtained which are included in the known tables. Ines Larice(184) gives an arithmetic method for deducing types of order n

when those of previous orders are known. The method is suggested by projection of the Veronese F_2^4 in S_5. As examples the types for $n=14$, 15 are determined. She also gives(185) two types already listed by Ruffini(252).

Montesano(209) considers the relative values of the α_i and makes a decided advance, e. g., the theorem

$$(n-1)\{\rho(n+1)-9(n-1)\}=\Sigma(r_i-r_k)^2$$

when $n>2$. He proves that there are no symmetric types other than $n=2$, $\alpha_1=3$; $n=5$, $\alpha_2=6$; $n=8$, $\alpha_3=7$; $n=17$, $\alpha_6=8$. In (211) he constructs new types by taking the product of a given type by a quadratic transformation. He finds that the four symmetric types and the Jonquières type are the only exceptions to $r_1+r_2+r_3=n+k(k>1)$. He also finds *asymmetric* types for which $r_i \neq r_k$ contrary to the opinion expressed by Clebsch(68), that of least order being $n=23$, $r_i=12$, 10, 9, 8, 7, 6, 5, 4, 3, 2. Many particular properties of the integers r_i are given, e. g., $r_1+r_2+r_4>n$ (except for the case $n=2$); $n-1 \geqq r_1 \geqq \frac{n+1}{3}$, the limits being attained only in the Jonquières case and the symmetric cases, respectively. He shows that if k of the r_i's are equal, then $k<9$ and gives for $\rho=9$ an infinite sequence of types. He also adds new types for $n=14$, 15 to the lists of Larice(184). Montesano(213) determines also the *semi-symmetric* types, i. e., those in which the F-points fall into two groups of like order [cf. also (214)]. Miss Hudson(143) considers solutions for n a number of the form $n=x\mu+\nu(\nu=0, 1, \ldots, x-1)$. For $x=2$, 3 the general types which can be stated in terms of such a division of n have been given in part by Cremona and Ruffini and in larger measure by Palatini. For $x=4$ Miss Hudson adds new types to those already known and finally determines types for general x. Mlodziejowski(202) derives the types for larger n from previous types by quadratic transformation and compiles tables for the first 21 orders. Later (202.1) he discusses the arithmetic solutions of the equations of condition and allied linear systems. For a very complete discussion consult Hudson[(144.1), Chap. IV].

S. Kantor(168) first recognizes the importance of ρ, the number of F-points, in the classification of types. He states (p. 273) that the number of types for $\rho=9$ is infinite and gives (p. 280) a table of the finite number of types for $\rho<9$ which includes the non-existent type $n=11$, $\alpha_3=5$, $\alpha_5=3$. His theorem (p. 297) that the numbers n, r can be expressed as rational functions of $\rho(\rho+1)/2$ parameters is not practically useful since the formulae "n'ont bésoin que d'une restriction relativement au caractere entier des coefficients." Nevertheless the only effective method thus far found for obtaining solutions n, r_i with actually existent nets is that of combining known transformations. For this method the group of characteristics introduced by Kantor[(168) Part IV] and treated in the next section is indispensable. In this way Coble(74), (p. 375) has proved that the infinite number of types for $\rho=9$ divide into $960.3^7.2$ classes such that the types in each class depend upon the unrestricted variation of 8 integers.

3. *Linear Transformations with Integer or Rational Coefficients Associated with Cremona Transformations. The Geometry of the Cremona Group.*—If the transformation T of order n has F-points in E_x of multiplicities $r_i (i=1\ldots\rho)$ and if T^{-1} has F-points in E_y of multiplicities $s_j (j=1, \ldots, \rho)$; also the P-curve of the point F_i in E_x passes a_{ij} times through the point F_j on E_y, then a curve C_x in E_x of order x_0 with multiplicity x_i at F_i is transformed by T into a curve C_y in E_y of order y_0 with multiplicity y_j at F_j where

$$(1) \qquad G\colon \begin{array}{l} y_0=nx_0-r_1x_1-r_2x_2\ldots-r_\rho x_\rho \\ y_1=s_1x_0-a_{11}x_1-a_{21}x_2\ldots-a_{\rho 1}x_\rho \\ y_2=s_2x_0-a_{12}x_1-a_{22}x_2\ldots-a_{\rho 2}x_\rho \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_\rho=s_\rho x_0-a_{1\rho}x_1-a_{2\rho}x_2\ldots-a_{\rho\rho}x_\rho \end{array}$$

If the behavior of the transforms C_x, C_y at pairs of ordinary corresponding points should also be indicated, the equations (1) may be extended by adding

$$\begin{array}{l} y_{\rho+1}=-(-1)x_{\rho+1} \\ y_{\rho+2}=-(-1)x_{\rho+2}. \end{array}$$

Since C_x is the transform of C_y by T^{-1} the transformation G^{-1} associated with T^{-1} has a form similar to (1) and arises from (1) by interchanging the coefficients r, s and the a_{ij} with the a_{ji}. Hence the determinant of the coefficients, D, has the value

$$(2) \qquad D=\begin{vmatrix} nr_i \\ s_ja_{ij} \end{vmatrix}=\pm 1.$$

If C_y is transformed by a new Cremona transformation T_1 into C_z and if G_1 is the linear transformation (1) associated with T_1 then C_x is transformed into C_z by the product TT_1 whose associated transformation is the product GG_1. Naturally the products TT_1 and GG_1 are not completely defined unless the distribution of the F-points of T_1 among the F-points (or ordinary points) of T^{-1} is given. Since T itself is a product of quadratic transformations, the linear transformation G may be generated by the element

$$\begin{array}{llllll} & y_0 = 2x_0 & -x_1 & -x_2 & -x_3 \\ (3) & y_1 = x_0 & & -x_2 & -x_3 \\ & y_2 = x_0 & -x_1 & & -x_3 \\ & y_3 = x_0 & -x_1 & -x_2 & \end{array}$$

and permutations of $x_1, \ldots, x_\rho$.

The linear transformation G appears first in G. Jung[(156)1887-8] but the first comprehensive investigation of it is due to S. Kantor(168) (pp. 293-316). He finds that G has the invariant linear and quadratic forms,

$$(4) \qquad \begin{array}{l} 3x_0 - x_1 - x_2 \ldots - x_\rho \\ x_0^2 - x_1^2 - x_2^2 \ldots - x_\rho^2 \end{array}$$

and observes that all the numerical relations, §1 (2), are a consequence of the invariance of these forms [cf. also Montesano,(210); Coble(74).] He calls G (or its system of coefficients) the "characteristic" of T and $x_0, x_1, \ldots, x_\rho$ the "singularity-complex" of C_x. Two curves C_x, C'_x with complexes x, x' meet in

$$x_0x'_0 - x_1x'_1 - \ldots x_\rho x'_\rho$$

ordinary points so that the invariance of the polarized form of the quadratic (4) is evident. A cubic curve on the F-points of a quadratic T is transformed into a cubic curve on the F-points of the quadratic T^{-1} whence the particular complex

$$(5) \qquad 3, 1, 1, \ldots, 1$$

is invariant under T itself. Hence its linear polar form (4) as to the quadratic form (4) is invariant. Hence if the complex x is transformed by G into the complex y then the complex

$$x_0 - 3,\ x_1 - 1,\ x_2 - 1,\ \ldots,\ x_\rho - 1$$

is transformed into the complex

$$y_0 - 3,\ y_1 - 1,\ y_2 - 1,\ \ldots,\ y_\rho - 1,$$

i. e., if C_x is transformed into C_y the canonical adjoints of C_x are transformed into the canonical adjoints of C_y. Kantor notes the existence of the generator (3) and gives an extended account of the characteristic equation of the linear transformation G with particular reference to the periodicity of G in connection with the periodicity of T. He erroneously ascribes to D the value $(-1)^\rho$. Since the complex (5) is invariant the characteristic equation of G always has one root $+1$. If it has ν roots $+1$ and ν linearly independent complexes Kantor says that G is of "rank ν."

Thus (3) is of rank 3 with invariant complexes 1, 1, 0, 0; 1, 0, 1, 0; 1, 0, 0, 1 upon which (5) is now dependent.

Montesano(210) shows that if a group of equal r_i's is coordinated to a like group of equal s_j's and if F_{i_1}, F_{i_2} are two F-points of the first group coordinated to the points F_{j_1}, F_{j_2} respectively of the second group by the scheme of Clebsch (§1) then $x_{i_1}-x_{i_2}=\pm(y_{i_1}-y_{i_2})$. Montesano(212) also discusses the behavior of the matrices n, r_i, s_j, a_{ij} in the operation of forming various products of Cremona transformations. The inequality

$$r_i+s_j \overline{<} m+a_{ij}$$

is proved by a hyperdimensional argument by Coble(74).

The linear transformation, G, used by Kantor almost exclusively in connection with periodic transformations and finite groups for which the F-points must satisfy restrictive conditions has been applied by Coble(74) to the study of general sets of discrete points of the plane. He defines two ordered sets of m points, P_m^2 in E_x with points $p_1, \ldots, p_m$ and Q_m^2 in E_y with points $q_1, \ldots, q_m$ to be *congruent* under the Cremona transformation T if ρ of the pairs p_i, q_i are the F-points of T, T^{-1} and the remaining $m-\rho$ of the pairs are corresponding pairs of ordinary points. Necessarily then $m \geqq \rho$. If $m=\rho$ the conditions that P_m^2 and Q_m^2 be congruent under T are the $2\rho-8$ projective conditions on the F-points of T; if $m=\rho+1$ these conditions are the $2\rho-6$ projective conditions which amount to a projective construction for T. In order to put into evidence the projective invariants of an ordered set P_m^2 the first four points are placed at the reference and unit points and the factors of proportionality in the remaining $m-4$ points are so adjusted that the last coordinate of each is the same. Then the set has $2(m-4)+1$ variable homogeneous coordinates (whose ratios are absolute projective constants of P_m^2) which are taken as the coordinates of a point P in a space $\Sigma_{2(m-4)}$. To the $m!$ ways of reordering the set P_m^2 there correspond $m!$ points P in $\Sigma_{2(m-4)}$ which are conjugate under a Cremona group $G_{m!}$ in $\Sigma_{2(m-4)}$ which is discussed in (73). To the passage from a set P_m^2 to a congruent set Q_m^2 there corresponds in $\Sigma_{2(m-4)}$ a Cremona transformation from P to Q. Thus all sets Q_m^2 congruent in any order to a given set P_m^2 are represented by points Q in Σ which form a conjugate set under a Cremona group $G_{m,2}$ in Σ for which the identity corresponds in the plane to the passage from P_m^2 to any projectively equivalent set. This discontinuous group $G_{m,2}$ (infinite for $m \geqq 9$) is isomorphic with the group of transformations G of type (1) on variables $x_0, x_1, \ldots, x_m$; a group $g_{m,2}$, generated by (3) and the permutations of $x_1, \ldots, x_m$. The isomorphism is simple unless congruence implies projectivity which is true for only three cases (for general position of P_m^2) : $m=5$, under quadratic transformation; $m=7$, under the Geiser transformation, $n=8$, $a_3=7$; $m=8$, under the Bertini transformation, $n=17$, $a_6=8$. The structure of the arithmetic group $g_{m,2}$ has been used for the study of $G_{m,2}$, and the properties of the related ternary Cremona transformations. The group $g_{7,2}$ isomorphic with the group of the double tangents of a plane quartic curve is discussed by Bramble(43).

There are 72 sets of points P_6^2 congruent in some order to one of the 72 sets; these are the sets such that cubic curves on P_6^2 map the plane on the *same* cubic surface; there are 288 sets P_7^2 congruent to a given set; these are the Aronhold sets of seven double points of a quartic envelope of class four. Kohn(182) had defined a "Wurf" of m points to be a set P_m^2 or any projective transform. He had considered the projective relations among the 72 Würfe P_6^2 attached to a cubic surface and the 288 Aronhold Würfe P_7^2 attached to a quartic envelope but did not recognize the existence of the Cremona transformations under which these Würfe are congruent. Conner(82) derives some of these transformations by projection from the space figure determined by the eight base points of a net of quadrics. Kasner(179) obtains the $G_{6!}$ in Σ_4 determined by the permutations of P_6^2 from the irrational invariants of a single-six of lines on a cubic surface. A group preliminary to $G_{m,2}$ is the cross-ratio group, due to E. H. Moore(216), a Cremona group which arises from the permutations of a set P_m^1 of m points on a line. Tinto(282.1) finds the transformations which lead to congruent sets P_6^2 as transformations of chords of two twisted cubic curves on a cubic surface. Further applications of the notion of congruence are found in §§ 9, 12.

Coble(74) (p. 370) has obtained a group $e_{m,2}$ isomorphic with $g_{m,2}$ and $G_{m,2}$ as follows. If P_m^2 is on an elliptic cubic C^3 with elliptic parameter u (such that $u_1+u_2+u_3\equiv 0$ is the collinear condition) a set Q_m^2 congruent to it under the quadratic transformation indicated by (3) is projective to a set Q_m^2 on C^3 itself. If $u_1, \ldots, u_m$ are the parameters of P_m^2 on C^3 the parameters $u'_1, \ldots, u'_m$ of Q_m^2 on C^3 are

$$(6)\qquad \begin{aligned} u'_i &\equiv u_i - \tfrac{2}{3}(u_1+u_2+u_3) \quad (i=1,\,2,\,3),\\ u_j &\equiv u_j + \tfrac{1}{3}(u_1+u_2+u_3) \quad (j=4,\,5,\,\ldots,\,m). \end{aligned}$$

The third-periods implied in $\frac{1}{3}(u_1+u_2+u_3)$ may be removed by a collineation on C^3 or they may be avoided altogether by taking P_m^2 on a cuspidal cubic with collinear condition $t_1+t_2+t_3=0$ [cf. Bramble(43)]. The transformation (6) together with permutations of $u_1, \ldots, u_m$ generate $l_{m,2}$. The elements have rational coefficients with at worst the denominator 3. It is in effect the form which $g_{m,2}$ takes when x_0 is eliminated by the use of the linear invariant (4). This group is especially useful in discussing the first infinite case P_9^2.

Klein in his Erlangen Program[(^{181}) §8, 1 (1893)] remarked that a geometry, in the sense in which he used the term, of the Cremona group had not been developed. Fano[(^{111}) § 21] calls attention to the discontinuous character of the Cremona group and questions whether there are properties of curves invariant under the Cremona group which are not also invariant under the birational group. Marletta(^{193}) finds conditions under which two curves in birational correspondence are also in Cremona correspondence. The conditions are expressed in terms of the existence of the canonical adjoints of successive indices. He also finds conditions for the reduction by Cremona transformation of the order of a given curve and gives types of curves of least order. Coble(^{77}) compares the meagre development of Cremonian geometry with that of the related birational and projective geometry. He suggests as a geometric element the set of points P_m^2 rather than the point. As dual element there may be used either the linear systems of curves defined at P_m^2 or the aggregate of P-curves defined by P_m^2(^{42}). As the basis for an invariant theory he uses the conditions on the set P_m^2 which express that two of the P-curves coincide. These are called the *discriminant conditions* attached to P_m^2. If P_m^2 and Q_m^2 are congruent under T then T transforms the discriminant conditions of P_m^2 into those of Q_m^2. Ota(^{231}) considers the projective geometry of a homaloidal system (net, web, etc.) as a more general type than the geometry of the original space. As a rule, however, the theorems are merely translations from one linear system to another and the difficulties with respect to F-points are but lightly touched.

4. *Construction of the F-Points; Construction and Description of Particular Transformations.*—Clebsch(^{68}) had noted that if the ρ F-points of T in E_x are given and four of the ρ F-points of T^{-1} in E_y are given the remaining $\rho-4$ F-points in E_y are determined. Thus for $n>2$ and therefore $\rho\geqq 5$ the 2ρ F-points of T must satisfy $2\rho-8$ projective conditions and T is then determined. For $n=2$, T is determined by the two sets of F-points p_1, p_2, p_3; q_1, q_2, q_3 and a given corresponding pair, p_4, q_4. The construction for T then arises from the fact that if p_5, q_5 is any other corresponding pair the five points p_1, p_2, p_3, p_4, p_5 in E_x are projective in order to the five points q_1, q_2, q_3, q_5, q_4 in E_y. In general then two problems arise: first, to obtain some statement for the conditions on the F-points; and second, when these are satisfied some statement of the conditions on a corresponding pair. All of these conditions may of course be implied in a definite geometric construction.

Many special types are considered under involutorial and periodic transformations, and under geometric applications.

De Jonquières(147) describes the transformation ($n=n$, $a_{n-1}=1$, $a_1=(2n-1)$) named after him and finds the inverse F-points and the P-curves [cf. also (145)]. Cremona(86) gives the following construction for this type. In space a line is drawn from a point x on a plane E_x across two fixed curves to meet a plane E_y in y. If one of the two curves is of order $n-1$ and the other a line $(n-2)$-secant to the curve, e. g., a plane curve of order $n-1$ with an $(n-2)$-fold point and a line on the $(n-2)$-fold point, or a $(1, n-2)$ curve on a quadric and a generator of the first kind, then the x, y are images in a special Jonquières transformation. Doehlemann(99) gives the latter construction for $n=3$. Coble(76) connects the two sets of $2(n-1)$ simple F-points with a set of $2(n-1)$ points in higher space. Schuttenhelm(256.1) discusses the Jonquières transformation particularly for $n=2, 3$.

The cubic transformation has a double F-point p_1 and four simple F-points $p_2, \ldots, p_5$ on E_x and similar F-points $q_1, q_2, \ldots, q_4$ on E_y such that the simple F-point $p_i(q_i)$ $(i=2, \ldots, 5)$ has for P-curve the line $\overline{q_1q_i}(\overline{p_1p_i})$ and the double F-point $p_1(q_1)$ has for P-curve the conic on all the points $q(p)$. The five points p are projective in order to the five points q. Bertini(31) shows that any cubic transformation between two planes E_x, E_y can be generated by transversals of a line and conic in space with one common point; or by a variable conic required to pass through a fixed point in each of the given planes and to meet a fixed conic in space in two points. Saltel(253) defines the "Arguesienne" (after Desargues) transformation as the pairs x, x' on a line through a fixed point P and on a conic of a fixed pencil. This cubic involution has a double F-point at P and simple F-points at the base points of the pencil. Various particular cases are discussed. This involution followed by a collineation is the general cubic transformation. Cardoso-Laynes(50) and Lacour(183) discuss metrically defined cubic transformations. Retali(248) considers the cubic transformation $z=re^{i\phi}$, $z'=r\cos^2\phi e^{i\phi}$ in the complex plane of z; and Monaco-Aprile(206) gives an algebraic account of it. Maschke(198.1) studies the determination of the transformation by assigned F-points.

The quartic transformation ($a_2=3$, $a_1=3$) is the product of two quadratic transformations in general position. Cotterill(83) gives the following construction. Conics on three points F_i and a point x cut a line l in pairs of an involution; conversely, conics on F_i which cut l in pairs of an involution meet in a point x. Let then F_i be a triangle in a plane π, F'_i a triangle in a plane π' which meets π in l; if conics on

F_i and x and conics on F'_i and x' meet l in pairs of the same involution, then x, x' are images in the quartic transformation. The F-points and P-curves are found and connected with Cremona's theory. Veronese(284) constructs the transformation by his V_2^4 in S_5 (the map of the plane by the linear system of conics) from two planes each of which cuts V_2^4 in three points. Doehlemann(98) gives the following construction: from x on a fixed plane α there is a unique chord across a fixed space cubic curve which meets a fixed plane β in y. If the curve breaks up into a conic and a secant line the transformation is cubic. Schmidt(253.2) discusses the determination of the transformation when fixed points are assigned.

The quintic transformation ($a_2=6$) has F-points $p_1, \ldots, p_6$ in E_x and $q_1, \ldots, q_6$ in E_y such that the P-curve of $p_i(q_i)$ $(i=1, \ldots, 6)$ is the conic on all the points $q(p)$ except $q_i(p_i)$. Müller(219) gives the following construction. If S is the collineation which sends $p_1, \ldots, p_4$ into $q_1, \ldots, q_4$; and σ the collineation which sends $p_2, \ldots, p_5$ into $q_2, \ldots, q_5$, the pencils of conics on these respective sets are projectively related by S and σ. If x is on a conic of each pencil in E_x, y is on a corresponding conic of each pencil in E_y. Kohn(182) obtains the points p, q as sections of halves of a double six of lines on a cubic surface; and also as the projections of six of the eight base point of a net of quadrics from each of the other two base points. Coble(72) gives other properties and (73) considers the ways in which the "associated" P_6^2, Q_6^2 may coincide. He notes also(76) that if p_7, p_8, p_9 form with P_6^2 the base points of a pencil of cubics a quadratic transformation with F-points at p_7, p_8, p_9 will send P_6^2 into Q_6^2. Schoute(255) defines lines AO and AO' as symmetric with respect to a triangle ABC if $\angle BAO=\angle O'AC$; given O the three lines on it from A, B, C have symmetric lines which meet at O where O, O' are pairs of a quadratic involution. If the lines on O are replaced by circles on O and a fixed point then O, O' are pairs of an involution of the fifth order. He determines the F-points and P-curves and notes that it is the transform of the quadratic involution by an inversion.

Le Paige(234) gives constructions for the F-points of the de Jonquières involutions $n=5, 6$; and later(235) for the transformations. These constructions are extended by Schoute(256) to transformations of higher order.

Veronese(284) states the theorem that every Cremona transformation of order n can be determined by two projections of the surface $F_2^{n^2}$ obtained by mapping the plane by all curves of order n. This follows immediately from the fact that if to the net of lines on E_x we add a fixed curve of order $n-1$, the two homaloidal nets on E_x, E_y each of order n, are in projective correspondence. Coble(74) gives constructions for all congruent sets P_m^2, Q_m^2 ($m \leqq 8$). This implies the construction of the F-points for $\rho=8$ and both the F-points and the transformation for $\rho<8$. In the two further symmetric cases ($n=8$, $a_3=7$; $n=17$, $a_6=8$), the two sets of F-points are projective.

Godeaux(123) discusses the transformation from x to x' where x, x' are the further intersections of a cubic of the pencil on $p_1, \ldots, p_8$ with a conic of the pencil on $p_1, \ldots, p_4$. He also(124) defines a transformation in terms of a net of order n with an $(n-1)$-fold point at O which contains a pencil with an $(n-3)$-fold point at O. If the line $\overline{xx'}$ is on O and the curve of the net on xx' touches $\overline{xx'}$ at x' then x, x' are images under a Cremona transformation of order $n-\nu-2$ where ν is the number of inflexional tangents at O of a general curve of the original net. Again Godeaux(125) takes a pencil of cuspidal cubics with necessarily fixed cusp and cusp tangent and discusses the transformation of order 7 between a point x and its tangential point x' on the same cubic.

5. *Isologous Curves; Fixed Points and Curves; Cyclic Sets.* When the planes E_x and E_y coincide and the point y becomes a point x' on E_x the Cremona transformation T from x to x', which we write in direct and inverse form as

$$(1) \qquad T: x'_i=f_i(x); \quad T^{-1}: x_i=\phi_i(x') \qquad (i=1, 2, 3),$$

acquires new properties. The F-points of T^{-1} may coincide in part or altogether with the F-points of T. For certain points x, x' will coincide with x. These are the fixed points of T and of T^{-1}. Two given transformations T_1, T_2 have now a definite product T_1T_2. We may also speak of the powers T^m of a given transformation T. A fixed point of T^m is either a fixed point of T, or a member of a set of m_1 (m_1 a factor of m) points permuted cyclically into each other by T and called a *cyclic set* of T of order m_1. If the general point of E_x is in such a cycle of precisely m points, then T^m is the identical transformation and T itself is periodic with period m. Finite groups of Cremona transformations now occur as well as infinite groups both discontinuous and continuous. These matters are considered in §§ 5, …, 9.

De Jonquières(147) defines the *isologous* curve of a point $P(p_1, p_2, p_3)$ to be the locus of points x which are on a line with P and the image x' of x and shows that the Jonquières transformation of order n has $n+2$ fixed points. Cremona(88) sets up the isologous net (for variable P) of the transformation T, a net of curves of order $n+1$ with an r-fold point at $F_r (r=1, \ldots, \rho)$ and on the $n+2$ fixed points of T, whose equation is $|pxf|=0$. Guccia(129) considers certain points and lines whose existence puts two conditions on the isologues and therefore are finite in number, e. g., a point R at which the isologue of P has a cusp. Guccia(130) develops a variety of points and lines projectively related to the isologues and states a number of theorems concerning them into which the fixed points and cyclic sets enter. He also(136) investigates the two isologous nets, $|pxf|=0$, $|qx'\phi'|=0$ which occur for T and T^{-1}. Doehlemann(100) finds that a curve of fixed points is a factor of each net. He distinguishes between fixed points at which every direction is fixed and fixed points at which but two directions are fixed.

A fixed curve of T is one which T transforms into itself. Thus Rosanes(251) finds that the quadratic involution $x'_i=1/x_i$ has a net and a web of invariant cubics. T may also have a curve of fixed points. Doehlemann(100) shows that, if T of order n has a curve of fixed points of order μ, then $n \gtrless (3\mu-1)/2$. He also finds the class of the envelope of lines joining x, x' as x' approaches x as a limit. Castelnuovo(53) proves that if T has an irreducible curve of fixed points of genus $p>1$ then either T is periodic of period 2, 3, or 4, or T is the transform of a Jonquières type. Conner(82) discusses the general quintic transformation $(a_2=6)$ which has a cubic curve of fixed points. Most of the involutions discussed in the next section have curves of fixed points of considerable interest.

Rosanes(251) notes that a quadratic transformation has two cyclic triads and remarks that the number of cyclic sets might be stated in general but that this number would have many exceptions for particular transformations. S. Kantor(163) was the first to give the formula

$$\pi_m=\frac{1}{m}\left[n^m-\Sigma n^{\frac{m}{p_1}}+\Sigma n^{\frac{m}{p_ip_j}}\ldots(-1)^\nu n^{\frac{m}{p_1\cdots p_\nu}}\right]$$

for the number π_m of cyclic sets of order $m=p_1^{m_1}p_2^{m_2}\ldots p_\nu^{m_\nu}$ (p_i a prime) of a transformation T of order n. For $n=2$ he had given this formula earlier(162). This result for $n=2$ had been read by Hirst at the Birmingham meeting of the British Association in 1865 but it remained unpublished until(140) 1881, a year after the appearance of Kantor's results. Doehlemann(101) considers the case for which T has ∞^1 cyclic pairs which lie on a curve of fixed points of T^2. This curve is of order at most $2n-1$ and in general this limit it attained only for $n=2$, a case treated in more detail. Miss Hudson(144) following out a particular case noted by Doehlemann(100) proves that if under non-involutorial transformation two curves of fixed points coincide their general points are of the second kind (i. e., every direction at a point is a fixed direction); and that if all the general points of a curve are of the second kind the curve can be regarded as a pair of adjacent curves of fixed points. Such a transformation may also have an isolated double invariant point.

The content of this section is given in detail by Hudson(144.1, Chap. V).

6. *Involutorial Cremona Transformations.*—If the transformation T sends a general point x into x' and at the same time sends x' into x, i. e., if $T^2=1$, then T is *involutorial,* or of *period two,* or an *involution.* Then $T=T^{-1}$ and the two inverse nets coincide. Thus only those transformations T whose inverse nets are of the same type can, under proper conditions, become involutorial. One of these conditions is that the two sets of F-points coincide but this is not usually sufficient.

The most important contribution to our knowledge of involutions T is due to Bertini who, recognizing that the number of projective types T is unlimited, sought to reduce them to a finite number of types by *transformation,* i. e., by the formation of products $T_1^{-1}TT_1$ where T_1 is any Cremona transformation. He shows that by proper choice of T_1 this transform reduces to one of the four types:

(a) the harmonic homology;
(b) the Jonquières involution;
(c) the Geiser involution;
(d) the Bertini involution.

The Jonquières involution is composed of pairs x, x' on a line with O and harmonic to the two further intersections of this line with an irreducible curve of order $p+2$ and genus p with a p-fold point at O. The Geiser involution is composed of pairs x, x' which form with seven given points $p_1, \ldots, p_7$ the base points of a pencil of cubics. The Bertini involution is composed of pairs x, x' which form with eight given double points the base points of a net of sextics.

Bertini(28) first proved that every involution of the Jonquières type $(n=n, a_{n-1}=1, a_1=2(n-1)$ could be transformed into either (a) or (b) and gave the general properties of such involutions. Later(29) he proved by the use of an invariant linear system (L) of curves derived from a P-curve which passes a_{ii} (an $a_{ii}>0$ always exists) times through its own F-point that an involution can be transformed into one of the four types (a), ..., (d). Here he first recognized the property of a web of sextics with eight given double points that the net of the web which passes through a point x also passes through a point x'. Two cases were not covered: first the case where the P-curves break up; and second the case $a_{ii}>3$ and every curve of (L) not invariant. For a review of these two papers see Darboux(92). Finally Bertini(30) shows that an involution with isolated F-points either has an $a_{ii}\overline{<}3$ or is transformable into one of lower order and this can be transformed into one of his four types. He shows also that if $a_1=0$, $a_2=0$ the P-curves *of greatest order* r_1 go through all the F-points; that if $a_1\neq 0$ then $a_1=3$ and the 3 P-curves of order r_1 go through all the F-points except one simple F-point; and that if $\alpha_1 = 0$, $\alpha_2 \neq 0$, then $\alpha_2 = 6$ and the 6 P-curves of order r_1 go through all the F-points except one double F-point. Bertini's study of the birationally distinct types of involutions forms the model for the study by S. Kantor of the birationally distinct types of periodic transformations and of finite groups (cf. § 8). They recur there and are discussed in the general accounts of these topics.

Geiser(120) first noted the involution (c) composed of the two further intersections x, x' of pencils of the net of cubics on seven points of the plane, P_7^2, with points $p_1, \ldots, p_7$. He observed that if x is at p_1, x' is on the cubic of the net with node at p_1; if x is on the line $\overline{p_1p_2}$, x' is on the conic through $p_3, \ldots, p_7$; if x is on a line, x' is on an octavic with triple points at P_7^2; and further that x, x' coincide on a sextic with nodes at P_7^2. Clebsch(69) showed that this sextic is the Jacobian of the net of cubics and mapped it by means of the net of cubics upon a general plane quartic curve; cf. also Milinowski(199) and Williams(287). Goring(127) finds that the involution is of class one; that its isologous curves are the cubics of the net; and that as x describes a line α, the line $\overline{xx'}$ envelopes a rational cubic envelope whose double tangent α has contacts at the pair x, x' on α. Emch(103) discusses problems of closure connected with the involution. Wallstaff(286.1) considers a degenerate case of order six. Bennett(25) discusses invariant linear systems under the four types of involutions.

The Bertini involution determined by 8 points P_8^2 is of order 17. The F-point p_1 has for P-curve the sextic with triple point at p_1 and nodes at the other points. Wiman(289) maps it into an involution on a cubic surface. It has a fixed point at p_9 the ninth base point of the pencil of cubics on P_8^2, each of which is an invariant cubic. It has a curve of fixed points, the locus of the ninth node of sextics with nodes at given P_8^2 [cf. Halphen(139)]. For general accounts see Doehlemann(97) (165-74), Sturm(272) (105-7), Coble(75) (251-4). Snyder(267) gives an algebraic exposition. Conner(82) shows that the Geiser and Bertini involutions in the plane are the projections of pairs of points of a space involution, which lie on an invariant quadric. The space involutions are the Geiser involution determined by six points and the Kantor involution determined by seven points.

The birational classification of involutions due to Bertini is supplemented by a projective classification due to Caporali. For an involution of order n Caporali(49) introduces the idea of the *class* ν, the number of pairs x, x' on a line. The locus of fixed points has the order $n - 2\nu$ and the genus p. An isologous curve has order $2\nu + 1$, genus π and triple points at the isolated fixed points. If K is the class of the envelope of lines on which a pair x, x' coincide, then $K+p+\pi=n$. Caporali notes that at a point on the locus of fixed points there are two fixed directions, one along the curve and one on a line of the envelope of class K whereas at isolated fixed points every direction is fixed. Bertini(33) proves the following theorems: Involutions of class zero are of Jonquières type. Involutions of Jonquières type and class not zero have two, one, or no lines of fixed points and are of classes $(n-2)/2$, $(n-1)/2$, or $n/2$ respectively. The construction of these types is given. Involutions of class one are the Geiser involution and its particular cases of lower order (9 types in all). Involutions of class two are either of Jonquières type; or of type $n=11$, $\alpha_5=4$, $\alpha_2=5$ with a pencil of fixed cubics on these 9 F-points or a specialization of this type (11 types in all). All involutions of class one except the general Geiser involution are transforms of involutions of class zero by one (or in a single case two) quadratic transformation. Involutions of class two are transforms of involutions of class zero by one quadratic transformation. Martinetti(194) lists the types of involutions of classes 3 and 4; and further(195) gives the construction for and thereby verifies the existence of the types listed. Berzolari(35) gives numerous theorems relating to the F-points and isologous curves for an involution of class ν and tabulates the involutions of class 5. Okken(229) adds one type to and eliminates two types from the list of Berzolari. Okken(228) uses the net of isologous curves to discuss the involutions of class 6. Berzolari(36) proves that if an involution has an r-fold F-point whose P-curve passes through it $r-3$ times and if $r\geqq 8$ then the involution belongs to one of 15 types whose class $\nu\leqq 7$. Martinetti(198) obtains lower limits for the genus of the general curve of the isologous net for involutions of various types. He uses a linear system with the same behavior at the F-points as the isologous net but otherwise unrestricted. Morgan(217) sets up equations of the four types of involutions given by Bertini and shows how the involutions of class $\nu\leqq 5$ can be transformed into these types. Conversely, he shows how involutions of any class can be obtained by transformation. Aprile(4) discusses some involutions of class 3.

If the points of a plane E_x are in (2, 1) rational correspondence with the points of a plane E_y so that to a point x of E_x there corresponds one point y of E_y, whereas to one point y of E_y there corresponds two points x, x' of E_x, then x, x' is a pair of a Cremona involution. Following Clebsch(69), de Paolis(240), (241), (242) studied these correspondences (for a résumé see (93)) which are reported in Chap. V § 1. The relation of the (2, 1) correspondence to involutions is recognized by Noether(226) and again by Lüroth(189) who maps the pairs of the various types of involutions on the points of a rational surface and thereby on the points of another plane. The latter object is accomplished by Bertini(34) by simpler geometric considerations. Castelnuovo(54) proves more generally that the sets of k points in any plane involution ($k=2$, the pairs of a Cremona involution) can be put into (1, 1) rational correspondence with the points of a plane. Boyd(41) discusses the (2, 1) correspondence associated with the Jonquières involution.

Palatini(236) uses the mapping of point pairs of a plane upon the points of an S_4 to derive a method for setting up involutorial Cremona transformations in the plane. Steinmetz(269) considers the properties and simple cases of the involutions determined by a net of curves with two variable intersections. Bobek(39) defines an aggregate of involutions by means of a pencil of cubic curves on the base points $P_9{}^2$. A rational curve C^ν of order ν with y_i-fold points at p_i is completely determined if

$$\sum_{i=1}^{9} y_i = 3\nu - 1, \qquad \sum_{i=1}^{9} y_i(y_i - 1) = (\nu - 1)(\nu - 2).$$

The C^ν meets a cubic curve in one point q_ν outside $P_9{}^2$. If then the cubic curves are projected each into itself from the point q_ν on it, an involution is determined of order $3\nu+5$, with an F-point of order $(\nu - 2y_i + 1)$ at p_i, and with a curve of fixed points of order $\nu+5$. These involutions are studied in detail. Stiner(270) defines an involution by means of a pencil of line conics and a point p. Pairs x, x' are such that the three lines $\overline{xx'}$, $\overline{xp}$, $\overline{x'p}$ touch the same conic of the pencil. This involution is used to construct a rational quintic curve with six given double points and two given simple points; and is applied also to other problems. Godeaux(123) considers an involution as cut out on the plane by a system L of lines and a system C of conics, an L and a C determining a pair x, x' of the involution. He divides involutions into four types according as the dimension of the systems L, C are 1, 1; 1, 2; 2, 1; 2, 2. F-points and P-curves for the first three types are given but the fourth is not completely discussed. Van den Berg(26.1) constructs a quintic involution with six double F-points by symmetric circles; cf. also Servais(260.1) and Vigarie(284.1). Crocchi(90.1), Schoute(254.1), and Chizzoni(63.1) study involutions constructed by means of pencils and nets. Emch(103.1) considers curves invariant under involutions.

In addition to the types of involutions described in § 4 others appear in the finite groups reported in § 8. A general account of involutions is given by Sturm(272) (pp. 95-130); and Hudson(144.1) (Chap. V, II).

For the determination of the birationally reduced types of periodic transformations and finite groups (§ 8) and of infinite continuous groups (§ 10) certain results in connection with linear systems are necessary and such systems are considered in the next section.

7. *The Reduction of Linear Systems of Curves.*—In the reduction of linear systems of curves to simpler typical forms by Cremona transformations an important theorem is that which states that a homaloidal net H_x of curves of order n, can be transformed by a sequence of quadratic transformations, $Q_1, Q_2, \ldots, Q_j$ into the net of straight lines. For if T is the Cremona transformation which converts lines into H_x, then the product $TQ_1, Q_2, \ldots, Q_j$ is a collineation C, and T itself is a product $(CQ_j^{-1}) \cdot Q_{j-1}^{-1}, \ldots, Q_1^{-1}$ of j quadratic transformations. If then any linear system can be transformed into a simpler form by T, it can in general be transformed into simpler form by quadratic transformations alone.

Cayley(**58**) states that Clifford surmised that every T can be expressed as a product of quadratic transformations and he verified this for $n=2, \ldots, 8$. Clifford's notes(**71**) indicate an examination of these cases. After an announcement(**223.1**) of the theorem, Noether(**224**) shows that the relations $\sum_{i=1}^{\rho} r_i^2 = n^2 - 1$, $\sum_{i=1}^{\rho} r_i = 3(n-1)$ attached to a homaloidal net with F-points of orders $r_1, \ldots, r_\rho$ lead to $r_1 + r_2 + r_3 > n$ and the reducibility of the net by quadratic transformation follows. Rosanes(**251**) gives a proof by induction from n to $n+1$ that $r_1 + r_2 + r_3 > n$ and draws the same conclusion; cf. also Bertini(**27**) and Mlodziejowski(**200**). Noether(**225**) extended the proof of the reducibility of the homaloidal net by a product of quadratic transformations to the case of coincident F-points. Nearly thirty years later Segre(**260**), after an analysis of Noether's work, pointed out that a quadratic transformation would not necessarily reduce the order of the net if, of the three F-points of highest order, the last two approach the first in different directions; nor if the last two approach the first in the same direction but not on the same linear branch; nor in certain cases on the same superlinear branch. Segre also made the same objection to the existing work on the reduction of linear systems. Castelnuovo(**57**) then proved that every Cremona transformation can be expressed as a product of Jonquières transformations, each itself a product of quadratic transformations. Alexander(**1**) raises objection to Castelnuovo's proof and gives a new proof of reducibility by means of quadratic transformations alone; cf. also Chisini(**63**). At certain stages of Alexander's process the order of the net may be raised so that Jonquières transformations may be necessary for repeated lowering of the order. Veronese(**284**) tried to prove the reducibility by using the projections of his V_2^4.

In the following account of the reducibility of linear systems to systems of least order by Cremona transformation the notation of Chapter III is used.

For rational curves ($p=0$), Noether(224) proves that a pencil is reducible to a pencil of lines. Guccia(131) shows that any system ($p=0$) of grade D is reducible to one of four types: (a) $n=(D+2)/2$, with an ordinary base-point of order $D/2$ and an isolated simple base-point; (b) $n=(D+\sigma+1)/2$ ($0 \leqq \sigma \leqq D-1$) with a base-point of order $(D+\sigma-1)/2$ with σ (distinct or coincident) fixed tangents; (c) a linear system of conics without base-points; (d) the net of lines. He also proves that a system ($p=0$) with ordinary base-points is reducible if $n>(D+2)/2$ (D even) or $n>(D+1)/2$ (D odd). Guccia(132) observes that Picard's proof of the theorem that the rational ruled surfaces and the Steiner quartic are the only surfaces with rational plane sections does not consider possible relations among the base-points; and he applies the results of the preceding paper to demonstrate the theorem.

For elliptic curves ($p=1$) Bertini(29) proves that a pencil ($k=1$) is reducible to a pencil of order $3r$ with 9 r-fold points; and a net ($k=2$, $D=2$) to a net of cubics on 7 points. For $k>1$ Guccia(133) shows that a system ($p=1$) can be reduced to one of the two types: (a) a system of cubics with 0, 1, …, 7 simple base-points distinct or coincident; (b) a system of quartics with two double base-points distinct or coincident; cf. also Segre(259) and Martinetti(197). Guccia(137) defends the validity of his methods against a criticism of G. Jung. Nalli(220) determines the types of least order for a pencil of elliptic curves which is transformed into itself by a Cremona involution.

For systems of genus 2 Bertini(29) proves that a web ($k=3$, $D=4$) is reducible to a web of sextics with 8 nodes, or a web of quartics with one node and 8 simple points. Segre(259) shows that every system ($p=2$, $k>2$) is reducible to a system of quartics with one node, or of quintics with one triple point and one node, or of sextics with 8 nodes. To this Guccia(134) notes the exception of a system of sextics with a common four-fold point with two pairs of coincident branches. This exception can be removed by Jonquières transformation. Martinetti(196) considers superabundant systems ($p=2$, $s>0$, $k>1$) and finds three irreducible types for all of which $k=2$, $s=1$ as follows: (a) quartics with a node O and 10 simple points, two on a line with O, the other 8 with O the base of a pencil of cubics; (b) quartics with a node and 8 simple points all on one cubic; (c) sextics with 8 nodes and 2 simple points all on a cubic. To this list de Franchis(117) adds the type: quintics with triple point p_1, double point p_2, and 11 simple points $p_3, \ldots, p_{13}$ such that p_2 is consecutive to p_1, and that $p_1, p_3, \ldots, p_{13}$ are consecutive; and that p_1, p_3, p_4 are on a line. De Franchis(115) shows that pencils of genus 2 are reducible to one of six types: (a) quartics with a node and 12 simple points; (b) the type described just above with an additional simple point p_{14} consecutive to p_{13}; (c) sextics with 8 nodes and 4 simple points; (d) septimics with one triple point and 10 nodes; (e) nonics with 8 triple, 2 double, and 1 simple point; (f) 13-ics with 1 five-fold and 9 four-fold points. The existence of these types is proved.

For systems of genus 3, both regular and superabundant, of any dimension de Franchis(116) lists the irreducible types.

Bertini(29) proves a general theorem on reducibility: a system $[k=a, p=(a-1), D=2(a-1)]$ is reducible if $a-1$ is less than the greatest order r_1 of a base-point; if $a-1=r_1$ the system is reducible except in the cases of (a) the linear system of curves of order n, genus $n-2$, with one common $(n-2)$-fold point and $2n$ simple points, and (b) the web of sextics with 8 nodes. The most comprehensive general contribution to reducibility is given by G. Jung(156). Complete and general systems for which $s=0$ are divided by Jung into *normal* and *derived* types. A derived type arises from a normal type by adding a simple base-point—thus normal types have no simple base-points. Normal types are divided into *monomial* types, (A) with no multiple base-point, (B) with one multiple base-point; *binomial* types (C) with just two multiple base-points; and *trinomial* types (D) with at least three multiple base-points for which $r_1+r_2+r_3 \leq n$. For every value of p of the form $(n-1)(n-2)/2$, there are systems of type (A); for given p and every solution in positive integers $r_1>1$ and i of $p=(i-1)r_1+\frac{1}{2}(i-1)(i-2)$ there are systems of type (B); and for each type (A), (B) there are derived types which in some cases are enumerated. The binomial types (C) are divided into two kinds: (1st) $r_1+r_2<n$; (2d) $r_1+r_2=n$; the trinomial types (D) into three kinds: (1st) $r_1=r_2=r_3$; (2d) $r_1>r_2=r_3$; (3d) $r_1>r_2>r_3$. For these similar diophantine equations are given which in the case (D) are subject to the condition of *geometric* compatability. A number of theorems resulting from this classification are given. Incomplete and special systems are then taken up and divided into: type 1° for which some arbitrary base-points determine the others *uniquely* (as 8 base-points of a pencil of cubics determine the ninth); and type 2° for which this is not so (as in the pencil $n=3r$ with 9 r-fold points). The memoir concludes with tables of general systems for $p=0, 1, 2, 3, 4$. In (157) Jung extends the considerations above to the case where the base-points are higher singularities and gives tables of types for $p=0, \ldots, 4$. He finds as upper limits for the degree and dimension for given p, $D \overline{<} 6p+2$, $k \overline{<} 5p+3$. Castelnuovo(51) reduces these limits to $D \overline{<} 4p+4$, $k \overline{<} 3p+5$. If these maxima are attained the system is reducible to either a system of quartics $(p=3)$ or to a hyperelliptic system of type: (a) curves of order $p+3$ with an ordinary $(p+1)$-fold point and a double point; or (b) curves of order $p+\mu+2$ $(\mu=0, 1, \ldots, p)$ with a $(p+\mu)$-fold point and μ double points consecutive to it. He modifies results of Segre(259) to read as follows: If a linear system determined by its base points has irreducible curves and if $k>p$ the system is complete; if $k>2$ and the plane involution of order D is composite (i. e., if curves on one point must pass through other points), then $k \leqq p+1$. In (52) Castelnuovo introduces a variety of new and effective ideas into the theory of linear systems such as the "virtual" or "effective" order of a base-point, and "virtual" and "effective dimension." Important theorems are obtained by the use of adjoint systems and the *characteristic* linear series cut out on one curve of the system by the others [cf. Chap. III].

Armenante(5) had early used a Cremona transformation to reduce to a minimum order the mapping curves of a rational surface. A résumé of the results on linear systems which are immediately applicable to the mapping of the plane upon surfaces in S_3 or S_n and their consequences in the theory of surfaces is given by Castelnuovo and Enriques(55) with a rather complete list of references to that time [cf. also Chap. XII].

The use of "successive canonical adjoints" in the study of linear systems is based on the theorem that if a system Σ is transformed by Cremona transformation into a system Σ', then the adjoints of order $n-3$ of a general curve f of Σ are transformed into those of the general curve f' of Σ'. Such adjoints were used by Brill-Noether(44), (227) but only with respect to the unique linear series g_{2p-2}^{p-1} which they cut out on f and f'. The extension of the covariance of the system of canonical adjoints to the entire plane under Cremona transformation is proved by Castelnuovo(52) though the fact had been stated and effectively used by S. Kantor(167), (168). Other instances of the value of this method are found in Castelnuovo(54), (53); in the determination of the types of finite and of continuous groups, cf. § 8 and § 10; and in the study of the rational double plane by Castelnuovo and Enriques(56); cf. also G. Jung(153).

After the revision by Segre and Castelnuovo of the proof of the reducibility of the homaloidal net to the net of straight lines by quadratic transformation, Ferretti(113) revised the discussion of linear systems and confirmed the results of Bertini, Guccia, Martinetti and de Franchis which were given above. He derived theorems which connect the order n of a reduced system, the dimension x_j of the j-th successive canonical adjoint, and the order of the highest base-point. Accounts of the work of Noether, Segre, Castelnuovo, and Ferretti are given along somewhat different lines by Franciosi(118) and Nencini(221).

A general account of the reduction of linear systems has been given by Fano(111).

S. Kantor remarks that the base-points of a homaloidal net or more general linear systems may not be given. The groups of points of like order may then be separated by rational processes but the determination of the points within such a group would involve algebraic irrationalities. He obtains(174) the reduced types of rational, elliptic, and hyperelliptic linear systems for which the reduction does not involve such irrationalities; and (175) the similar problem involved in the expression of a Cremona transformation as a product of simpler types. Kantor also(178) considers the expression of T as a product of simpler types when only integer coefficients occur in the homogeneous equations of T, T^{-1}. He finds that the factors may be of sixteen different types and draws some conclusions as to the irrationalities involved in the expression of T as a product of quadratic factors.

8. *Periodic Transformations and Finite Groups.*—A transformation T is periodic and of period r if T^r is the lowest power of T for which $T^r=1$, the identical collineation. A finite group is composed of periodic elements and the powers of a periodic element form a cyclic finite group so that the methods applicable to periodic transformations and to finite groups must be in the main the same. Groups may be classified according to Autonne with respect to the maximum order n of the transformations T which compose them. This would lead to an infinite number of types since this maximum order would rise indefinitely under continued random Cremona transformation of the group. Or they may be classified according to S. Kantor as equivalent under Cremona transformation to one of a finite number of types. They may be regarded as known if such an aggregate of types has been exhibited. A particular type may involve one or more arbitrary integers as in the case of the Jonquières type among the four types of involutions derived by Bertini.

S. Kantor(164) gives an account of periodic quadratic transformations. Let A, B, C be F-points of Q; A', B', C' be F-points of Q^{-1}; A'_i, B'_i, C'_i $(i=1, 2, \ldots)$ be the transforms of A', B', C' by $Q, Q^2, \ldots$; and A_i, B_i, C_i the transforms of A, B, C by $(Q^{-1})^i$. If Q^r is the identical collineation, then A'_i, B'_j, C'_k coincide eventually with A or B or C. For points on the line BC pass successively into directions at A', at A'_1, at A'_2, etc. If eventually A'_i is not at $A, B,$ or $C,$ this set of directions cannot become again the points of a line. This necessary condition Kantor later calls the "principle of *enchainement* (*Verkettung*) of the F-points." Obviously, all the points $A'_i, \ldots, A_i, \ldots$ are F-points of Q or of one of its powers. Five particular cases are considered: (1) $A=A'$, $B=B'$, $C=C'$, the usual involution with four fixed points; (2) $A=A'$,

$B=C'$, $C=B'$, an inversion with pole A and conic of fixed points, or such an inversion combined with a rotation (center A and circular points B, C) of period n; (3) $A'=C$, $B'=A$, $C'=B$, of period 6 with one fixed point f and a cycle of three points which with f is a 4-point with diagonal triangle ABC; (4) $A'=A$, $B'=B$, of period r, equal to the least multiple of r_1, r_2 if the pencils of lines on A, B are cyclically transformed into themselves with periods r_1, r_2 respectively; (5) $A'=B$, $B'=A$, $C'_r=C$, then $Q^{r+2}=1$ (r even) or $Q^{(2r+4)}=1$ (r odd). The case $A'_2=A$, $B'_2=B$, $C'_2=C$ is not periodic since no reduction in the order of successive transforms of a line occurs. The case $A'=A$, $B'_k=B$, $C'_r=C$ is pronounced periodic but the intermediate cases are not examined so that a given necessary and sufficient condition for periodicity is evidently inadequate. Snyder(²⁶⁶) discusses algebraically the case (4) and gives canonical forms more specific than those of Autonne(⁶) both for distinct and for coincident F-points.

Autonne[(⁹), abstracts (⁶), (⁸)] defines a quadratic group to be one all of whose elements are quadratic. He divides these into three classes: (1) a G_{120} with F-points at four points; (2) groups with two points a, a', F-points both of Q and of Q^{-1}, which may or may not be interchanged, which arise from finite groups of projectivities in the pencils on a, a'; (3) a class with coincident a, a'. The orders of the possible groups are not indicated. G. Jung(¹⁵⁸) contributes an addition to this paper. The G_{120} is included in the series of cross-ratio groups of E. H. Moore(²¹⁶) and is exhaustively treated by Slaught(²⁶²) who gives fundamental regions for it and discusses its invariants. P. Field(¹¹⁴) gives the modifications of Slaught's work which occur when two or four of the points are conjugate imaginaries. Autonne(⁷) defines the order of a Cremona group to be the maximum order of a transformation. A semi-cubic group is a cubic group for which the product of any two cubic elements is quadratic or linear. He gives canonical forms of semi-cubic groups in two categories of which the first contains groups of orders n, 10, 8, 24, 24; and the second groups of orders 16, 64, 96, 160. Autonne[(¹¹), abstract (¹⁰)] considers cubic groups with distinct F-points and fixed double F-point O. The group G contains an invariant subgroup Γ which effects the identical permutation on the lines through O. Seven types of groups Γ are listed which are enlarged to 15 additional groups G. In (¹⁴) the discussion is extended to the case of coincident simple F-points. Autonne(¹²) in a preliminary article considers substitutions on the homogeneous contragredient variables x_1, x_2, x_3; u_1, u_2, u_3 and requires that they be *contact* transformations, i. e., that $(xu)=0$, $(dxu)=0$, $(xdu)=0$ shall entail $(x'u')=0$, $(dx'u')=0$, $(x'du')=0$. If linear they are collineations or correlations. In an extended series of memoirs [(17), (20), (21); abstracts (13), (15), (16), (22)] he considers rational and reversible "cremonienne" transformations of the form

$$x'_i=\phi_i(x^a, u^b), \qquad u'_i=\psi(x^c, u^d)$$

of the orders a, b, c, d indicated in x, u which are subject to the conditions of contact above. The ∞^4 elements x, u of the plane are transformed among themselves in such a way that the ∞^3 *principal elements* for which $(xu)=0$ are invariant. Thus the transformations indicated are not Cremona transformations in general but rather *birational* transformations of the quadric variety $(xu)=0$. If this variety be mapped upon the points of an S_3 [cf. (21) pp. 451-2 or 409-11], the cremonienne gives rise to a particular type of Cremona transformation which satisfies a differential condition whose geometric equivalent is that the aggregate of curves whose tangents belong to a fixed linear complex be invariant. In [(23), (24); abstracts (18), (19)] this theory is applied to differential equations of the first order with rational coefficients in connection with the view of Clebsch of such an equation as the principal coincidence of a connex $(ax)^m(u\alpha)^n=0$.

S. Kantor(165), developing methods of his prize memoir (not then published), observes that if a rational surface, mapped on a plane, admits a (1, 1) periodic correspondence (usually a periodic collineation) then this correspondence is the map of a periodic Cremona transformation in the plane and that the various mappings of the surface on the plane lead to various transforms of this periodic plane transformation. In (166) he notes types of periodic quadratic transformations. In (167) Kantor proves that every curve which admits a (1, 1) birational correspondence B can be birationally transformed into a curve on which B is effected by a Cremona transformation which is either (a) a homography; or (b) a Jonquières transformation with $O'=O$ or (c) a transformation with 6, 7 or 8 points in its characteristic. He further(171) considers the possible number of correspondences B on a hyperelliptic curve of order n with an $(n-2)$-fold point and shows that if B is periodic it is effected by a periodic Jonquières transformation. Montesano(207) defines a finite group by means of four points $p_1, \ldots, p_4$ and a line α. The four inversions I_k with center p_k and fixed conic with self polar triangle p_l, p_m, p_n and p_k, α as pole and polar satisfy the relations $I_iI_j=I_jI_i=I_kI_l$ and generate a G_8.

S. Kantor in the memoir(168) crowned by the Naples Academy in 1883 seeks to determine the types of periodic transformations which are not reducible by Cremona transformation to simpler types. Caporali(49.1) gives a résumé of this memoir. A preliminary discussion of periodic collineations and their invariant curves and of the various correspondences existent on particular cubics and pencils of cubic curves is applied later to the construction of transformations. The various "enchainments" of the F-points are then discussed as in (164) and those whose characteristics (cf. § 3) are not periodic are discarded. The periodic characteristics are then tested for geometric compatibility, and of the resulting transformations those only are retained which are not reducible by transformation of the type $Q_1^{-1}QQ_1$ to periodic collineations or to quadratic types already listed. There result 10 periodic types with periods 6, 9, 12, 12, 14, 15, 18, 20, 24, 30. A similar process for $n=3$ and $n=4$ yields 5 cubic types of periods 6, 6, 6, 10, 12; and 3 biquadratic types of periods 3, 8, 6. An arithmetic study of the general characteristic then leads to the conclusion that all periodic ones are birationally equivalent to a Jonquières type or to one with a characteristic that contains at most 8 points. The final conclusion is that all periodic Cremona transformations can be transformed into either: (1) a periodic collineation; or (2) a Jonquières transformation of one of three kinds, each kind having two integer parameters; or (3) one of 28 particular types of order $n \leqq 17$ of which 18 are mentioned above. A lengthy excerpt of this memoir has been given by Kantor himself (170). The arithmetic discussion of the characteristics is replaced later (172) by a consideration of the invariant systems of curves, a method which also ensures the geometric existence of the characteristic. However, Castelnuovo (54) had proved earlier that every cyclic Cremona transformation can be reduced by transformation to one that transforms into itself either (a) the net of lines (a collineation); or (b) a pencil of lines (a Jonquières transformation); or (c) a system of $\infty^r (r>1)$ of cubics with $(9-r)$ base-points; or (d) a web of sextics with eight double base-points. This theorem applied to finite groups rather than to cyclic elements is fundamental in the determination by Kantor (161) of the types of finite groups. Kantor lists 53 types of which, apart from the collineation groups, six are ascribed to Autonne and 40 are new. A more elegant exposition along precisely the same lines is given by Wiman (289) who makes a number of additions and corrections to Kantor's list. It appears by the use of successive canonical adjoints that every finite group is equivalent to: either (1) a collineation group; or (2) a Jonquières group with fixed line pencil; or (3) a group of quadratic transformations with an invariant pair of line pencils; or (4), (5), (6), (7), (8) a group with an invariant linear system of cubic curves on 3, 4, 5, 6, 7 points respectively; or (9) a group with an invariant web of sextics with 8 nodes. Of the later cases the type (7) is handled by mapping the plane on a cubic surface by means of the invariant web of cubics. The possible collineation groups of the surface furnish the required groups in the plane. In type (8) the invariant net of cubics maps the plane on a double plane with a quartic curve of branch points, C^4. The collineation groups which C^4 may admit furnish the Cremona groups desired. Cox(84) shows how the transition from collineation group to Cremona group may be carried out practically. In type (9) the invariant web of sextics maps the plane upon a quadric cone in S_3 on which there is a space sextic curve, $C^6(p=4)$, of branch points. The collineation groups which C^6 may admit furnishes the required Cremona groups. Sharpe(261) discusses the character of these Cremona groups.

Wiman(288) discusses the groups of birational transformations, some of which are Cremona transformations, of curves of genera $p=4$, 5, 6; cf. for $p=4$ Miss Van Benschoten(26) and for $p=5$, McKelvey(191). Snyder(264) obtains the groups of collineations and quadratic transformations which leave a curve of genus 6 unaltered. Miss Tappan(277) determines the finite groups, at most quadratic, which leave sextic curves not transformable to lower order unaltered. Miss Allen(2) in listing involutions of degree 3 determined by nets of genus 0, 1, 2 finds certain cyclic sets of quadratic transformations of period 3. Coble(72) considers a group G_{120} of quintic transformations determined by five points of the plane whose invariant theory (in dual form) had been given by Clebsch(70). The group is a transform of Autonne's G_{120}. Ciani(65) discusses ternary quartic curves which admit finite groups of quadratic transformations. Ciani(64) studies the quadratic groups of orders, 8, 48, 96 which arise when two projective pencils of four lines each are unaltered according as the four lines are general, harmonic, or equianharmonic.

Cherubino(59) considers finite groups of the form

$$x'=\frac{a_i x+b_i}{c_i x+d_i}, \qquad y'=\frac{y}{(c_i x+d_i)^{p+1}} \qquad (i=0,\ 1,\ \ldots,\ k-1)$$

and their hyperelliptic invariant curves largely from the view-point of function-theory and group-theory. These transformations are of Jonquières type with an invariant line pencil. In (60) these groups are extended to include elements of the form $y'=y\alpha_i(x)/\delta_i(x)$ which are called *reduced* types with respect to the more general *semi-projective* group, $y'=\{\alpha_i(x)y+\beta_i(x)\}/\{\gamma_i(x)y+\delta_i(x)\}$, i. e., one in which the transformation on one variable y is projective. An additional transformation on z', z similar to that on y', y extends the transformation to S_3, etc. In S_3 the transformations are of the generalized Jonquières type studied by Enriques and Fano(106) and by Fano(110) (cf. § 10). Cherubino studies the curves and surfaces which admit the reduced and semi-projective group. He further(61), (62) extends these ideas to include the anti-birational elements of Comessatti[(79); cf. § 11] obtained by adjoining $x'=\bar{x}$, $y'=\bar{y}$. The extended groups are said to be of the second kind.

9. *Infinite Discontinuous Groups.*—S. Kantor(171) considers the infinite group of Jonquières transformations which have a hyperelliptic curve of order n with an $(n-2)$-fold point as a fixed curve with particular reference, however, to the construction of such transformations and with no mention of group properties. He also(176) calls attention to the existence of sub-groups of the Cremona group which are of the same general character as the group itself, e. g., the group which has an invariant elliptic curve and the Jonquières group which has a given line pencil for invariant. Bydzovsky(45) discusses the transformations which have a given cubic curve as an invariant and finds that for any system of F-points of T on C^3 there are 18 transformations which carry C^3 into itself. Indeed, if T' has F-points on C^3 it transforms C^3 into C'^3 which can be transformed back into C^3 by 18 collineations K so that $T'K=T$. This idea is used by Coble(74) to obtain the group isomorphic with the group of characteristics (cf. § 3). Coble[(74), p. 178] finds that a pencil of plane cubics can be transformed into only 960 projectively distinct types; a pencil of plane sextics with 9 nodes into only $2^8.960$ projectively distinct types; in general a pencil of order $3r$ with 9 r-fold points into only a finite number of types. Hence in all of these cases there is an infinite discontinuous group of Cremona transformations under which the pencil is invariant. Invariants of such groups are also discussed (pp. 384-5). Coble(75) finds that a general rational sextic can be transformed into only $2^{13}.31.51$ projectively distinct rational sextics. Thus there is an infinite discontinuous group which transforms the sextic into itself. The group is generated by the Bertini involutions determined by any 8 of the 10 nodes. The associated group of characteristics is congruent to the identity modulo 2. The group determines a group of linear transformations on the parameter of the sextic which is generated by the involutions whose double pair is the Jacobian of two pairs of nodal parameters. A method for exhibiting infinite series formally invariant under the group is set forth by Coble[(77), pp. 359-61]. Coble[(76), p. 16] remarks that a Veronese surface V_2^4 in S_5 is transformed into a Veronese surface by a regular transformation (cf. Chap. 7, § 3) which in the plane corresponds to the quintic transformation $(n=5, a_2=6)$. Thus there arises in the plane an infinite discontinuous group generated by one such quintic transformation and collineations.

10. *Continuous Groups.*—Though it was long known that the Cremona group contained continuous subgroups such as the group of inversions, Enriques(104) was the first to study the continuous c-parameter groups

G of Cremona transformations. He showed that G must have an invariant linear system of curves of order m and dimension $k \geqq 1$. By taking adjoints of order $m-3$ a new invariant linear system is obtained, etc.; until finally it appears that G has an invariant linear system of dimension $k \geqq 1$ of rational or elliptic curves which may be transformed into one of the normal types of Guccia (cf. § 7). Of these types only three admit continuous groups as follows: (a) the group (∞^8) of collineations or a subgroup of it; (b) the group (∞^6) of quadratic transformations which transforms into itself the system of conics on two points (or the two pencils of lines on the two points—the inversive group) or a subgroup of it; (c) the group (∞^{n+5}) of Jonquières transformations of order n which transform into itself the linear system $(k=n+1)$ of curves of order n with a fixed $(n-1)$-fold point and $(n-1)$ fixed tangents thereat or a subgroup of it. By using these invariant linear systems for each type as a mapping system Enriques proves that the geometry in the plane which has as a principal group [cf. Klein(181)] a continuous group of Cremona transformations depending upon a finite number of parameters coincides with the projective geometry on (a) the plane; (b) a ruled quadric; or (c) a rational normal cone of order n in S_{n+1}; or with a particular case of one of these geometries. Enriques then(105) studies the invariant subgroups of his type (c) since Lie had already proved that (a) is simple and (b) is known to contain only two invariant subgroups, each (∞^3) of which leaves the lines on one of the two points unaltered. From the normal cone in S_{n+1} he finds three invariant collineation groups in G_{n+5} all of whose elements have the vertex O of the cone as a fixed point. The first group G_{n+4} consists of the elements of G_{n+5} which have a second fixed point at O. The second group G_{n+2} consists of the elements of G_{n+5} perspective at O (homologies) each of which has a plane of fixed points. The third group G_{n+1} is the subgroup of permutable elations whose plane of fixed points is on O. These groups translate to the plane as follows. If μ is the invariant linear system of G_{n+5} with $(n-1)$-fold point at O the general element of G_{n+5} has a fixed curve in μ which does not contain fixed lines on O. The elements of G_{n+4} have fixed curves in μ which have at least two fixed lines on O as a part. The elements of G_{n+2} leave every line on O unaltered. The elements of G_{n+1} effect on each line through O a parabolic collineation. By taking the dual of the hypercone in S_{n+1} as a curve N of order n at infinity, Enriques finds that the geometry of the plane under G_{n+5} can be interpreted as a new extension to S_{n+1} of ordinary metric euclidean geometry in S_2, for which the curve N is the absolute.

Fano(108) notes that all the continuous Cremona groups of S_k will be determined if all the projective groups in $S_i\,(i \geqq k)$ which have a rational invariant V_k have been determined. Having obtained (107) these projective groups he finds in this way for $k=2$ the types given by Enriques. Mohrmann in his dissertation (203), and more fully in (204), studies by algebraic methods the collineation group of the normal cone of order n in S_{n+1} attached to Enriques group (c). He finds an essential difference in the cases n odd, n even; and gives finite equations both for the collineation group and for the Jonquières group. Later (205) he studies the geometry upon the normal surfaces of the groups of Enriques by suitable algebraic methods. Bohlmann (40) verifies that every continuous group of quadratic transformations is Enriques' group (b) or group (c) for $n=2$. Kasner (177) develops a comprehensive theory of algebraic invariants under the inversive group as a medium for the study of projective geometry on a quadric surface. This theory of double binary forms in digredient variables is pursued in (178). Newson (222), (223) gives an account of the continuous inversive group using group-theoretic and geometric methods. Fano[(112), pp. 339-43] has given a general report on continuous Cremona groups.

11. *Geometric Applications.*—We consider first certain applications in which the Cremona correspondence is not specifically from plane to plane. Cremona (88) establishes a correspondence between two bundles of lines by setting up a Cremona transformation from plane E_x to plane E_y in space. The lines ax, $a'y$ on two fixed centers a, a' correspond. He shows that the locus of the points of intersection of corresponding lines is a curve of order $n+2$, genus $n-1$, with certain Cayley numbers. Reye (249) discusses the quadratic transformation between bundles of lines, and planar fields of either points or lines, and develops related theorems concerning quadrics. He uses these in the determination of the transformation itself by pairs of F-points or of corresponding points. Darboux (90.2) uses a quadratic transformation for the construction of a quadric on nine points. Hirst (141) discusses the congruence of order $n+2$ and class n obtained by drawing lines through corresponding points of a Cremona transformation T of order n between two planes α, β in space. The two planes are singular planes of the congruence and the F-points of T, T^{-1} are singular points of the congruence. The focal surface is of order $4n$ and class $4(n-1)$. G. Jung (151) shows that if a bundle of lines on O and a bundle of planes on O' are put in Cremona correspondence of order n they generate a monoid of order $n+1$; or if $O'=O$ the lines in their corresponding planes form a cone of order $n+1$, the planes envelop a curve of class $n+1$. This idea is developed further by Jung (152) and by Visalli (235). Lo Piano (245) sets two bundles of lines in Cremona correspondence and through a fixed point P passes a line across two corresponding lines. The locus of the fourth harmonic

of P as to the two crossings is a rational surface of order $n+2$ which is studied in general and for $n=2, \ldots, 5$. Tinto(282) remarks that the chords of a space cubic curve form a ternary domain and considers the effect in space of a Cremona transformation between two such domains. Visalli(286) discusses the complex of lines which pass through a point x of a plane α and meet a line ξ of a plane β when α, β are connected by a Cremona correlation. Zahradnik(293) uses a quadratic correlation to construct the rational cubic and quartic. Lazzeri(186) discusses the Cremona correlation and the two curves of respectively order $n+1$ and class $n+1$ for which image point and line are incident. He proves that only for $n=2$ can incidence always occur; cf. Ameseder(3) and Sturm(274). The only correlation, other than the polarity in a conic ($n=1$), which is a polarity occurs for $n=2$, point and polar line as to a triangle; cf. Hain(138). Oppenheimer(230) uses Ameseder's null-system to construct triangles inscribed in a given cubic curve C^3 whose sides pass through a point of C^3. Beyel(37) uses two quadratic null-systems to generate the plane cubic of order 3 and class 3 and brings out certain properties of it. Tuck(283) uses the quadratic reciprocity to construct triangles inscribed and circumscribed to a cubic curve. Ota(232) uses the quadratic correlation for geometric purposes.

Certain geometric studies relate to the transformations themselves. Thus G. Jung(154) takes a Cremona transformation T of order n which sends x into x', x' into x'' where $\overline{xx'}$ is ξ, $\overline{x'x''}$ is η, and $\overline{xx''}$ is ζ, and studies the two $(1, n)$ correspondences between x', η and between x', ξ. In (155) he discusses the $(1, 2n)$ correspondence between x', ζ and makes applications to the transformations of orders up to six. This article connects closely with that of Guccia[(130), cf. § 5]. Pannelli(239) extends the idea of Jung to the l-th power of T and discusses the n^l+2 fixed points which break up into the cycles of Kantor. Kanda(160) makes some general remarks on Cremona transformations and their product in a Grassmann nomenclature. He defines two products T_1T_2 and $T_1 \times T_2$, of which the first is the usual product, and the second sends x into the line $\overline{x_1x_2}$ when T_i sends x into $x_i (i=1, 2)$. Kantor(169) recapitulates some of the geometric results of his prize memoir relating to birational transformations of a cubic into itself [cf. Geuss(121); and Martinetti(196.1)]. Each such transformation is in general contained in ∞^2 quadratic transformations. Montesano(208) considers the possibility of putting two planes in Cremona correspondence in ∞^1 ways so that a point of the one plane corresponds to the ∞^1 points of a rational curve in the other. He obtains a space construction for the required figure.

Cremona transformations are frequently used in geometric constructions both of particular sets of points and of loci. Müller(219) applies the transformation of the fifth order ($a_2=6$) to the problem of projectivity formulated by Sturm(273). Müller also gives a construction for the 9-th base-point of a pencil of cubics [cf. Oppenheimer(230)]. The construction of a quadric on 9 given points is given by Reye(249), Sturm(273), and London(188). London also gives a construction for the space quartic curve on 8 points and for the 8 base-points of a net of quadrics. Saltel(253) uses the Arguesienne transformation to obtain constructions for various curves of higher order which are transforms of constructible curves. Pecl(244) uses the cubic transformation,

$$x'=ax^3/(bx^2+cy^2), \qquad y'/x'=y/x$$

to construct rational curves. Zahradnik(291), (292), (290) uses the cubic transformation $x'=xy^2/(x^2+y^2)$, $y'/x'=x/y$ or $(x'+iy')(x+iy)=ixy$ to construct and discuss rational curves and certain special curves such as the lemniscate and cissoid [cf. Morgan(218), and Saltel(253.1)]. Thalberg(279) applies involutorial transformations to the study of plane curves, particularly the hyperelliptic curves of Jonquières type and special types of quartics.

Cremona transformations sometimes occur incidentally in connection with birational transformations. Miss Scott(257) shows that the maximum reduction in the order of the transform of a curve by birational transformation is secured by the use of adjoints of order ν where $n-3\leqq\nu<n$ and that the transforming net becomes a homaloidal net if these adjoints are rational. Snyder(263) shows that curves of sufficiently high genus for given order can be transformed into curves of the same order by collineations only. In (265) he considers the lowest order of possible transforms of a given curve and finds that this transform of lowest order can usually be obtained by Cremona transformation.

Many papers deal with the plane quartic curve of general or of special type. Müller(219) obtains a quartic as the locus of points from which two given sets of four points appear to be projective. Godt(126) indicated the connection of the quadratic transformation in a single plane with the theory of the plane quartic; cf. Clebsch-Lindemann[(67), vol. 3, pp. 442-52]. Timerding(281) elaborated this theme by pointing out that with a general Cremona transformation T in a single plane, there is determined a form $\theta(u, v)$ of degree $2(n-1)$ in each of u, v which for given u is the line equation of the transform of u by T and for given v the transform of v by T^{-1}. For $u=v$ there is a locus of lines of class $4(n-1)$, which touch their corresponding curves. For $n=2$ this is a quartic of class 4 with double-points of the three F-points and the four fixed points of T. From this Timerding derives the usual Aronhold theory of the double tangents. Pyrkosch(247) discusses a pencil of quartics with three fixed nodes as the transform of a pencil of conics. Ciani(65) discusses quartics which admit quadratic transformations. Bydzovsky(47) discusses the quadratic transformations which leave an elliptic plane quartic unaltered. There are nine, seven of which form with the identity a group. He inquires(46) as to the conditions under which this number may reduce because some coincide or reduce to collineations; and later(48) considers special cases with more quadratic transformations. Thalberg(278) derives a variety of geometric properties of quartics with at least one node by the use of Cremona transformations.

Of isolated geometric applications we note the following: W. Stahl(263) observes that, if p_7, p_8 are two of the base-points of a net of quadrics, a line λ_x on p_7 is on a unique quadric of the net and meets a cross-generator λ_y which is on p_8. The correspondence between λ_x, λ_y is of type $(n=4, a_3=1, a_1=6)$. E. H. Moore(215) discusses in a general way the manner of introducing or removing either the parameters of, or the conditions inherent in, a given geometric figure by the use of Cremona transformations and illustrates his remarks by considering various transforms of six points in multiple perspective relation. Dewulf(95) employs a special Jonquières transformation of order n to obtain results of metric interest. Miss Scott(258) uses Cremona transformations to prove certain theorems concerning the number of circuits (complete branches) of a plane curve of genus 0 or 1. Poincaré(246) uses Cremona transformations in the study of rational points on given curves—for the most part the elliptic cubic. Leib(187) uses a quadratic involution to determine and interpret more complicated invariants of two triangles from simpler ones. Conner(81) makes free use of various Cremona transformations in the study of multiple correspondences determined by the rational plane quintic curve; cf. also (80). Miss Hudson(142) considers the possible forms into which a rational plane sextic curve can be transformed when one of the discriminant conditions on its nodes vanishes; Coble(75) considers the general case. Emch(102) discusses the curves of order $3n$ invariant under a quadratic involution and gives a particular account of the sextics. Thalberg(280) uses the same involution to prove that the $4r$ corner tangents of a curve of order $3r$ with r-fold points at the four corners and three diagonal points of a quadrangle touch a curve of class $r=2, 3, 4$. Sturm[(272), p. 20-31; (271), p. 508] gives applications of ternary transformations in space geometry.

Comessatti(79) makes an important departure. He defines an *antibirational* transformation in a plane π to be the transformation obtained by combining the conjugate transformation $z'=\bar{z}$, $y'=\bar{y}$ with a Cremona transformation. In the mapping of a real rational surface F upon π, imaginary elements will in general occur so that two conjugate points of the real surface do not in general map upon conjugate points of π but rather upon pairs of an involutorial anti-birational T. If two real surfaces F, F' thus gives rise to involutions T, T', the surfaces F, F' are *similar* (the one a transform of the other under a *real* transformation) if, and only if, the T, T' are *similar* (the one a transform of the other by a plane Cremona transformation). The involutions T are reduced to five types: (a) the conjugate transformation; (b) the anti-quadratic transformation without fixed points; and (c), (d), (e) products of (a) and real Jonquières, Geiser, and Bertini involutions. Therefrom real rational surfaces divide into three types: (a) those with a pencil of real curves; (b) those not under (a) but with a real net of elliptic curves, $D=2$; (c) those not under (a), (b) but with a real web of curves of genus 2, $D=4$. These surfaces have (a) any number of real folds; (b) four real folds; (c) five real folds. The groups of real transformations of the surface into itself are given as groups of antibirational transformations in π.

12. *Algebraic and Other Applications.*—One would suppose that Cremona transformations would frequently occur in the great variety of algebraic applications in which rational representations are desired since by proper procedure the denominator of such expressions may be removed. Nevertheless such applications are rare. Coble(72) is the first to use finite Cremona groups in the solution of equations. After a preliminary study of a G_{120} defined by five points in the plane, the group is transformed into Autonne's quadratic group g_{120}. In this form, and as a particular case of E. H. Moore's(216) series of cross-ratio groups, Slaught(262) had determined a set of fundamental regions and had given the invariantive forms. Coble obtains these invariantive forms by using Clebsch's[(66), p. 353] typical representation of the binary quintic and states the form-problem of the Cremona group. The solution of the quintic in terms of the solution of this form-problem is given. To solve the form-problem it is noted that, under the even subgroup g_{60} of g_{120}, the simplest invariant linear system of curves (the system of cubics on four points) divides into two invariant nets which experience under g_{60} the linear transformations of Klein's[(180) Chap. 4, Part 2] group of the A's and group of the B's respectively. The solution of the form-problem of the Cremona group is given in terms of the solution of the form-problem of the linear group of the A's which Klein had indicated in terms of the ikosahedral irrationality. The Cremona group appears as the natural bond between the collineation group of the A's and the original group of the permutation of the roots of the quintic [cf. (73), p. 198]. Further applications of this kind are given in Chap. 8, § 12.

Schottky(254) shows that, when one of the even abelian theta functions ($p=4$) vanishes for the zero values of the arguments (which happens when the normal curve is on a quadric cone rather than a general quadric), there will exist certain three (rather than four) term relations among the products of odd theta's which serve to define a set of 8 points in the plane. The properties of the functions are then used to set up an especially simple form of the curve of fixed points of order nine and genus four of the Bertini involution defined by the eight points and to obtain the equations which map this curve upon the normal space sextic as Bertini and Noether had done. The coordinates of the eight points thus appear as modular functions of genus four. Frobenius(119) carries through a similar procedure with functions of genus three for the double tangents of the planar quartic determined by seven points of the plane. Neither recognizes that under integer linear transformation of the periods these planar sets of points pass into other sets which are *congruent* to the original set of points under proper Cremona transformation as defined in § 3. The discriminant conditions of the ternary sets of points are given by the vanishing of the even functions for the zero arguments.

Ouivet(238) applies Cremona and in particular quadratic transformations to the determination of the general integral of a differential equation of degree and order one.

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CHAPTER V

MULTIPLE CORRESPONDENCES BETWEEN TWO PLANES

1. *The* (1, 2) *Correspondences between* S'_2 *and* S_2.—The first generalization of the (1, 1) correspondence as established by the Cremona transformations between two planes is the (1, 2) case, which will now be considered. A general synthetic discussion is given by R. Sturm(99).

Let the simple plane be denoted by (x), the double plane by (x'). To a point P_1 in (x) corresponds P' in (x'), but to the latter correspond P_1 and P_2 in (x), either of which uniquely determines the other. Thus, associated with the (1, 2) correspondence is an involutorial birational transformation I_2 in the plane (x). The earlier history of plane involutions will be found in Chapter IV, but the memoirs most closely related to the double plane will be mentioned here: C. F. Geiser(41) considered the involution defined by a net of cubic curves with seven basis points. A pencil formed by an additional point P also has another basis point Q. This involution is of order 8. It has the seven basis points for fundamental points of order 3, the fundamental cubic belonging to P_i has a node at P_i and belongs to the net. The invariant points define a curve of order 6, genus 3, having double points at the basis points. This is known as the Geiser involution; it will be designated by (G). A. Cayley(31) outlined a (1, 2) transformation between two planes.

E. de Jonquières(49) constructed another involution by means of a fixed curve $C_n : O^{n-2}$ and a pencil of lines through O. A point P is transformed into P', the harmonic conjugate of P on OP as to the two residual points in which OP meets C_n. This involution is of order n, has O as a basis point of order $n-1$, and the points of tangency of the tangents to C_n from O as simple fundamental points. This is known as the perspective Jonquières type; it will be designated by (J).

The product of (J) by a certain harmonic homology (H) may be an involution if proper restrictions are imposed on (J). The product is called the non-perspective Jonquières type. It may also be transformed into a harmonic homology, as was shown by E. Bertini(6). This led Bertini to inquire whether all plane involutions could be reduced to a finite number of types when transformed through Cremona transformations(7). In addition to the three types (H), (G), (J), already known, one more was found. It is of order 17 and has eight fundamental points of order 6. The locus of invariant points is a curve of order 9, having the fundamental points for triple points. Through the eight points as double points a ∞^3 system of sextic curves can be passed, but a net within the web determined by one additional basis point has a second, which is the conjugate of P in the involution. This is called the Bertini involution; it will be denoted by (B). The equations of the involution of type (B) were obtained by V. Snyder(93) who also gives various particular cases. E. Caporali(23) proposed to classify involutions by their *class,* that is, the number of pairs of distinct conjugate points lying on an arbitrary straight line. The class is not invariant under birational transformation. Constructions of involutions of any given class were given by K. Bobek(12.1).

For the early literature on attempts to classify involutions according to class see Doehlemann(38). An extensive more recent study of those of class six was made by P. A. Okken(79). H. Liebmann(67) showed that the four types can be obtained by projecting a quadric upon a plane stereographically, and then from points not on the quadric by means of a linear congruence of lines.

The first systematic treatment of (1, 2) correspondences between the points of two planes was given by A. Clebsch(33). The locus of points $L'=0$ in (x'), the two images of which coincide, is called the curve of *branch-points* (or branch curve) and the image curve in (x), $K=0$ is called the curve of *coincidences* (also called double curve).

The equations of the transformation may be written in the form

$$x'_i=R_i(x)$$
$$x_i=M_i(x')+N_i(x')\sqrt{L'},$$

wherein R_iM_i, N_i are rational polynomials in (x'). A non-self-conjugate locus in (x) is transformed into a curve touching $L'=0$ at every common point except fundamental points. The case in which $L'=0$ is a conic is treated in detail.

When P' describes a locus in (x'), its images P_1, P_2 in (x) may describe the same locus, or different loci, conjugate to each other in the associated involution defined by P_1, P_2. The necessary and sufficient condition that P_1, P_2 describe separate loci is that the locus described by P' shall touch $L'=0$ at every common point except the fundamental points. The problem arose incidentally in connection with that of the bisection of the periods of an abelian function. If $L'=0$ is a conic, I_2 is H. If $L'=0$ is the general quartic, I_2 is G. See also L. Cremona and G. Battaglini(36).

R. de Paolis(82) studied the (1, 2) correspondences directly, using only algebraic methods. When P_1, P_2 describe the same curve image of a rational curve of (x'), the curve is hyperelliptic, and is invariant (not point by point) under I. The points P_1, P_2 describe the characteristic g_2^1 of the curve.

The limiting position of the line P_1P_2, as $P_1 \rightarrow P_2$ at a point on K is called the *principal direction* at P_1. It is not, in general, the tangent to K.

If a curve C in (x) has P_1^r and its conjugate P_2 to multiplicity s, $s \neq r$, P_2^s, it is not self-conjugate, and its image C' in (x) has P'^{r+s}, and the conjugate of C has $P_1^s P_2^r$. If a non-conjugate C meets K in r points, not tangent to the principal directions at those points, C' touches L' in the corresponding points. If a non-self-conjugate C has r-point contact with K, C' has $2r$-point contact with L' at the corresponding point. If a non-self-conjugate C intersects K in the principal direction, C' has a cusp at the corresponding point on L'. If a non-self-conjugate C has a multiple point of order r on K, C' has r branches all touching L' at the corresponding point.

To a net of rational curves in (x') corresponds a net of self-conjugate hyperelliptic curves in (x).

The basis points of the net $|\phi|$, images of the ∞^2 lines of (x'), are the fundamental points of (x). If P_i is of multiplicity i, then

$$\Sigma i^2 = n^2 - 2,$$

in which n is the order of a curve ϕ. The number of conditions imposed by the basis points is

$$h = \frac{n(n+3)}{2} + p - 3.$$

For $p > 1$, the basis points are not independent. Here p, the genus of a general curve ϕ, is called the genus of the transformation.

The lines of (x) are transformed into a non-linear system of rational curves of order n in (x').

The order of the associated involution of conjugate points P_1, P_2 in (x) is $N = n^2 - 1 - \Sigma i'^2$ in which i' is the order of the basis point P'_i, of the system $[\phi']$, images of the lines of (x).

The order of K is $\nu = 3(n-1) - \Sigma i'$.

Each curve of the system $[\phi']$ has $\dfrac{N-\nu}{2}$ double points.

2. *Fundamental Elements.*—A basis point P_i of multiplicity i of the system $|\phi|$ has for image in (x') a rational curve of order i.

This same point may also be fundamental in I, i. e., conjugate to a rational curve, [First kind]; or it may be conjugate to another fundamental point of the same multiplicity, having the same image curve in (x'), [Second kind]; or it may be self-conjugate [Third kind]. If a fundamental curve meets each ϕ' in i points, the image point in (x) is i-fold in I.

Fundamental curves of the second kind in (x') do not meet curves ϕ' except at fundamental points. In (x), fundamental curves of the first and second kinds are rational, those of the third kind are hyperelliptic.

The fundamental curves of the simple plane together with K constitute the jacobian of the net $|\phi|$. The g_2^1 on each ϕ has $2(p+1)$ coincidences, hence L' is of order $2(p+1)$.

If any C' touches L', its image C has a node on K at the image point.

To tangent lines of L' correspond nodal curves of the net $|\phi|$. The inflexional tangents to L' correspond to curves ϕ having cusps on K. A basis point P_i of order i on $|\phi|$ is of multiplicity $3i$ on the jacobian of $|\phi|$. R. de Paolis(83) considers the following particular case. Given a conic $A(x)=0$, and a point (y) in its plane. The polar of (y) as to $A(x)=0$ is $A(x, y)=0$. Put $x'_i=y_iA(x)-2x_iA(x, y)$. Between (x'), (x) this defines a $(1, 2)$ correspondence. The jacobian of the net $|\phi|$ consists of K, of $A(x)=0$ and of the line $A(x, y)=0$. The involution I is the general involutorial quadratic inversion.

This transformation is then interpreted in non-euclidean geometry.

In a third paper R. de Paolis(84) studies the general cubic $(1, 2)$ transformation of genus 1. The net $|\phi|$ now consists of the cubics through 7 arbitrary basis points. L' is the general quartic, and the seven lines, images of the cubics of the net having nodes at the basis points are bitangents of L', forming an Aronhold septuple. The twenty-eight bitangents and the sixty-three systems of contact conics and the system of contact cubics of L' are all obtained. The involution I is the Geiser involution of order 8; all the basis points are fundamental of the first kind. The curve of coincidences, invariant points of I, is a sextic of genus 3, having the basis points for double points.

3. *Derivation of the Types.*—E. Bertini(10) obtained all the irreducible forms by means of the types of plane involutions, and by projecting from space.

H is obtained by projecting from a cone stereographically; J by projecting ∞^3 systems of curves from a quadric; G by projecting F_3 on a double plane, and by mapping it on another plane birationally; B by ∞^3C_6 on a quadric cone mapped stereographically. See also E. Bertini(7).

M. Noether(76) showed that the $(1, 2)$ transformations not associated with harmonic homology in the simple plane are of three types, according as the curve of branch points is $C_{2n}: P^{2n-2}$ or a general C_4 or $C_6: 2P^3$. An outline of the same procedure was given by the same author(77). The method is intimately connected with that of mapping rational surfaces on a simple and on a double plane.

The (1, 2) point correspondence between the planes (x') and (x) associated with (H) may be expressed in the form $x'_2x_3-x'_1x_2=0$, $x'_1x_2x_3-x'_2x_1^2=0$. The invariant or coincident points constitute the line $x_1=0$. That associated with (G) has the form

$$x'_1C_1+x'_2C_2+x'_3C_3=0,$$
$$x'_1x_1+x'_2x_2+x'_3x_3=0,$$

in which each $C_i=0$ is a general conic. Each cubic of the net formed by linear combinations of $x_iC_k-x_kC_i=0$ is transformed into itself. The curve of invariant points is the jacobian of the net $K_6(r_1^2\ldots r_7^2)$, in which r_i is a simple basis point. The curve of branch points in (x') in (1, 1) correspondence with K_6, is the general quartic L'_4, which expresses the condition that the line in (x) touches the associated conic. Every line of (x) contains one pair of conjugate points. From this standpoint the system of the bitangents, the contact conics, cubics, etc., of L'_4 can all be determined directly by elementary methods. Their two images in (x) are conjugate as to (G). That associated with (B) may be expressed by the equations

$$x'_1\phi_1+x'_2\phi_2=0,$$
$$x'_1f+x'_3\phi_1(k_1\phi_1+k_2\phi_2)=0,$$

in which $\phi_i=0$ is a general cubic curve, and $f=0$ is a proper sextic having eight of the points h_i common to $\phi_1=0$, $\phi_2=0$ for double points. The curve of invariant points is the jacobian of the system $\phi_1=0$, $\phi_2=0$, $f=0$. It is of order 9, and has eight h_i as triple points, $K_9(h_1^3\ldots h_8^3)$. The curve of branch points is also a sextic L'_6 having three branches touching each other at a common point. An arbitrary line in (x) contains four pairs of conjugate points; i. e., the simplest form of (B) is of class 4. Finally, the (1, 2) correspondence of form (J) can be expressed by $x'_1x_2-x'_2x_1=0$, $x'_3M_1-u'M_2=0$, in which $M_i=0$ is a curve of order m with an $(m-2)$-fold point at (0, 0, 1); u' is linear in (x'). The jacobian is the curve of coincidences, a hyperelliptic curve having an $(n-2)$-fold point at (0, 0, 1). This involution is of class zero. This is the easiest way to find and to classify the plane involutions of order 2; to find all possible ways in which the curves of a net can have two variable points of intersection. The question of reducibility or of the equivalence of two given cases is not more difficult by this method than by that of Bertini. The seven basis points r_i have for images in (x') seven bitangents of L'_4; a point of a bitangent has two images, one of which is the basis point r_i, and the other describes the cubic of the net having a node at r_i. The point and the nodal cubic are conjugate in the involution (G).

An interesting case of (J) is that in which all the curves of the net have contact of the second order on each branch at the multiple point. Its equation is

$$\lambda_1(-u_{n-1}x_3+u_n)+\lambda_2 u_{n-1}x_1+\lambda_3 u_{n-1}x_2=0$$

in which u_i is binary in x_1, x_2 of order i. All the basis points are now coincident at $(0, 0, 1)$.

In (**41.1**) L. Godeaux gives an incomplete construction of the types of plane involutions by means of systems of conics and of lines. When both are pencils, type 1, a particular case of the perspective (J) results; the general (J) is called type 2. The author's type 3 is also reducible to (J). The fourth type, that generated by the field of lines and a net of conics, is not completed. As shown above, this is (G). An outline of the algebraic procedure and the derivation of the equations is given by J. Neuberg(**75.1**).

Another particular I_2 is interpreted dually by Stiner(**93.1**). Given four fixed lines, no three concurrent, and a point Q. A point P determines a conic touching the four given lines and PQ. It has a second tangent from P and from Q, and these intersect in P'. The I_2 is P, P'. It is employed to construct $C_5: 6P^2 2P^1$.

A (1, 2) transformation between two planes is outlined in a study of the correspondence between the points of S_4 and the pairs of points of S_2 by F. Palatini(**79.1**).

The curves of coincident points characterize the type of the involution. Hence it follows that any surface mappable on a double plane is, or is not, rational according as its curve of branch points can be reduced to one of the above curves.

Thus, the mapping of a quartic surface with a double line upon the plane, and the birational equivalence of various maps were developed in detail by J. Cardinaal(**24**) and (**25**).

A. Schwarz(**87**) obtained a (1, 2) correspondence between (x), (x') as follows: let V_1, V'_1 be vertices of two projective pencils of lines in (x), (x') respectively, and V_2, V'_2 be vertices of two pencils in (2, 1) correspondence. A point P_1 in (x) determines a line of (V_1) and of (V_2), hence a point P' in (x'). But P' has two images of (x), collinear with V_1. I is the perspective quadratic inversion. The genus of the (1, 2) transformation is 0. The net $|\phi|$ consists of cubics having a common node at V_2 and three simple collinear basis points. It is a particularization of the Geiser net.

Another construction of a (1, 2) correspondence of genus 0 was given by V. Retali(**85**) in which the simple and the double plane are superposed. Given a conic C and a fixed point R. With any point P' is associated the double points P_1P_2 of the quadratic involution on the line PR, having given the two pairs of conjugates P_iR_i, PR, C. The net $|\phi|$ consists of conics.

M. Noether(78) gave the configurations in (x') associated with the (B) and (J) involutions, analogous to what had already been done for the types (G) and (H). The curve $L'=0$ is a sextic having three branches tangent at the same point in B. The lines of (x) are transformed into sextics having this same three branch contact and having four variable double points. In (J), $L'=0$ is a sextic with a 4-fold point, for $p=4$.

P. P. Boyd(18) did the same thing for type (J), of the general case. F. M. Morgan(74) gave the necessary and sufficient criteria for reducing a $(1, 2)$ transformation of type (G) to the canonical form. The reductions due to particular positions of the basis elements were given by V. Snyder(96).

All the involutions found by E. Bertini(7) are rational; that is, the pairs of points P_1, P_2 can be mapped birationally on the points of a double plane, but certain restrictions were imposed on the basis points of the nets $|\phi|$. These restrictions were removed by G. Castelnuovo and F. Enriques(27). They showed that whenever a surface can be mapped on a double plane in such a way that the curve of branch points is of the form of $L'=0$ as already found, in the $(1, 2)$ transformations associated with the involutions (G), (B) and (J), the surface was necessarily rational, and conversely. The necessary and sufficient condition that a double plane is rational had been formulated in the same way by M. Noether(76).

V. Retali(86) considers a $(1, 2)$ plane transformation of order 3 and genus 0. Compare V. Snyder(95).

4. $(1, k)$ *Correspondences between Two Planes* $k>2$.—When the points of a plane can be arranged in sets of k, such that any point will determine uniquely the entire set to which it belongs, the sets are said to form an involution of order k. If the sets can be mapped birationally upon the points of a second plane (x'), the involution I_k is said to belong to the plane (x').

Numerous constructions for such involutions were given by J. S. Vanecek and M. N. Vanecek(102) for surfaces and for planes. Plane involutions may be generated by two projective pencils of curves. F. Chizzoni(32) considered a pencil $|\phi|$ of curves of order n, and a pencil $|\psi|$ of curves of order n'. A curve ϕ and its associate curve ψ have N variable intersections. Any point P determines a curve ϕ and a curve ψ, hence all N points of intersection. Let R_n be a basis point of order r_h of $|\phi|$, and of order r'_h of $|\psi|$. Then

$$N=nn'-\Sigma r_h r'_h, \quad n^2=\Sigma r_h^2, \quad n'^2=\Sigma r'^2_h.$$

A point R_h basis for both $|\phi|$ and $|\psi|$ is a fundamental point of the first kind; if for either one only, of the second kind.

Suppose ζ_i is a curve having no variable point on any curve of $|\phi|$ or $|\psi|$, unless the latter becomes composite. Let ζ_i have R_h to multiplicity η_{ih}, and be of order z_i. Then

$$nz_i = \Sigma r_h \eta_{ih}, \quad n'z_i = \Sigma r'_h \eta_{ih}.$$

Any point of ζ_i has the whole curve ζ_i for its image. Two curves ζ_i, ζ_k intersect in fundamental points of the third kind, apart from those of the first kind and of the second kind.

A large number of relations is derived, but no new results in reducibility. $N=1$, $N=2$ were known before, and the forms $N=2$ confirmed. Particular cases of a I_5 were given by C. le Paige(66).

Constructions by means of a net of curves of order n through $\frac{n(n+3)}{2}-2$ fixed points were given by J. de Vries(115). The involution is of order $\frac{n(n-3)}{2}+2$. The locus of intersection of corresponding curves of two pencils of the net in (p, q) correspondence is of order $n(p+q)$. A somewhat similar treatment is given by C. P. Steinmetz(97). The paper contains several illustrations, but no new results.

G. Castelnuovo(30) extended his studies of involutions on a given curve(29) to apply to a plane field. Given a plane pencil of lines with vertex O. Any line meets a fixed C_n in n points. Consider the polar group of each set by the procedure of the earlier memoir. The new groups all lie on the first polar of C_n as to O, and they depend on $\infty^1 C_n$, giving rise to I_n in the plane. All the points of each group are collinear.

Particular examples of plane involutions were given by J. de Vries(108), (109), and (112). In (110) an I_4 and a related I_3 are discussed.

A cubic involution I_3 is derived by W. van der Woude(101) and (100). The triads of I_3 are vertices of self-polar triangles as to a fixed conic γ. If P describes a line a, its associates describe a quartic curve having a node at the pole of a as to γ. I_3 has ten singular points; their polars are 10 singular lines. Together they form a Desargues $(10_3, 10_3)$ configuration.

J. de Vries(111) obtains a Cremona transformation between the points and lines of a plane by means of an involution I_3 of class 1. Each line is associated with the opposite vertex of the triad to which it belongs. Associate each point P of a line r with the point P^* in which p meets r. When P^* is given, there are n points P, n being the class of the rational envelope of p, when P describes a line. Further properties were found in (113.1) by the same author. If $n=1$, the Desargues $(10_3, 10_3)$ configuration results. Already discussed by W. van der Woude(101).

An I_3 is determined by the intersections with a fixed plane of the congruence of space cubic curves through 5 fixed points. This is called Reye's cubic involution(86.1). Further properties are given by J. de Vries(114.1). J. de Vries(113) constructs I_4 by means of two projective nets of conics. If they each have a basis point, and these coincide, I_3 results. When P describes a line, its associates describe a curve of order 15, with 8-fold points at the singular points. The curve of coincidence is of order 9. The image of the common basis point is a C_8 with a 5-fold point. The I_3 is of class 3. See also P. Visalli(103) and (104).

G. Jung(52) studied two planes (x), (x') in $(k, 1)$ correspondence, using notation and method similar to that of R. de Paolis(83) and (82).

If the image in (x) of a line of (x') is of genus p, then $p=k-1-c$, where c may be positive, zero or negative, according as the system $|\phi|$ is partial, complete, or the basis points of the net are dependent. If $c_1 \sim \phi'$, ϕ' touches L' in v points, and has $\frac{N-v}{2}$ variable double points. The associated involution I_N of (x) has the order $N=m^2-\Sigma r'-1$, and the locus of coincidences K is of order $\nu=3(m-1)-\Sigma r'$.

In (53) the same author considers involutions associated with a given Cremona transformation of order n of two superposed planes. Let $|\phi|$, $|\phi'|$ be the two Cremona nets, having V_i, V'_i for fundamental points of orders r_i, r'_i. Designate the $n+2$ invariant points by u.

Join a point $(x)=P$ to its image P' by a straight line and regard it as a line of the plane (x'). Its image is a curve of ϕ_n and the line PP' now regarded as in (x). To the points of the multiple plane correspond the net of isologues P_{n+1} of the simple plane. The correspondence is of order $n+1$ and of genus $n-1$. The simple plane contains no fundamental points. To an arbitrary line of (x) corresponds a curve of order $n(n+2)$. A similar transformation exists by interchanging (x) and (x'). The Geiser involution is included in this category when $n=2$.

The author points out that the table given by R. de Paolis is incomplete. Three more forms should be added.

Further details are given by G. Jung(60). The same method is followed further by P. Visalli(107).

C. F. E. Björling(18) raises the question of reducibility, and obtains numerous necessary inequalities, but no final results.

M. Pannelli(80) generalizes the method of G. Jung(53) by replacing the given Cremona transformation k times. The greatest difference between this and the simpler scheme lies in the number and complexity of the fundamental elements in the simple plane.

F. Borghese(13) extends to superposed planes in $(1, \nu)$ correspondence the question answered for Cremona correspondences by S. Kantor(57), namely, to determine the number of invariant points, and how many points return to their initial positions when the correspondence is repeated k times. If the curve of coincidences is of order k and has fundamental points to multiplicity λ_i, while the other plane has these same points to multiplicity θ_i, the number of isolated invariant points is

$$n+\nu+1+k(k-3)-\Sigma\lambda_i(\lambda_i-1)=\Sigma\theta_i$$

in which n is the order of ϕ, the image of a line of the multiple plane.

The number of cyclic groups of N elements is equal to the number of cyclic groups of N points in the two uniform correspondences, one of order n and the other of order ν.

That every involution on a straight line is rational was first proved by J. Lüroth(69) and later(68) the same author proved independently that every plane involution of order 2 and those of order 3 which arise from Cremona transformations of period 3 are rational. The general theorem that every plane involution of any order is rational was proved by G. Castelnuovo(26). That the corresponding theorem in S_3 is not true was confirmed by G. Aprile(1), by means of a concrete example. The general theory of plane involutions was developed by F. Deruyts(37).

P. Visalli(105) determined the properties of the congruence of lines joining corresponding points of two planes in $(1, \nu)$ correspondence. If n is order of ϕ, image of a line of the multiple plane, the congruence is of order $n+\nu+1$, class n. The two given planes are singular planes and all fundamental points of each are singular. Other singular planes may exist. The congruence and its focal surface are both rational. This is a direct generalization of T. A. Hirst(42), (43), (44) and of I. Conti(34) and (35).

C. A. Scott(89), (88), (90) studies $(1, k)$ correspondences from the equations $x'_i=\phi_i(x)$, wherein any two curves of the net $|\phi|$ have k variable intersections. Let the images of P' be $P_1, \ldots, P_k$. A curve $f(x)=0$ is transformed into $f(x')=0$, but when P' describes $f'=0$, P_1 describes $f=0$ and $P_2, \ldots, P_k$ describe its conjugate. K, L' are defined as in (1, 2) correspondences but when $P_1=P_2$ describes K, the points $P_3, \ldots, P_k$ describe the co-jacobian. The simple plane is divided into compartments, any one of which is the image of the (x') plane. Fundamental points are of three kinds, as given by R. de Paolis(82). C. A. Scott(91) considers various cases which cannot arise in the (1, 2) correspondence. Systems with dependent elements of determination are considered and the resulting effect on the fundamental elements determined. The theory is intimately connected with that of linear systems of plane curves. See further G. Castelnuovo(28); E. Bertini(8), (9), (11); G. Jung(51); E. Caporali(21); F. S. Macaulay(70), and (71).

G. Ferretti(40) showed how to construct involutions or order n and of class 0 and 1. All the preceding memoirs concerned with the class of the involution take $n=2$ except G. Castelnuovo(26) who considers n general. By means of projections from hyperspace, it is found that if I_n is of class 0, it is an I_3 or a perspective Jonquières involution. If $\nu=1$, $n=8$ and the curves of lowest order which can generate I_n are cubics.

A particular (1, 2) point correspondence was given by J. Neuberg(75) in which lines are transformed into conics.

The types of I_3 as expressed by two equations implicit in (x), (x') were found by A. M. Howe(48) to be five in number.

(1)	(x) lines	$\Sigma x_i x'_i=0$	(x') lines
	lines	$\Sigma x'_i \phi_{3,1}=0$	lines
(2)	lines of pencil	$x_1 x'_1+x_2 x'_2=0$	lines of pencil
	$C_n : P^{n-3}$	$\Sigma x'_i C_{n,i}=0$	lines pers. Jon.
(3)	conics 1 basis pt.	$\Sigma x'_i C_{2,i}=0$	lines
	conics	$\Sigma x'_i C_{2,i}=0$	lines
(4)	cubics: 6 pts.	$\Sigma x'_i C_{3,i}=0$	lines
		$\Sigma x'_i C_{3,i}=0$	lines
(5)	pencil $C_3 : 9P$	$x'_1 C_{3,1}+x'_2 C_{3,2}=0$	lines
	$C_9 : 8P^3$	$\Sigma x'_i \psi_{9,i}=0$	lines

followed by an enumeration of the cyclic types.

A similar enumeration of the non-cyclic I_4 was made by T. R. Hollcroft(47). He found ten irreducible types.

Surfaces having equations of the form $z^n=f(x, y)$ have rational (cyclic) involutions belonging to the plane.

A. Bottari(14) determined when the surface is rational. The results are:

(a) For arbitrary n: $f=0$ consists of $n+2$ lines, n through 0.
For $n=3$:
(b) An arbitrary cubic.
(c) C_6 with two tacnodal branches $=8P^2$.
(d) $n=5$.
An arbitrary cubic and an inflexional tangent.

In the second paper(15), with fewer restrictions on n, twenty-five additional types are found. For $n>30$, only types (a) and (b) are possible. When $f=0$ is irreducible, further results were obtained by M. Botasso(17).

5. *Cyclic Involutions I_n in S_2.*—Cyclic involutions I_n are generated by Cremona transformations of period n. When P' describes a straight line, each point P_k in (x) describes a curve which is invariant under the Cremona transformation, all the points P_k are on the same curve. The curve K, if such exists, is the curve of invariant points of the Cremona transformation. All the points $P_1 = P_2 \ldots P_n$ are points of K.

A. Emch(39) gives a construction of systems of cubic curves by means of involutorial quadratic transformations.

L. Autonne(2) obtains the necessary and sufficient conditions that a quadratic plane transformation shall be periodic, and that each power shall be quadratic, except the identity. It is expressed in terms of cyclic projectivities of the two pencils, the vertices of which are fundamental for both planes. See S. Kantor(55), (56), (58), (59), (60) and (61), A. Wiman(118), and V. Snyder(94). For applications of cyclic quadratic transformations to involutions on quartic curves, see Bydzovsky(20).

6. *Multiple Correspondences (m, n) between Two Planes.*—When both indices (m, n) are greater than one, the correspondence between two planes may not be rational, but a number of cases should be noted.

Given two planes π, π' in (m, n) point correspondence. Consider both immersed in the same S_4, in such a way that the point common to π, π' is not self-corresponding, and connect by straight lines every point of each plane with all of its corresponding points in the other. Then intersect the joining lines with an arbitrary S_3. The locus of the points of intersection is a surface in $(1, n)$ correspondence with π, and in $(1, m)$ correspondence with π'. Conversely, if a surface is in $(1, n')$ point correspondence with one plane and $(1, m')$ with another, it defines a

$$\left(\frac{m'}{\mu}, \frac{n'}{\mu}\right)(\mu \geqq 1)$$

correspondence between the two planes.

C. Stéphanos(98) considered a point-line transformation defined by the connex $f(x^m, u^n) = 0$. If the point x is fixed, each u touches its envelope in a point y; similarly, if u is fixed, each x of the locus has a tangent v. The totality (y, v) constitute another connex called the conjugate of the given one.

P. Visalli(106) considered two harmonic homologies in different planes in projection. The pairs of points P'_1, P'_2 have the same conjugates P_1, P_2 and P'_1, P'_2 as their only images. This is a very particular case of a (2, 2) correspondence. An arbitrary line goes into a conic, but a line through the center of homology is transformed into a line through the center of homology in the other plane. Such a correspondence is called a compound involution.

A particular (n, n') compound involution by means of pairs of projective pencils of curves was given by C. Wiener(116).

G. Burali-Forti(19) constructed a (2, 2) correspondence as the product of a (2, 1) correspondence between the planes σ, σ'' and a (1, 2) correspondence between σ'' and σ'. Let the former be called (σ), wherein σ is the simple plane, and the latter (σ'), wherein σ' is the simple plane.

Let P''_r be a fundamental point of σ'' of multiplicity i''_r in (σ) and P_r be a fundamental point of multiplicity i_s in (σ). Similarly, let P''_s be of multiplicity $i''_{r'}$, in (σ'), and $P'_{s'}$ be a fundamental point of σ' of multiplicity $i'_{s'}$ in (σ'). Let N be the order of the associated involution (σ) and N' of (σ').

Let K be of order v in (σ), and p be the genus of (σ). Let K' be of order v' in (σ') and p' be the genus of (σ'). From the two sets of formulas for the components, those for the (2, 2) correspondence between σ and σ' are obtained. This is also a compound involution, but more general than that of Visalli. $\Sigma i_r''^2 = n^2-(N+1)$; $\Sigma i_s^2 = n^2-2$; $\Sigma i_r'' = 3(n-1)$; $\Sigma i_s = 3n+2(p-2)$ and similarly for the (σ').

A more general (2, 2) correspondence was given by G. Marletta(72), namely, the general (non-involutorial) one in which the images of lines in either plane are conics in the other. A point P in π has P'_1, P'_2 for images in π'. The images of P'_1 are P and P_1; those of P'_2 are P and P_2, hence associated with P is a pair of points P_1, P_2 in the same plane. This symmetric (2, 2) is called the conjugate transformation of the given one. A construction by means of projection in S_4 is outlined. The correspondence may be established by two conics with projective tangents, or by projecting a quadric from two arbitrary points O, O', and representing the points stereographically. When O, O' are conjugate, the Visalli case results.

A similar study is made by Marletta(73) of (2, 2) correspondences in which straight lines in either plane have cubics for images in the other. If $p=p'=0$, the order of the symmetric (2, 2) in each plane is 5 and is also rational. Curves of coincident points are rational cubics. If $p=p'=1$, the correspondence can be realized by mapping a general cubic surface doubly on a plane from any two points on it. It may also be generated by a pencil of lines and a projective pencil of cubic curves, all in the same plane.

A study of general (m_1, m_2) correspondence was made by R. Baldus(3). Three kinds of fundamental curves exist: (1) A basis point in k_1 leaves of the m_1-fold plane π_1 has for images k_1 rational curves, every point of which has one image at given basis point, while the other m_1-1 images describe curves. (2) A fundamental curve in π_2 may belong to $h_2>1$

superposed planes, having h_2 distinct points in π_1 as its image. (3) The image points may be coincident in π_1, without defining a principal direction. In this case the fundamental curve in π_2 may not be rational. Such singularities may arise by projecting surfaces having a double curve on a double plane, for example quartics from two double points. Every involutorial (m_1, m_2) correspondence between two planes can be expressed as the product of three correspondences $(m_1, 1)$, $(1, 1)$, $(1, m_2)$.

C. Wiesing(117) generated an involutorial (2, 2) correspondence by means of two projective nets of conics in different planes, each with two fixed basis points.

Further constructions for the compound involution were given by V. R. Barraco(4). Given a pencil μ of curves of order m, and a system v of curves of index 2, that is, such that two curves of the system pass through a given point. Suppose any curve of μ meets any curve of v in two points. Given A, the residual intersections of the two v curves with the μ curves fixed by it are A_1, A_2. Now start with A_1, determining A and A'_1.

The pairs (AA_1), (AA_2), (A_1, A'_1) are *conjugate* points, (A_1, A_2), (A, A'_1) are associated points, both forming involutorial (2, 2) correspondences. The formulas obtained can be derived readily by the method of F. Chizzoni(32).

Using a notation similar to his, and denoting the fundamental curves of the third kind by Z_{ij}, it is found that the conjugate of a straight line is of order $4mn-2\Sigma x_i^2-2$, passes with multiplicity $2r'_h m+2r_h n-2\Sigma\rho_{ik}x_i$ through R_h and with multiplicity $2z_i$ through Z_{ij}. The conjugate of the basis point R_h is of order $2mr'_h+2nr_h-2\Sigma\rho_{ik}z_i$, has R_h to multiplicity $4r_h r'_h-2\Sigma\rho_{ik}+2$, Z_{ij} to multiplicity $2\rho_{ij}$, and other basis points R_k to multiplicity $2r_h r'_k+2r_k r'_h-2\Sigma\rho_{ih}\rho_{ik}$. The order of the associated transformation is

$$(4mn-2\Sigma z_i^2-2)^2-(2nr_k+2mr'_h-2\Sigma\rho_{ik}z_i)^2-\Sigma\Sigma(2\rho_{ik})^2-2,$$

a formula previously found by G. Marletta(72).

The particular case $m=1$, then $n=2$, and finally $m=1$ and $n=2$ is considered in detail, confirming results already obtained by Marletta.

A calculus of composition for (m, n) correspondences was developed by A. Kanda(54).

T. Kubota(62), (63), (64), (65) constructs a (2, 2) correspondence by means of a conic γ and two points A_1, A_2 on it. From any point P in the plane draw PA_1, PA_2 meeting γ in C_1, C_2. Let Q be the pole of C_1C_2. This defines a (1, 2) correspondence. In (65.1) the same author adds one more type of involutorial (2, 2) correspondence. It may be defined by the equations

$$x'_1=x_1, \qquad x'_2=x_2, \qquad x'_3=-x_3\pm 2r\sqrt{x_1^2+x_2^2}.$$

Every quadratic (2, 2) correspondence can be constructed by the product of two such (1, 2) correspondence and collineations.

An involutorial (2, 2) correspondence of class 1 was constructed by J. de Vries(114) by means of a pencil of conics and a fixed conic γ. A point P is associated with the two points in which its polar as to γ intersects the conic of the pencil through it; γ is the curve of coincidences. Similarly, given a fixed cubic, a point P, its associates are the intersection of its first and second polars as to the cubic. All the inflexional tangents are singular lines. The curve of coincidences consists of the cubic and its Hessian.

The last paper contains a correction of a statement by H. Bateman(5) that the Laguerre inversion corresponds to the Lorenz transformations in the theory of relativity.

The classification of (2, 2) correspondences from the forms of the defining equations was made by F. R. Sharpe and V. Snyder(92).

There are five types of non-involutorial involutions. Let the equations be of the form

$$\Sigma a_i(x')u_i(x)=0, \qquad \Sigma b_i(x')v_i(x)=0.$$

The five types may be represented by the scheme:

	$a_i(x')$	$u_i(x)$	$b_i(x')$	$v_i(x)$
1.	conic	line	line	conic
2.	conic	conic	line	line
3.	conic	line	conic	conic
4.	conic	conic	conic	conic
5.	C_n	C_n	line pencil	line pencil

There are six types of compound involutions, obtained by the combinations G, G; G, J; G, B; J, J; B, B; J, B.

The necessary and sufficient conditions that a (2, 2) correspondence is a compound involution is that its defining equations have the form

$$\frac{F_1(x)}{F_1(x')}=\frac{F_2(x)}{F_2(x')}=\frac{F_3(x)}{F_3(x')}.$$

A similar classification of the general (2, 3) types was made by T. R. Hollcroft(45); twelve types are given. The types of compound (2, 3) involutions were also given by Hollcroft(46): he enumerates fifteen types. In (47.1) he completes the other (m, n) correspondences, for $m, n\leqq 4$, both the general ones and the compound involutions. No essentially new features appear in general correspondences with one or both indices greater than 4.

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CHAPTER VI

INVOLUTIONS ON RATIONAL CURVES

The geometry on a rational curve C_n of order n in S_r, $r \geqq n$ is intimately associated with that of linear involutions $I_m{}^s$ of order m and of dimension s. It is only from that standpoint that the subject will be considered here. See W. F. Meyer(133), Apolarität und rationale Curven, Tübingen, 1883, and the following articles in the Encyklopädie der mathematischen Wissenschaften:

F. Dingeldey(65) Kegelschnitte und Kegelschnittsysteme.

O. Staude(165) Flächen 2. Ordnung und ihre Systeme und Durchdringskurven, Section X.

G. Kohn and G. Loria(96) Specialle ebene algebraische Kurven, Section IX.

G. Loria(122) Spezielle ebene algebraische Kurven von höherer als der vierten Ordnung, Section BI.

Moreover, valuable summaries are found in E. Pascal's(140) Repertorium der höheren Mathematik, second edition, vol. 2, part I (1910), Chap. XII, F. Dingeldey(66) Kegelschnittsysteme; Chapter XIV, L. Berzolari(16) Die Geometrie auf einer ebenen algebraischen Kurve, Section II; Chapter XIX, E. Ciani and H. Wieleitner(38) Projektive Spezialisierung von Kurven vierter und dritter Ordnung, sections 4 and 5; Chapter XXII, H. Wieleitner(238) Metrisch spezialisierte ebene Kurven, sections 1 and 2. And in vol. 22, part 2, (1922) Chapter XXIX. Also, O. Staude(166) Die Raumkurven dritter und vierter Ordnung; Chapter XXXVII, and L. Berzolari(18) Besondere algebraische Raumkurven.

1. *Involutions on* C_1.—Let x define a point on a straight line.

Given two polynomials $f_n(x)=0$, $g_n(x)=0$ and a parameter λ. The equation $f_n(x)+\lambda g_n(x)=0$ defines an *involution* of order n and of dimension 1. Between the values of λ and of x there exists a $(1, n)$ correspondence. The $2(n-1)$ values of λ for which two of the associated x coincide are called *limiting points* or *branch points;* the associated double points x are called *coincidences.* Some of the properties of (1, 2) correspondences between C_1, C'_1 were given by M. Chasles(34).

I_4 on S_1 cannot be expressed as a square by any rational transformation. P. Th. Pepin(139).

Two involutions of order $n>2$ on the same line do not have a common group unless relations exist among the coefficients in the defining equations.

The curves of a pencil of curves of order n meet an arbitrary line in the groups of an involution.

Two involutions I_n, I'_n are said to be projective when λ, λ' are projective. In two projective involutions there are $n+n'$ groups, each of which has a point in common with its image in the other. E. de Jonquières (88) gives numerous applications. Given I_3, I_2 projective, to construct $\phi=0$, the locus of their common elements. L. Cremona (51) summarizes results then known. P. Cassani (29) generalized the problem of Desargues, by determining the locus of the double points of an I_2 on a line defined by a pencil of conics, as the line describes a pencil. See also E. Waelsch (183).

2. *Involutions on a Conic.*—By means of the parametric representation the same properties of I_n can be interpreted on a conic or on any rational curve. Thus, Em. Weyr (196) proved that if a complete n-side is circumscribed about a conic and inscribed in a curve of order $n-1$, then there are ∞^1 such n-sides describing a rational I_n. Such an involution is fixed when one group and $n-1$ further pairs of points belonging to the same group g_n are given. The I_3 of osculating circles of an ellipse was found by V. Janni (83). The triangle formed by each group has constant area.

Em. Weyr (225) proved that given an I_n on a C_2, lines joining pairs of associated points envelop a curve of class $n-1$, called the *involution curve.* For application to (2, 2) correspondences see Em. Weyr (189). For particular (m, n) correspondences see M. Chasles (35).

Em. Weyr (192), (193) proved the theorem: Given two C_2 having double contact, an infinite number of triangles can be inscribed in one, circumscribed about the other, forming I_3^1 with points of contact for triple elements. Equilateral triangles and a pair of concentric circles furnish an example.

R. de Paolis (188) considered two involutions I_m, I_n on the same C_1 and showed that without knowing the double points of either an I_{m+n}^{m+n-1} can be constructed such that the $m+n$ elements of multiplicity $m+n$ are the double points of the given projective involutions.

Em. Weyr (229) studied an I_2 on a parabola by elementary methods. He also (190) proved that if I_n^1 has two n-fold points, its elements form projective series with the n-fold points for fixed elements.

Em. Weyr (226) gives the two following elementary applications. Given a point P on a conic. Inscribe the right-angled triangles in C_2, right angle at P. The hypotenuse always passes through a fixed point on normal to C_2 at P.

Construct triangles with one vertex at P, such that the normal bisects the angle at P. The base passes through a fixed point on the tangent to C_2 at P.

Em. Weyr(227) contains a summary of known results. Given a pencil of conics. They describe on a fixed C_2 through a basis point an I_3^1. By considering composite conics of the pencil, the I_3^1 is constructed from the two triads.

C. Le Paige(101) defines an *evolution* as follows: Given the pairs aa', bb', cc' of an I_2, the harmonic conjugate of a as to b', c', etc., forms an *evolution*, with the other five points a', b, b', c, c'.

P. Serret(159) gives constructions for involutions of order 2, then generalizes to find the tangents to a rational curve of class n.

C. Le Paige(109) expresses the double points and the branch points of an I_3^1 in terms of the Clebsch-Aronhold notation; the latter being the Hessian of the Jacobian of the two defining cubic forms.

C. Le Paige(108) showed that the branch points of two cubic involutions with common double elements together are three tetrads of an I_4. Had been touched upon by Em. Weyr(205) and (191).

Em. Weyr(219) showed that if two cubically involutorial conics, that is, each of which is the involution conic of an I_3^1 on the other, one involution of points, the other of tangents, touch each other, their radii of curvature are as 1 : 4, the conic having the point involution having the larger curvature. If two conics are in double contact and their curvatures are as 1 : 4 at one of them, then the same ratio of their curvatures exists at the other.

C. Le Paige(110) gives a construction of the problem: Given three triads of I_3^1 on a C_2, and two points, find the third point of the group determined by them. See Em. Weyr(210) who had already solved the problem for I_n^1. The conics through three fixed points determine an I_3^2 on a conic through one of them. Suggestions for further constructions are given but no complete solution is provided. The method is analytic; the Clebsch-Aronhold notation is used.

C. Le Paige(111) considers *conjugate* involutions. Given an I_3^1, the groups common to two certain I_3^2 derived from it form a conjugate I_3^1. As an application, given three triangles inscribed in a conic; they determine three inscribed conics. These conics have a common tangent.

Em. Weyr(220) considers an I_4^1 of tangents to a fixed C_2. The vertices of the quadrilaterals generate a general cubic curve J_3 as involution curve. The three pairs of opposite vertices are conjugate points in the three non-central involutions on J_3. The six vertices of any tetrad and the vertices of any triangle circumscribed to C_2 are basis points of a pencil of cubics. An I_4^1 is completely and uniquely defined by three triple elements.

Weill(186) gives further interpretations of I_3^1. Given two conics referred to their common self-conjugate triangle, one C_2 with one variable coefficient. A triangle inscribed in the first, circumscribed to the second defines the I_3. The locus of the Brianchon point is another C_2. Various further properties are mentioned with an outline of possible generalizations.

C. Servais(160). The branch points of two conjugate I_3, associated with the same double point, are harmonic conjugates of the double elements of the cyclic projectivity defined by the other three double elements.

Em. Weyr(218) considers an I_n of tangents on C_2. It is determined uniquely by two complete groups, but a finite number of different I_n^1 may have $2(n-1)$ common pairs. A general C_4 is not an involution curve (as a pentagon cannot be inscribed). If one can, ∞^1 can, and the sides of a pentagon are always tangent to a fixed C_2. If C_4 has a node, and the nodal tangents make with the lines joining the node to the points of contact of any bitangents a harmonic set, C_4 is an involution curve. Similarly, C_4 is an involution curve if the tangents at three collinear points form an inscribed triangle.

Em. Weyr(221) discusses various degenerations of I_4^1 on C_2. If I_4^1 is defined by one tetrad and two double elements, the involution cubic J_3 is nodal. A cuspidal cubic cannot be an involution curve of any I_4^1 of tangents to C_2.

E. Waelsch(184) discussed the I_4 associated with polar groups on C_2.

Properties of C_2 expressed by means of invariants and covariants of binary forms were given by G. Pittarelli(143).

If I_4^1 has two pairs of double elements, J_3 is composite. If two simple circumscribing quadrilaterals to C_2 have a common diagonal, their eight vertices lie on a conic C''_2, in I_4 relation to C_2. If I_4^1 contains three groups of only double elements, J_3 consists of three lines, sides of a self-polar triangle as to C_2. It is the diagonal triangle of every tetrad of I_4^1. The seven systems of contact conics with a given rational C_4 are derived.

3. *General Properties of I_n^1, Independent of Carrier.*—de la Gournerie(79) gives a special I_4^1 defined as follows: Given four points on a line. They determine three quadratic involutions, each of which has a center. These three centers are determined for every group of a quartic involution I_4. These points together with the point at infinity may make a group of an involution, the same for all. One other group determines the involution. Three groups are composed of the two double points of the three quadratic involutions.

Given three collinear points O', O'', O'''. Construct the segment $e'f'$ with its center at O', and harmonic as to O'', O'''. Similarly for $e''f''$,

$e'''f'''$. Take three quadratic involutions with $e^{(i)}f^{(i)}$ as double points. Take any point x, get its conjugate as to each group of the special I_4. Application is made to the spiral lines on the torus and to the solution of the quartic equation.

Em. Weyr(190) proves that if $I_n{}^1$ has two n-fold elements, its elements form projective series with the n-fold elements as double elements. Particular applications by Em. Weyr(192), (193) and (195).

It was shown by G. Kohn(91) that the theory of the harmonic center can be founded on that of involutions on S_1. A point m is the harmonic center of degree r as to a point system $a_1 \ldots a_n$ with O as pole, if the a_i all belong to an I_n having O as r-fold point and m is a point of multiplicity $n-r+1$. Then O is harmonic center of the a_i of degree $n-r$ with m as pole. This principle was employed to construct the satellite curve of O as to a plane curve C_n. Draw a C_1 through O meeting C_n in $|a|$. Construct the I_n having $|a|$ for one group, and O^n for a second. Construct the double points $m_1 \ldots m_{n-1}$ and their harmonic centers. As C_1 describes the pencil O, these points describe a curve, one component of which is the first polar of O as to C_n; the other is the $C_{(n-1)(n-2)}$, satellite of O as to C_n. The theory is developed further by G. Kohn(92).

G. Maisano(126) considers a particular $I_4{}^1$ with three groups having a triple element. The three simple elements define the cubic covariant of the cubic form given by the three triple elements.

G. Maisano(127). An $I_n{}^1$ has $2(n-1)$ double elements arranged in $n-1$ pairs, each of which belongs to a group of I_n. Applications to $I_4{}^1$.

C. Le Paige(105) gives various additional theorems on self-conjugate involutions. The determination of the branch points of I_3 when the coincidences are known.

C. Servais(160) provides alternate proofs for the properties of conjugate $I_3{}^1$. See C. Le Paige(104) and (112); Em. Weyr(214). The four double elements of a self-conjugate $I_3{}^1$ are equianharmonic. Another involution is defined by interchanging branch points and double points.

J. de Vries(174) gives an algebraic proof and a construction for completing I_n defined in various ways.

W. F. Osgood(135) gives an alternate geometric proof of the (1, 1) nature of the correspondence between points on the curve and values of the parameter. An illustrative example is given. See also J. Cardinaal(27).

K. Bartel(7) discusses possible constructions by means of pencils of lines in involution. Two $I_n{}^1$ generate C_n with P_n at each vertex. More details for $n=2$. K. Bartel(8) continues the same study for $n=4$. No new results.

Em. Weyr(204). Four double elements d_i of I_3^1 form with branch elements v_i pairs d_iv_i. Any d_i is a double element in the projectivity formed by the other three pairs.

4. *Plane Rational Cubic Curves.*—H. Durège(70) discussed the problem of closure in the following form: Given a point A on a plane rational cubic, find the tangential of A, etc. Parametric form so chosen that the sum of the parameters is zero for collinear points on cuspidal curves, while for nodal cubics, the sum equals the product. Applied to inscribed and circumscribed polygons. Compare A. Clebsch(39).

In (198) the same author (Weyr) constructs a pencil of cubics with the node and five simple basis points given. The nodal tangents form an I_2, hence there are two cuspidal C_3 in the pencil. In the same paper is found an outline of constructing I_n^1 when two complete groups are given. In (205) a pencil of C_3 with fixed nodal tangents is given; the lines from the node to the inflexions describe an I_3^1 having the fixed nodal tangents for triple elements. A number of known results are proved by a particular parametric representation.

Em. Weyr(191) studies the nodal cubic parametrically. By choosing 0, ∞ for the values of the parameter at the node three points are collinear when the product of the parameters is constant. Let A, B be two points on C_3. Start with a point P_1 and find P_2 on C_3 such that P_1AP_2 are collinear; P_2BP_3 collinear P_3AP_4, etc. If P_{2n+1} is identical with P_1, the figure $P_1P_2\ldots P_{2n}$ is an inscribed Steiner polygon. The necessary and sufficient condition that a Steiner polygon of $2n$ sides is associated with two points t_1, t_2 is that $t_1^n=t_2^n$. Elementary properties are given by Em. Weyr(230). Particular cases were discussed by K. Zahradnik(247), (250), (248) and (245).

Em. Weyr(210) maps a nodal C_3 on C_2. Project C_3 from the node upon a conic through the node perspectively. Cotangential points are mapped on an I_2 with images of the node for double points. The C_3 is incidentally mapped on the line joining the images of the node. Parametric representation is given. Another mapping by means of cross-ratio is discussed.

Em. Weyr(212) studies the systems of contact C_2. A large number of known theorems are proved by other methods. Every contact C_2 of C_3 is an involution C_2 of a definite I_3^1.

Em. Weyr(215) considers the system of conics in the triangle of inflexional tangents and touching an arbitrary line. Each has two additional tangents common with C_3, hence there is a (1, 2) correspondence between the conics and the tangents.

Since the fourth basis line is arbitrary, each line of the plane is in (1, 1) correspondence with an involution of tangents on C_3. The discussion is partly analytic.

M. Treibscher(169) constructs a curve of class 3 by means of a pencil of C_2 and an I_2 on the line at infinity. The envelopes of the axes and of the asymptotes are determined. Every curve of class 3 having the line at infinity for double tangent can be generated in this way, as the envelope of the asymptotes of a pencil of conics.

H. Drasch(69) generates a nodal cubic by a projective pencil of class 2, and one of class 1. Dual in space treated by H. Schroeter(157); numerous known properties are again proved by this method.

G. Pittarelli(144) discusses the invariant form of the parametric representation of the rational cubic curve.

G. Zecca(253) replaces each point on a rational plane C_3 by its first tangential and obtains the condition for closure after k applications. The same process is applied to a nodal space C_4, transforming each point into the residual intersection with the osculating plane.

G. Loria(121) showed that a nodal cubic has a continuous system of $2n$-gons for n odd, and none for n even. There are none on the cuspidal cubic.

J. de Vries(171) and (172) gives other methods of generating plane nodal cubic curves.

K. Zahradnik(252) showed that various curves can be generated as cissoids, including the right circular cissoid, Sluse's conchoid, the right strophoid, the cissoid of Diocles, the trisectrix, and others.

J. E. Wright(243) constructs the nodal cubics through eight given points and discusses their reality. If all basis points are real, not more than two acnodal cubics are in the pencil. In general $R-1=C-A$. $R=$number real basis points, $C=$number crunodal cubics, A number acnodal cubics.

K. Zahradnik(246) gives a general method of constructing nodal cubics. The method is a combination of inversion and duality.

L. Crelier(47) develops a synthetic study of the locus of intersection of corresponding lines of two pencils in (1, 2) correspondence.

E. Cavalli(33) gives another method of generating rational plane cubics. W. van de Woude(242) discusses quadrilaterals circumscribed about a given conic and inscribed in a cubic. A number of dual interpretations are included in the memoir of C. Servais(161) which discusses the plane quartic curve of class 3.

Involutions on C_3, $p=0$ are treated parametrically by R. M. Winger(240). The idea of the satellite line of the plane rational cubic is developed by the same author(239), with an outline of its application to non-singular cubics.

R. M. Winger(241) obtains further relations among the intersections of a general C_n and a rational C_3.

5. *Space Cubic Curves, and Various Plane Mappings.*—L. Cremona(50) studies C_3 by means of two lemmas: "if K_2 is the polar conic of a point θ as to a trilateral, vertices *abc*, each tangent to K_2 from θ forms with the lines θa, θb, θc an equianharmonic system" and "two projective pencils, one of lines, one of pairs of conjugate lines of an involution, with common vertex θ, having $\theta\omega_1$, $\theta\omega_2$ for self-conjugate rays and θa, θb, θc common to the two systems, if each of first gives with the second an equianharmonic system, then $\theta\omega_1$, $\theta\omega_2$ in the second correspond to $\theta\omega_2$, $\theta\omega_1$ in the first." They are applied to the section of C_3 by the planes of a pencil, forming an I_3. Through any point of space pass two quadric cones through C_3, and any line touches four such cones.

Let σ be the pole of the section of one of these cones by a given plane, and call the vertex of the cone S. The line $S\sigma$ describes a quadric H. This representation he calls hyperbolic projection of C_3 on the plane.

L. Cremona(53). Given a space C_3, developable $\Sigma_3{}^4$, any osculating plane of C_3 meets Σ in a C_2 and the generator counted twice. Through any point of the plane pass two other osculating planes, intersecting in A, in general not in the given plane π. Given another osculating plane π_1 and the associated conic $C_2{}^1$, generator P^1, and line A^1. The line A generates a ruled surface as the point (π, A^1) describes a curve. If the given curve is a conic, the ruled surface is of order 4 having C_3 for double curve. Particular cases when C_3 is a cubical parabola are discussed.

Em. Weyr(194). The system of quadrics through C_2 and three coplanar points of a given space C_3 form a pencil meeting C_3 in $I_3{}^1$. The other basis C_2 of $|H|$ is in the plane of the given triad. The quadrics of the pencil touching C_3 and the 16 having contact of second order are determined. Various particular positions of the basis elements are considered.

R. Sturm(167), by use of two apolar binary forms, obtains the equation of the linear complex to which C_3 belongs, and derives many of its properties. Continuation of Th. Reye(146) and (147), where the word "apolar" was first used. J. Rosanes(149) uses "conjugate" instead of "apolar."

L. Cremona(49) first gave the parametric form

$$x_1 : x_2 : x_3 : x_4 = t^3 : t^2 : t : 1.$$

See also Joachimsthal(84).

E. d'Ovidio(136) applies the Clebsch-Aronhold notation and the theory of binary invariants to various involutions on space cubic curves. No new results were obtained nor in (142) by G. Pittarelli. Continued by E. d'Ovidio(137) and by F. Gerbaldi(74) in which the methods of E. d'Ovidio(137) are applied to a system of two projectively related space cubics.

C. Le Paige(111) shows that if C_3 is met by any three planes of an I_3^2, and the bisecant to C_3 drawn through their common point, its points on C_3 belong to the three simple I_3^1 determined by the three triads.

Em. Weyr(223). The I_3 defined by a pencil of planes has its double points at the four points of contact of the tangent planes through the line, axis of the pencil. The lines joining conjugate points meet the axis in points of the I_3. Conversely, every I_3^1 on C_3 is cut by the planes of a pencil. The dual interpretation allows the null-system to which C_3 belongs to be developed. Weyr shows in a new way that the locus of the vertex of K_2 containing six basis points of C_3 is the Weddle surface, which is invariant (but not point for point) under the cubic involution defined by the bisecants of C_3.

Em. Weyr(224) considers an I_n^1 on C_3. The involution (ruled) surface is of order $2(n-1)$, having C_3 to multiplicity $n-1$, and the planes connecting points by threes define the involution curve of order $\frac{(n-1)(n-2)}{2}$. The particular case of $n=4$ is studied, including a number of special cases.

C. Le Paige(113) showed that if three planes of a tetrahedron turn about a triad of a cyclic I_3 on a space C_3, the fourth vertex describes a quadric surface.

A. Cantone(26) reproduces by synthetic methods known theorems on harmonic division of bisecants of a space cubic.

E. Caspary(28) introduces a new symbolic notation to express the condition that points on C_3 are collinear, coplanar, etc. A new proof of the existence of the Weddle surface is given.

A. C. Dixon(67) gives a synthetic and algebraic proof of known elementary theorems, followed by consideration of special cases, showing the connection with division of elliptic functions. The only involution employed is an I_2. In A. C. Dixon(68) the only transformation employed is point-plane duality.

E. Waelsch(184) interprets binary quintic forms on C_3. Given an I_4^1 of points. The tangents at the points of each tetrad have two transversals, conjugate in the linear complex to which C_3 belongs. As the tetrad describes C_3, the transversals generate a quadric surface. When the transversals coincide, the tetrad is equianharmonic. The polar triads of groups of I_4^1 form an I_3^1 which cut C_3 by planes of a pencil.

G. Kohn(95) shows that cubics having contact with the developable of a given cubic in one or more points can be found by means of certain (2, 2) correspondences.

G. Gallucci(73) arranges the points of C_3 on ∞^2 orthocentric tetrahedra.

J. Fairson(71) gives a geometric interpretation of the invariants of binary forms, applied to the space C_3.

S. Jolles(87) presents an alternative synthetic proof of the harmonic pole of a line as to the triangle formed by points of a plane as to C_3; it is the null-point of the plane with respect to the complex determined by C_3.

J. de Vries(180) employs the cubic involutorial transformation to systems of cubic curves, images of lines of a congruence.

E. K. Wakeford(185). Given six mutually skew lines in S_3. There are six C_3 having all as bisecants, and one $C_3{}^1$ having them as axes of two osculating planes. There are ∞^1 tetrahedra inscribed in C_3, circumscribed to $C_3{}^1$. If three C_3 have six common bisecants, they have seven. If seven lines are common bisecants of two cubics, there exists a third having all seven for bisecants.

6. *Plane Rational Quartics.*—From Em. Weyr(197). Given a lemniscate C_4 with P_2 at O. From any point X on C_4 can be drawn four tangents. The points of contact are equianharmonic on $C_1 \perp OX$, and distant $\dfrac{OX}{2}$ from O. The envelope of C_1 is an equilateral hyperbola, similar to that of which C_4 is the pedal. Through each point Y pass three osculating circles, points of osculation on $C_1 \perp OY$. This defines $I_3{}^1$ with two triple elements at O. The osculating circles at four con-cyclic points have con-cyclic residual intersections.

Any triad of parallel tangents to the cardioid meet the double tangent in three points which are projected from M, one-fourth distance from the cusp to the point of intersection of the curve and its cuspidal tangent, by lines making angles of 60° with each other. Pencil in $I_3{}^1$. Em. Weyr(195).

Given a plane C_4 with three nodes, all the nodal tangents inflexional. If a line through a node D meets C_4 in A and B, and the line joining the other nodes in D', these four points $DABD'$ are harmonic. From D' can be drawn six tangents, points of contact of which lie on three lines through D describing an $I_3{}^1$ with tangents at D as triple elements. The theorem was applied to the lemniscate. Em. Weyr(200). Compare Em. Weyr(233) and K. Zahradnik(249) and (251) for applications to the cardioid.

Em. Dewulf and P. H. Schoute in (64) determine the number of rational quartics through two given P_2 and $7P_1$. They consider a pencil of cubics and invert by quadratic inversion into quartics—hence obtain 12 solutions.

C. Bobek(19) gives the generation of plane rational C_4 by means of projective pencils of conics through three common basis points. The treatment is synthetic. He considers the case in which all nodal tangents are inflexional. A table of fifty forms is provided.

Any rational C_4 has $3I_2{}^1$, $\infty^2 I_4{}^1$, $\infty^1 I_3{}^1$. Curve mapped on C_2, equivalent to quadratic inversion through the nodes. Em. Weyr(217).

A. Ameseder(4) shows that the trinodal quartic curve is determined by five conditions besides the positions of the nodes.

C. Le Paige(107) employing a method given by C. Le Paige(106) constructed a C_4 when fourteen points were given.

P. H. Schoute(153). If the nodal tangents to a trinodal C_4 are inflexional, the points of contact of tangents from any point of a curve are collinear. The envelope is a C_4 having nodes at the vertices of a self polar triangle. Same paper in substance in (155). See also Em. Weyr(236), and E. Laguerre(99) and (98).

C. Le Paige(115). Given two perspective triangles. Keep one fixed; as center describes circle, axis envelopes a rational curve, class 4.

See also P. H. Schoute(154) and (151) and (152); G. Kohn(94).

M. Lazarski(100) constructs the C_4 with a triple point by means of a (1, 3) line correspondence.

W. Gross(80) showed that the C_2 through six points of inflexion of a rational C_4, the C_2 through the points of contact of four double tangents, and the C_2 touched by the six inflexional tangents all belong to the same pencil. The method was that of simultaneous invariants of binary forms.

G. Kohn(93) gives another proof of the projective property of the lemniscate, and states that the first proof is due to Em. Weyr(231). Given C_4 with three cusps, cuspidal tangents meeting at P. Any line through P meets C_4 in the four points equianharmonic as to C_1 and as to C_4. Dual theorem. By quadratic inversion, consider a triangle and an inscribed C_2, P as Brianchon point. Any C_2 circumscribing triangle and passing through P meets first conic in points equianharmonic on both.

J. de Vries(173) studies involutions on C_4 with a triple point; also (176).

A. Adler(3) proves that the contour of a cylindroid on a plane normal to the double directrix is the three-cusped hypocydoid.

G. Majcen(125) discusses a particular quartic.

J. N. Rice(148) obtains further properties of inscribed and circumscribed triangles.

7. *Rational* C_4 *in* S_3.—L. Cremona(48) gives an exhaustive synthetic study of the general case, obtaining many results found by others later. For early history, compare Repertorium, vol. II 2, pp. 946-952(18).

Em. Weyr(199) employed parametric representation in an invariant form: he gives the condition that four points shall be coplanar, three collinear points forming an I_3^1. Through any point of C_4 passes a plane meeting it in three other points, the osculating planes at which pass through the first point, forming an I_3^1.

Em. Weyr(192) shows that the curve $x_1=t^4$, $x_2=t^3$, $x_3=t$, $x_4=1$ has two linear inflexions. Four points are coplanar if $\Sigma t_i t_k=0$. The I_3^1 formed by the trisecants has the inflexions for triple elements. E. Bertini(10) proves that the general C_4 has three bisecants which form the intersection of the osculating planes at their points on C_4, and they meet in a point. A new I_4^1 is determined as follows: Given a point P, pass a plane through P and each of the axes of osculating planes; each determines one point on C_4. These points with P form a group of I_4^1. If C_4 has two linear inflexions, it has ∞^1 axes of osculating planes. If the four stationary points are equianharmonic, so are the residual intersections of the four tangent-secants. In this case C_4 is the intersection of a ruled cubic and a polar quadric. A. Armenante(5) and (6) reproduces the substance of L. Cremona(48) in algebraic form, by use of the Clebsch-Aronhold method. He also gives an alternate proof that every C_4 has three concurrent axes of osculating planes, E. Bertini(10). He also proves that the points of a Steiner surface and the bisecants of C_4 are in (1, 1) correspondence, from which many properties of the surface are derived.

Em. Weyr(233) derives properties of the I_3^1 of trisecants; continuation in (235) and (198). He also(201) treats the cuspidal C_4 parametrically. By letting $t=\infty$ represent the cusp and $t=0$ the point of contact of the stationary plane, four points t_i are coplanar when $\Sigma t_i=0$. The points of contact of the double tangent planes are harmonic as to the cusp and stationary point. The lines joining these all pass through the vertex of a second quadric cone containing C_4. The residual points of intersection with C_4 of the osculating planes of four coplanar points are also coplanar.

The general C_4, $p=0$ is mapped on a conic by Em. Weyr(202) and (203).

The C_4 lies on a quadric surface F_2, meeting the plane of the given C_2 in a conic $\overline{C}_2$, and C_2 itself in four points, through each of which passes a trisecant t of C_4. From a point on t, project C_4 on the plane of C_2 into $\overline{C}_4$ with P_3 on C_2, hence C_2, $\overline{C}_4$ are perspective. The system (t) gives rise to I_3 on C_2. Many properties of C_4 can be obtained by this process. If the osculating planes of one triad t, C_4 meet on C_4, the same property holds for every triad. The results obtained synthetically are then compared with those obtained analytically by E. Bertini(10), with frequent reference to Em. Weyr(199) who gives a new proof of theorem that every C_4, $p=0$ has three concurrent axes of osculating planes. If C_4 is projected from this point upon a plane, the trinodal quartic has inflexional tangents for all its nodal tangents.

Em. Weyr(206) continues the parametric representation to map a nodal C_4 on C_2. Four points t_i are coplanar if $t_1t_2t_3t_4=K$, the node being 0 and ∞. The osculating planes to C_4 from a point on it form a group of I_3^1, as the point describes the curve. The node represents two cubic elements.

All the I_2^1 on a general C_4 gives rise to all the cubic ruled surfaces on which C_4 lies, having a bisecant for double directrix. Em. Weyr(207).

The cuspidal C_4 is mapped on a conic by Em. Weyr(208) by methods analogous to those used by the same author in (201), (203). Method of I_3^1 determined by trisecants does not apply if C_4 has a double point or a cusp. The method here employed is applied to C_4 with a double point in (209) being a direct continuation of Em. Weyr(206). The fourth point on C_4 in a plane through three given points on it is determined, including the residual intersection of an osculating plane.

If A_1, A_2, A_3, A_4 are the points of contact of the four stationary planes of a general non-singular C_4, then the points of contact x_1, x_2, x_3, of the three osculating planes from any point y on C_4 are the three harmonic centers of y as to the tetrad $A_1A_2A_3A_4$. Em. Weyr(218) continuing (216).

The I_4^1 discussed by C. Le Paige(109) now breaks up into three I_2. The covariant T (of Le Paige) gives the three pairs of points on the axes of osculating planes.

A. Adler(2) employs similar methods to study C_4 with one or with two linear inflexions. When C_4 has one, there exist ∞^1 axes of osculating planes, and when C_4 has two, the osculating planes on each trisecant belong to a pencil. The curve now belongs to a linear complex. A. Adler(2). A. Adler(1) gives new proofs of several well known theorems, including that of the existence of three concurrent axes of osculating planes. The C_2 on which C_4 is mapped and the involution conic of the I_3^1, image of the trisecants, are polar reciprocal as to a third conic. The three concurrent axes are double lines of ruled cubic surfaces on which C_4 lies; the simple directrices complete a tetrahedron. C_4 is invariant under the three axial involutions determined by opposite edges of the tetrahedron.

St. Jolles(85) defines osculants and discusses congruences of bisecants of a given C_4.

A. Brambilla(23) associates the lines r of a cubic surface with the conics of a quadric whose planes contain r. The pole of r as to the conic describes a cubic.

A. Brambilla(22) gives a particular case of a problem in closure, of polygons inscribed in a space nodal quartic.

In (24) A. Brambilla applies the $I_3{}^1$ to the generators of a cubic ruled surface on which the quartic curves lie.

G. Marletta(128) reproduced proofs of all known general results as to a C_4 of $p=0$, and many particular ones, by projecting the normal C_4 of S_4 into S_3. His method is synthetic.

G. Frauenfelder(72) considers a pencil of C_4 with two common linear inflexions and inflexional tangents. He shows that they are asymptotic lines on a ruled cubic surface having the line joining the inflexions for double directrix. Every generator meets C_4 in a pair of points harmonic as to the points of inflexion, which are pinch points on the surface. See L. Cremona(52), G. F. Halphen(81), V. Snyder(163).

S. Jolles(86) gives a collection of Weyr's and various related methods of generating rational space C_4. The dissertation of Schrek(158) uses Weyr's method and related methods to present a general discussion of the space C_4. No new results are given. Polygons connected with C_4, $p=0$ in S_3 were also considered by P. Locchi(119).

8. *Plane Rational Curves of Order Higher Than 4.*—For the general theory see article by G. Loria(122). Em. Weyr(228) employs involutions in the plane to prove the theorem: Through B arbitrary points in the plane and γ further points on a rational C_n can be passed $(6-B-\gamma)(2n+B+\gamma-5)$ conics having $6-B-\gamma$ consecutive points in common.

By means of I_n of points on a line he(234) generates any C_n with an $(n-1)$-fold point linearly.

He also(232) considers (n, n) symmetric correspondence, more general than $I^1{}_{n+1}$, with numerous particular cases. He gives a proof that C_4 of S_3 has three concurrent axes of osculating planes.

J. Lüroth(123) proved the important theorem that if the points of a curve can be expressed rationally in terms of a parameter, then they can also be put into (1, 1) correspondence with a rational parameter. A rational curve cannot contain irrational involutions.

G. Garbieri(78) develops a general method of applying binary forms to express involutions on plane and space rational curves.

M. Genty(75) starts with the parametric representation $x_i=f_i(t)$ and derives all the ordinary known properties. No new results are obtained.

Em. Weyr(235) employs the principles of enumerative geometry to confirm a number of known results.

P. H. Schoute(156) gives a new derivation of quadratic inversion, by successive application of which any rational curve may be constructed.

E. de Jonquières(89) constructs a plane C_8 with six given double points and two simple points by means of two projective pencils of cubics, with an extraneous line. Fundamental involutions on the rational plane C_5 are discussed by J. de Vries(179). In (178) J. de Vries considers I_s^1 on a rational plane C_n, and considers the systems of conics through five associated points, $s>5$. In this system of conics $4(n-2)(2n-1)(s-1)$ touch the given C_n.

Another proof of Lüroth's theorem was given by E. Netto(134).

J. E. Wright(244) gives an alternate proof that C_n in S_2 has

$$\tfrac{1}{2}(n-1)(n-2)P_2,$$

if $x_i=f_{n,i}(t)$, where $f_{n,i}$ is an integral polynomial.

J. R. Conner(45) constructs three linearly independent apolar binary quintics to those defining C_5 in S_2; they define a *conjugate* quintic curve. The I_5^2 is discussed. The invariant theory of rational C_5 in S_2 is identical with the covariant theory of six lines in a plane.

J. I. Tracy(168) shows that on a quintic curve with a four-fold point, there exist three quadratic involutions determined by pairing the tangents at the multiple point. The involution conic of each is discussed.

J. R. Conner(46) proves that a rational C_m uniquely determines and is uniquely determined by a single binary form of order $3(m-2)$, except as to projections. Any involution I_n^1 may be replaced by a single binary form of order $(r+1)(n-r)$.

A. Plamitzer(145) studies two projective involutions, I_m^1 on a rational C_μ, I_n^1 on a rational C_ν. Lines joining associated points envelop a curve of class $m\nu+n\mu$ and of genus $(m-1)(n-1)$. Application to particular case $\mu=\nu$, $C_\mu=C_\nu$, and to ruled surfaces of order $m\nu+n\mu$.

A. B. Coble(42) establishes relations between the ten nodes of a rational C_6 and those of the quartic surface known as Cayley's symmetroid. Two curves are equivalent under Cremona transformations only when their double points are congruent as a whole. The same property is shown to exist on the symmetroid. See also H. P. Hudson(82) who showed that C_6 with 10 double points cannot in general be reduced to a curve of lower order by Cremona transformations. Further properties given by J. R. Conner(43).

9. *Rational Curves C_μ, ($\mu>4$) in S_k, $k>2$.*—Em. Weyr(187) proves the theorem: Through q points in S_3, and b points on a rational C_n in S_3, there are $k(2n-k-b+1)$ quadrics of a pencil which have k point contact with C_n, wherein $b+q+k=10$.

J. Rosanes(149) shows by means of conjugate binary polynomials that every C_n can be expressed by the sum of $\frac{n(n-1)}{2}$ linear expressions, each raised to nth power if $n>3$, and to four if $n=3$.

G. Korndörfer(97) gives the parametric representation of rational curves in S_3 by means of abelian integrals.

G. Castelnuovo(31) makes a systematic study of involutions by means of hyperplanar sections of the normal C_n in S_n. All earlier theorems are derived from this point of view. Also G. A. Bordiga(20) for $n=4$.

There are $\binom{n-m}{m+1}$ spaces S_m which meet a normal rational C_n of S_n in $m+1$ points, each meeting an S_{n-2m-1} of S_n $\left(m<\frac{n-1}{2}\right)$ in one point; through $i+1$ points of C_n pass $\binom{n-i-m}{m}$ S_{m+1} meeting C_n in $m+i+1$ points and meeting an S_{n-2m-i} of S_n in one point. The S_i having $i+1$ consecutive points on C_n envelope a variety V_{i+1} of order $(i+1)(n-i)$.

The tangents to C_n meet a fixed osculating S_{n-1} of C_n in points of C_{n-1} projective with given C_n.

The curve is self-dual. Each point referred to its own osculating S_{n-i} defines an involution on C_n.

The S_{n-1} through S_{n-k-1} as basis meet C_n in groups of I_n^k. The vertex S_{n-k-1} is called the *central space* of I_n^k. I_n^k is determined by $k+1$ groups of n elements. Each group is determined by any k points belonging to it. If two normal curves C_n of S_n, C_m of S_m are projectively equivalent, to I_m^k on C_m corresponds an I_n^k on C_n. If to each group of I_m^k there be added $n-m$ fixed points, I_n^k on C_n results and conversely. If in I_n^k ρ elements be fixed, $I_{n-\rho}^{k-\rho}$ is called the *adjoint* involution of the fixed points. In I_n^k are $(k+1)(n-k)$ elements of order $k+1$. r points constitute a *neutral* group of r points, if $k-r+1$ other points are necessary to fix a group of I_n^k.

$k-2$ arbitrary elements of I_m^k appear in $\frac{(n-k+1)(n-k)}{2}$ neutral groups of k elements.

The $n-r$ points, which with r neutral points make a group of I_n^k constitute an I_{n-r}^{k-r+1}.

If ρ points are such that $k-r+\rho$ fix a group of I_n^k, they constitute a group called the *neutral of species* ρ. If $r+s=n$, I_n^r, I_n^s have in common I_n^{r+s-n}. If $r+s+1<n$, they are both contained in I_n^{r+s+1}.

In I_n^{n-1} the R_{n-1} containing an n-fold element is called a *principal group*. If n is odd, the n groups, each consisting of an n-fold element, constitute a group of I_n^{n-1}. Let S_{n-k-1} be the central space of I_n^k; if in the points of a group the osculating S_{n-1} be constructed, they will have S_k in common, called the *axial space* of I_n^k. It is conjugate to S_{n-k-1} in the polar system defined by C_n. I_n^k, I_n^{n-k-1} are called *conjugate involutions.*

If I_n^2 contains I_n^k, the conjugate of the former is contained in the conjugate of the latter.

The $(k+1)$-fold elements of I_n^k coincide with the $(n-k)$-fold elements of its conjugate I_n^{n-k-1}.

If two conjugate involutions I_n^k, $I_n^{n-k-1}\left(k<\frac{n-1}{2}\right)$ are both in I_n^{n-k}, then the first is contained in the second.

There are $m+1$ I_{2m+1}^{2m-1} which contain four given neutral groups each of $m+1$ elements, of the first species.

G. Loria(^{180}) develops a general parametric representation of the rational C_n in S_{n-1}, and discusses the groups of linear transformations under which they are invariant. The generalization from C_4 in S_3 is pointed out.

A. Brill(^{25}) discusses fundamental involutions I_n^{n-p-1} on C_n of S_p, further developed by W. Stahl(^{164}) with particular reference to rational C_4 in S_3. No further contribution to transformation theory is included.

L. Berzolari(^{11}) gives a full historical résumé of all the results concerning rational C_n in S_3 and shows how the theory can be developed by means of combinants of four simultaneous binary quartics.

F. Derutys(^{61}) and (^{59}). The number of groups of $n-1$ neutral elements of I_n^k which contain an element of multiplicity $n-2k$ is $2(n-k)(n-2k)$.

An I_n^k contains $2ab(2p-3)\binom{n-k+1}{p}\binom{n-k}{p-2}$ groups of $k+p-2$ neutral elements, each consisting of two associated, multiple elements of orders a, b where

$$a+b=k-p+2.$$

If three elements, of orders a, b, c where $a+b+c=k-p+3$ are contained in I_n^k, the number is $2abc(2p-3)(2p-4)\binom{n-k+1}{p}\binom{n-k}{p-2}$.

In general, for orders a_1, a_2, $\ldots$, a_ρ, $\rho=2p-2$, where $\Sigma a_i=p+k-\rho$, the number is

$$2\prod_{i=3}^{2p-1}(2n-i)\prod_{i-1}^{2p-2}(a_i)\binom{n-k+1}{p}\binom{n-k}{p-2}.$$

J. de Vries(^{177}). Given a C_n in S_3 the tangents of which form an I_p^1. The lines meeting two associated lines belong to a complex of order $(2n-3)(p-1)$. The lines meeting three associated tangents form a congruence of order and class $(n-2)(p-1)(p-2)$.

G. Marletta(129) shows that C_5 of S_3 with four stationary planes has one and only one quadrisecant if no two pairs of points of contact of stationary planes, or no three such points approach coincidence. G. Marletta(130) projects a normal C_n of S_n into S_3 or S_2; fundamental involutions are projectively equivalent. Particular attention is given to C_n with two tangents of $n-1$ point contact. See also G. Castelnuovo(30), L. Berzolari(13), A. Brambilla(21), V. Snyder(162).

J. R. Conner(44) gives a generalization of quadric inversion, $x_i = \dfrac{1}{x'_i}$ in S_n, and gives a discussion of I_n^{n-1} on C_n of S_n.

G. Marletta(132) studies the rational C_n in S_{2r+1}, having two S_r with $n-r$ point contact. It has ∞^1 osculating axes which form a ruled surface of order $(n-2r-1)(n-1)$. For $n=4$, $r-1$. See L. Cremona(54). Particular attention to C_n of S_3 with four points of hyperosculation. L. Berzolari(15) gives an elementary analytic discussion of C_n in S_3 having four points of hyperosculation. All the tangent lines to C_n meet the singular planes in projective groups. Further properties are found, by G. Marletta(131) by the use of the theorems given by E. Ciani(36), L. Berzolari(12) and (14), E. Ciani(37).

B. Kalicun(90) applies an I_5^1 to generate a rational quintic scroll with a four-fold directrix line.

J. F. de Vries(170) gives a general theory of normal C_4 in S_4. Polygons of $n+1$ sides inscribed in C_n of S_n are considered by H. S. White(237).

10. *Rational Involutions I_n^k on Rational Carriers.*—Em. Weyr(211). Given $k+1$ polynomials $f_1(x)=0$ in x, each of order n. The equation $\Sigma\lambda_i f_i(x)=0$ defines I_n^k. The neutral elements are those which belong to more than k groups. Plücker's equations concerning characteristics of rational curves in S_2, S_3, ... are derived. The results obtained were known before, but the method is partly new. Algebraically, a neutral group of h elements ϕ_n is a factor of more than one $f_i(x)$. Em. Weyr(222). The curves or surfaces of a linear system of order n, dimension k cut a C_1 in I_n^k, containing $\dfrac{(n-1)(n-2)}{2}$ pairs of neutral elements. Applications are made to cubic curves.

Em. Weyr(188) proves the theorem: The twelve faces of any three trihedra inscribed in a C_3 of S_3 touch the same quadric surface which contains three bisecants of C_3. This quadric is the involution surface of I_4^2 formed by such tetrahedra. Various applications were made by W. F. Meyer(133).

C. Le Paige(106) constructs an I_4^3 on C_3 of S_3 when four groups and three points on a fifth are given. Previously solved by P. Serret(159), used by P. Schlesinger(150), and approached by C. Le Paige(114). By the same method, a plane $C_4 p=3$, is constructed when fourteen points are given. Can be generalized to construct C_n in S_2.

C. Le Paige(102) finds by a recurring process, the number of groups of $k+k'$ points common to $I_m{}^k$ and $I_{m'}{}^{k'}$ to be

$$\frac{(m-k)(m-k-1)\dots(m-k-k'+1)}{\underline{|k'}} = \frac{(m'-k')(m'-k'-1)\dots(m'-k'-k+1)}{\underline{|k}},$$

which define another involution. Application: Given two triangles and a conic. There exist two circumscribing conics, one about each, in a pencil with the given conic.

F. Deruyts(56) represents the groups of $I_n{}^{n-1}$ by points of S_n, composite $I_n{}^1$ by normal C_n of S_n. The $|S^n{}_{n-1}|$ through the representative point meets C_n in $I_n{}^{n-1}$.

J. Rosanes(149) proves the theorem, the groups of n points which express a binary form as the sum of n n^{th} powers constitute an $I_n{}^{n-1}$. In this, an $I_n{}^{n-m}$ has $n-k(m+1)-m$ neutral groups of $n-k$ elements $m<\dfrac{n-m}{m+1}$, which form an $I_{n-k}^{n-(1+m)k-m}$.

F. Deruyts(55) continues the same representation. G. Castelnuovo(32) considers C_n and an S_{n-k-1} in S_n.

The $\infty^k S_{n-1}$ having S_{n-k-1} as vertex cut C_n in groups of an $I_n{}^k$.

Lerch(118) showed that in $I_n{}^k$ there are points of multiplicity r_1+1, $\dots$, r^2+1, where $\Sigma r_i=k$ and $k+\rho=n$, $\rho!\binom{n-k}{\rho}\prod_{i=1}^{\rho}(r_i+1)$.

F. Deruyts(55) gives an alternate proof. C. Le Paige(103) proves that $I_n{}^k$ may have multiply neutral groups. In $I_n{}^{n-1}$ the doubly neutral groups form an I_{n-2}^{n-5}, and the groups which leave k elements undetermined form and I_{n-k}^{n-2k-1}.

M. Genty(76) proved that $I_n{}^k$ has $(k+1)(n-1)$ groups, each having an element of multiplicity $k+1$. If $n=2k$, there are $\dfrac{2^k(n-k)!}{k'(n-2k)!}$ groups containing k double elements. Previously proved by Em. Weyr and cited by Guccia. $I_n{}^k$ contains $(k'+1)(k-k'+1)(n-k)(n-k-1)$ groups having one element of multiplicity $k'+1$ and another of multiplicity $k-k'+1$.

M. Genty(77) considers I_n^k and a series of numbers k_i, last. $i=l$ and shows that there exist a number $\prod_{i=1}^{S}(k_i+1)\frac{(n-k)\,!}{(n-k-S)\,!}$ of groups consisting of S elements of multiplicity k_i+1. If q of the k_i are alike, the number must be divided by q^l.

F. Deruyts(57) determines the number of neutral elements common to certain given involutions: $I^3_{n_1}$, $I^3_{n_2}$ each contain an infinite number of neutral triads. The number of common neutral couples is $4\binom{n_1-2}{2}\binom{n_2-2}{2}$. The involutions $I^3_{n_1}$, $I^4_{n_2}$ have $2\binom{n_1-2}{2}\binom{n_2-2}{2}$ neutral triads in common. $I^4_{n_1}$, $I^4_{n_2}$ have $\binom{n_1-3}{2}\binom{n_2-3}{2}$ neutral tetrads in common.

For a general theory of rational involutions, see F. Deruyts(63).

Further formulas are given by F. Deruyts(58) concerning the maximum number of multiple elements and of common neutral elements. He also(59) shows that in $I_n^k k-p$ arbitrary elements can be associated with $\binom{n-k+1}{p-1}^2\cdot\frac{n-k-p+2}{p(n-k+1)}$ groups of $2p-2$ elements forming this many groups of $k+p-2$ neutral elements.

He(60) proved further that if k points of an I_n^k are in S_{n-3}, two more points are necessary to determine a group G_n having k points of this kind. Similarly for i^{th} kind, namely when i additional points are necessary to fix a group.

The number of G_k of k neutral elements of the i^{th} kind in I_n^k is equal to the number of groups of k elements common to $i+1$ I_n^{k-1} which have in common an I_n^{k-i-1}.

He(62) showed, too, that the faces of three complete polygons of n vertices inscribed in a rational space C^m are tangent to the same surface of class $(n-2)\binom{m-1}{2}$. There exist ∞^2 similar figures.

M. Lelieuvre(117) gives a general formulation of involutions and applies his results to construction of Poncelet polygons. See(116).

J. de Vries(181) considers the various I^2_3 on C_3 of S_2. Each I_3^2 has a neutral pair, called the *associated* points. They form a symmetric (2, 2) correspondence. Application to a non-Cremona transformation of order 5. J. de Vries(175).

K. Bartels(9) applies involutions to the construction of certain ruled surfaces.

J. de Vries(182) obtains an I_3^2 from the $|C_3|$ through six points in S_2. The involution curve is a C_2. The curve of coincidences is a C_6 with six singular lines and six singular points.

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CHAPTER VII

CORRESPONDENCES ON NON-RATIONAL CURVES

1. *Chasles's Principle of Correspondence.*—The relation established by an algebraic equation between two non-homogeneous variables λ, μ is called a *correspondence.* If $f(\lambda, \mu)=0$ is of degree m in λ, and of degree n in μ, the correspondence between λ and μ is denoted by the symbol (m, n). When λ, μ are regarded as points on straight lines, the equation $f(\lambda, \mu)=0$ defines an algebraic curve in generalized cartesian coordinates. If λ is the parameter defining a point on one rational curve and μ another, then $f(\lambda, \mu)=0$ defines an (m, n) correspondence between the points of the two curves. If λ, μ define points on the same rational curve, then the correspondence is between points on this curve. If $m=1$ or $n=1$, the correspondence is called an *involution* and if $m=1$ and $n=1$, it is called *projective.* In any case the correspondence is fixed when $mn+m+n$ pairs of corresponding elements are given.

Let λ describe C, μ describe C'. On C there are $2m(n-1)$ branch points, and on C' there are $2n(m-1)$.

When $C=C'$, there are $m+n$ points such that one of the corresponding elements coincides with the given point. Thus *an (m, n) correspondence on a rational carrier has $m+n$ coincidences.* This theorem is known as the Chasles principle of correspondence, M. Chasles(^{80}), but it had been used by various authors previously. See in particular E. de Jonquiéres(^{164}) and (^{167}). L. Cremona casually mentioned it in (^{97}) and (^{98}). A careful history of the theorem is given by C. Segre(^{309}). A much earlier application was given by J. Steiner(^{338}), in determining the number of normals, n^2, to a general C_n from an arbitrary point. If the given curve describes a pencil, the foot of the normal from a fixed point describes a curve of order $2n$. The same principle is then applied to the evolute, the Plücker numbers of which are determined.

Later developments are given by Em. Weyr(^{410}), and by M. Chasles(^{74}), who extends the results of Plücker to cases in which the line at infinity of the plane of the given curve is a singular element. See also M. Chasles(^{83}). A particular case was given by M. Chasles in (^{78}). A detailed application to correspondences on conics is given by M. Chasles(^{75}) with the remark that all the results may be extended to any rational curves. Other applications are given by Chasles in (^{84}).

This theory was extended by M. Pieri(272) who showed that the locus of the foot of the normals from P to the $|C_n|$ of a pencil is of order $2n$, passes simply through P and the n^2 basis points of the pencil, through the $3(n-1)^2$ points which are nodes of particular curves of the pencil, through the $2(n-1)$ points of contact of certain curves of the pencil with the line at infinity, and through the circular points.

A (1, 3) correspondence was discussed by Em. Weyr(408).

The theorem that two general algebraic curves of order p, p' intersect in PP' points was proved by M. Chasles(73) by the principle of correspondence.

G. Fouret(125) determines the number of contacts of a system of curves of signature μ, ν (μ through a given point, ν touching a given line) with a curve of order m, class n is $n\mu+m\nu$, confirming a known result. [A. Brill(45) and Fouret in (49).] If the given curve has an r-fold point at each imaginary circular point at infinity, the number of curves (μ, ν) which meet it at a given angle is $(m+n-2r)\mu+m\nu$.

This was further developed by E. Dewulf in (107), (108) and (106). The inclined polar of a point P as to an algebraic curve C_n is defined as the locus of a point M such that the ordinary polar of M as to the curve makes a constant angle with MP. Through P can be drawn n^2 lines meeting C_n at a given angle. The envelope of the inclined polars of C_n, as to C_n is of class nn'.

H. Schubert(296) discusses loci defined by (1, 2) correspondences on rational curves; compare also Em. Weyr(386).

G. Peano(265) obtains the enumeration of the complete system of invariant forms of (1, 1), (1, 2) and (2, 2) correspondences on rational carriers. The (2, 2) case was partially obtained by A. Capelli(52). See also G. Peano(266). Em. Weyr(405) applied the same principle to the determination of one-dimensional cyclic projectivities.

G. Loria(223) derives the ordinary properties of curves generated by (m, n) correspondence between rational carriers; see also R. Sturm(341). For the application to nodal cubics, see Pittarelli(274).

A metrical application is given by K. Vahlen(364). Two curves of orders g_0, h_0, classes g_1, h_1 have $g_0h_1+g_1h_0+g_1h_1$ common normals. The method is developed in (363). A few new relations are developed incidentally by F. Hardcastle(141). Plücker's numbers are re-derived by means of the coincidences of an involution by O. Zimmermann(425). A similar application to the determination of foci is made in (426). A historical résumé is given by C. Segre(313).

J. de Vries(376) shows that if an (n, n) correspondence between superposed carriers has $\frac{n}{2}(n+1)$ involutorial pairs, the correspondence is involutorial. If two (2, 2) correspondences have the same tetrad of branch points they are involutorial.

L. I. Neikirk(244) showed that if an (m, n) correspondence between x and y has a single correspondence of the type $(x_i : y_i)$, $i=1, \ldots, m$; $i=1, \ldots, n$ then $f(x, y)=0$ can be reduced to the form

$$\phi(x)+\lambda\theta(x)=0, \qquad \psi(y)+\lambda\omega(y)=0,$$

in which all the functions are rational, the first two of degree m, the other two of degree n.

J. de Vries(377) obtains further properties of (2, 2) correspondences, extending results obtained by Em. Weyr(387) and (393), and by J. de Vries(374).

2. *Inscribed and Steiner Polygons.*—For the early history of Poncelet polygons, that is, those having their vertices on one conic and their sides touching another, and the various degenerate cases, see Pascal(264).

Application to inscribed and circumscribed polygons of a conic was made by A. Cayley(69) especially for symmetric (2, 2) correspondences, and the result expressed more definitely in (70). If x, u are in symmetric (2, 2) correspondence and u, y define the same correspondence, then x, y will be a symmetric (2, 2) correspondence, but one relation must be satisfied in order that it be the same as the preceding ones.

Inscribed and circumscribed polygons of conics were discussed by E. Czuber(100) by means of two cones of revolution with a common axis and vertex.

F. Gerbaldi(130) discusses a cyclic symmetric (2, 2) correspondence, and applies the results to Poncelet polygons to prove the theorem:

Given two proper conics C_1, C_2 the condition in order that a polygon (and therefore ∞^1) exists, having its p vertices on C_1 and each side touching C_2 is expressed by the vanishing of an invariant of orders $2\left(\frac{p^2}{4}-1\right)$, $\frac{p^2}{4}-1$ for p even, and of orders $\frac{1}{2}(p^2-1)$, $\frac{1}{4}(p^2-1)$ when p is odd. See also G. Loria(221) and (219).

E. Beltrami(23) [also in D. Chelini(85)] obtains further conditions for inscribed and circumscribed polygons in conics, and generalizes the procedure to apply to nodal and space cubics, and outlines further generalization.

A Steiner polygon is one in which the odd sides pass through one fixed point, and the even sides pass through another fixed point.

Buchheim(50) gives a treatment of Steiner polygons by means of abelian functions. Further properties of Steiner polygons were found by C. Küpper(202) and (204). By means of parametric representation in terms of elliptic functions, H. Oppenheimer(259) obtained various expressions for closure. See W. K. Clifford(94) for a particular Steiner polygon.

E. Panzi(260) derives the extended form of Plücker's formulas by direct application of Chasles's principle. The first formula for the simple case had been derived by G. Loria(222) that giving the inflexions, by J. Bischoff(33). All the simple cases are obtained by C. Segre(305).

3. *Rational Involutions on Curves.*—Given two polynomials $\phi_1(x)$, $\phi_2(x)$ of degree n in x, the equation

$$\phi_1(x)+\lambda\phi_2(x)=0$$

defines an *involution of order n.* It is a $(1, n)$ correspondence between λ and x. When x is the parameter of a rational curve C, and λ is the parameter on a second curve C', then the involution on C is said to belong to C'. The lines joining the pairs of points of each group G_n envelop a curve called the involution curve J of the given involution I_n. The theory of quadratic involutions will be presupposed.

Em. Weyr(411) develops the theory of cubic involutions, and shows that if a, b are triple elements, the equation defining it may be put in the form

$$(x-a)^3+\lambda(x-b)^3=0$$

or if $a=0$, $b=\infty$, in the form

$$x^3=\lambda.$$

When x describes a conic C_2, J_2 is another conic. The positions of the branch-points and coincidences are determined by the relative positions of C_2, J_2. Numerous special applications are given.

In (409) Em. Weyr studies the I_4 on a C_2; the involution curve is now of class 3. When the given involution is expressed by tangents to a conic, the involution curve is of order 3. I_4 is completely defined by two of its groups. Another form of I_4 is given by A. Ameseder(3) by means of a pencil of planes and an I_2 of planes tangent to a quadric cone. The locus is a ruled surface of order 4, having the axis of the pencil and a conic in the involution plane for double curves. The treatment of further development is synthetic in A. Ameseder(4). A systematic treatment of cubic involutions is given by C. Le Paige(210), with particular reference to cross-ratios and polar groups. Particular metric involutions of order n are treated by G. Fouret(123). The properties of I_2, I_3, …, I_8 on conics, pencils of conics, cubics and pencils of cubics are given by B. D. Fraser(126), in a review of Kötter's synthetic geometry of algebraic curves.

An I_4 on C_4 is discussed by J. de Vries(379), including the derivation of the 63 systems of contact C_2 with a non-singular C_4. All the I_4 on C_4

determined by pencils of curves have an involution curve of class 6 and have 12 coincidences. See A. Ameseder(7). Unless relations exist among the coefficients, the I_s on C_g have $2(g+s-1)$ coincidences. L. Berzolari(32) treats cubic involutions and their conjugates on a nodal cubic curve by means of binary forms. Problems in determinations of coincidences were solved by J. de Vries(371) for various particular cases.

A rational involution I_n is uniquely determined when three partial groups containing n_1, n_2, n_3 points, where $n_1+n_2+n_3=2n+1$, are given. An application of involutions to certain transformations of plane curves was made by M. D'Ocagne(109). O. Tognoli(354) considers two rational involutions I_{nm}, $I'_{nm'}$ on a fixed C_n by means of two pencils $|C_m|$, $|C_{m'}|$. When the latter are projective, there are $n(m+m')$ coincidences. The envelope of lines joining corresponding points and the dual are discussed.

4. *Involutions of Higher Dimensions.*—If $\phi_i(x)$, $i=1, 2, \ldots, s+1$ are $s+1$ polynomials, then $\sum\limits_{i=1}^{s+1}\lambda_i\phi_i(x)=0$ defines an involution of order n and dimension s, designated by the symbol I_n^s. The I_4^3 on straight lines were employed by C. Le Paige(213) to find the sixth points of C_2, C_3 when 5 points are given on the latter, and to construct a C_4 through 14 points. A general discussion of the properties of several I_n on the same carrier are given by Em. Weyr(388).

It was shown by G. Castelnuovo(58) that on any elliptic curve exist $\infty^1 I_n^{n-1}$, each of which has n^2 n-fold points. The same author(55) determines the number N of rational involutions of order n and dimension q which exist on a curve of given genus p. If $p-1-(n-q)=Q$, then

$$N=\frac{1!\,2!\ldots q!\,1!\,2!\ldots Q!\,p!}{1!\,2!\ldots(q+Q+1)!}.$$

The number of involutions g_n^q on a rational curve having p neutral couples is N.

Em. Weyr(389) proved that I_n^{n-1} on a carrier of genus 1 has n^2 n-fold elements.

A number of derived I_n^{n-1} on elliptic curves were also obtained by Em. Weyr(403) and (402), defined by the intersections of an elliptic C_3 with groups of curves. Let C_2 meet C_3 in $0_1 0_2 0_3 a_1 a_2 a_3$. The three C_2 through $0_1 0_2 0_3$ touching C_3 in a_1, a_2, a_3 respectively, meet C_3 in a'_1, a'_2, a'_3. The six points $a_1 a_2 a_3 a'_1 a'_2 a'_3$ lie on a conic. If the conics having five point contact with C_3 at 0_1, 0_2, 0_3, a_1, a_2, a_3 respectively, meet C_3 again in $0'_1, \ldots, a'_3$ then these latter six points also lie on a conic. If one of the four double elements of an I_2 be considered as a triple element of an I_3, the other three form a triad. If one of the k^2 k-fold elements of an

I_k^{k-1} is chosen as a p-fold element of an I_p^{p-1} then the involution determined by this latter group, is one of the involutions by means of which I_k^{k-1} is transformed into itself.

G. Castelnuovo(⁶⁰) showed that an involution of order n and dimension $r>1$ belonging to an algebraic curve is always linear except (possibly) when each of its groups is made up of r arbitrary groups of an I_k^1, where $k=\frac{n}{r}$.

G. Humbert(¹⁵⁵) showed that the I_n^k on curves of genus p were of two kinds when $k>1$

(a) If $k=n$, then $p=0$
(b) If $1<k<n$, then I_n^k is rational.

For curves with arbitrary moduli, no other involutions exist. If $k=1$, the g_n^1 is either rational or, if irrational, is of genus $\omega<p$, with $2(p-1)-2n(\omega-1)$ branch points. The irrational γ_n^1 can exist only on singular curves, that is, with certain relations among their coefficients. These results were known before but new proofs were given by Humbert. Simpler proofs were given by M. De Franchis(¹⁰²) and (¹⁰³). It was shown by G. Kohn(¹⁸⁴) that the groups of $n+1$ elements formed by one element of a rational carrier and the n elements corresponding to it in a symmetric (n, n) correspondence belong to an $I^n{}_{n+1}$ when n is odd.

5. *Rational Involutions on Curves of Genus* $p>0$.—H. M. Jeffery(¹⁶²) and (¹⁶³) showed that if one group of I_4 on C_3, $p=1$ is known, the involution may be constructed. C. Küpper(²⁰³) discusses an I_n on C_3.

A. Cayley(⁷¹) observed that (1, 1) correspondences may become $(k, 1)$ by altering the definition. For example, the tangent to the common evolute of an ellipse and its parallel is normal to the parallel at two points when the latter is not composite, thus defining a (1, 2) correspondence.

C. Bobek(³⁶), (³⁸) and (³⁵) studied various properties of curves having an I_2, or a g_2^1, that is, hyperelliptic. These were extended to curves possessing a g_3^1 in (³⁹ᵃ). The groups of g_3 on C_m of genus p are cut out by curves ϕ_{m-4} of order $m-4$, which pass through the double points of C_m. $p\geqq\frac{m+2}{2}$. If C_m has two g_3^1, $p\leqq 4$. If $p>m-2$, each group G_3 lies on a straight line. If C_m has a ϕ_{m-5} which pass through the double points, g_3^1 is impossible. If more than $m-p-2$ groups G_3 are composed of collinear points all must be. See also C. Küpper(²⁰⁶). This theorem was generalized in (¹⁹⁹) to G_k^1 on C_m with a point of multiplicity $m-k$. J. de Vries(³⁷²) studied the g_2^1 on the C_5 with 4 double points. F. Amodeo(¹⁰) and (¹¹) considers g_k^1 and divides them into first and second species according as the points of each group are or are not collinear. Ways of obtaining them are obtained by C. Küpper(200).

F. Amodeo(14) and (12) defines an I_k^1 to be of species s if the lines joining points of each G_k by pairs envelop a curve of class s (involution curve). C. Küpper(194) shows that if a C_m of genus p contains a g_k^1 it cannot have adjoints ϕ_{m-k-2}. Every adjoint through P contains the group G_k of g_k^1 through P. If p is a maximum, C_m contains a point of multiplicity $m-k$ and each G_k is composed of collinear points. F. Amodeo(15) obtains the class of the involution curve of g_k^1.

Em. Weyr(401) applied the same methods to rational involutions on curves of genus 1. An I_m^1 and an I_n^1 have $(m-1)(n-1)-1$ common pairs of elements. On a curve of genus p, the number is

$$(m-1)(n-1)-p.$$

The number of double elements of I_m^1 is $2(m+p-1)$. Every I_4^2 on a curve of genus 1 can be represented by the plane sections of a space quartic curve through a fixed point, and every I_4^3 by all the plane sections of C_4. These results are employed by Em. Weyr(402) to determine Steiner polygons.

An I_n^{n-1} is uniquely fixed by one of its groups, Em. Weyr(398). Previously proven for $n=2$; see (395), (394) and (396).

The two fundamental points of a Steiner polygon of $2n$ vertices on C_3, $p=1$, are n-fold points of I_n^{n-1}, (389).

The method was developed into a systematic symbolical calculus by Em. Weyr(404). The same procedure may be expressed in terms of the parametric representation of points on a curve of genus 1 by means of elliptic functions. [E. Czuber(99).]

A C_4 with one double point is hyperelliptic, $p=2$. The characteristic g_2^1 is defined by the pencil of lines through the node. The transformation of interchanging the points of each pair is a perspective quadratic inversion. The conic of invariant points passes through the points of contact of the six tangents from the double point. E. Bertini(29); H. Picquet(271); E. Caporali(54) and (53). The conic is called the Bertini conic of the nodal C_4. If C_4 has two double points, a Bertini conic is associated with each as vertex of a perspective quadratic inversion, and passes through the other double point. E. Ciani(89). In case the C_4 of $p=2$ has inflexional tangents at the node the Bertini conic is composite and the curve is invariant under harmonic homology. These projective inversions are all particular cases of the Jonquières perspective involution.

It was proved by H. A. Schwarz(300) that a point on any curve of genus 2 can be expressed rationally in terms of λ and $\sqrt{f_6(\lambda)}$.

J. de Vries(378) studies the I_2 on the hyperelliptic C_5, $p=2$, by means of the pencil of conics through the nodes, and the I_3 cut on C_5 by the lines each joining a pair of conjugates in I_2. The I_3 is cut by $|C_3|$ through the nodes and one pair in I_2.

L. Zajaczkowski(415) applies the fundamental elliptic involution to generate certain ruled surfaces.

J. de Vries(375) corrects a theorem of Em. Weyr on Poncelet polygons. See " Beiträge zur Curvenlehre " of the latter.

6. *Cayley-Brill Principle of Correspondence.*—See Pascal(263). The generalization of the Chasles principle of correspondence to curves of genus greater than zero was first made by A. Cayley(66). Particular cases were recognized by M. Chasles(82) and by E. de Jonquières(165). The number of coincidences of an (a, b) correspondence on a curve of genus p is $a+b+2\gamma p$. The proof was not complete, but the result was confirmed by four specific examples. See also A. Cayley(168) and (169). The first proofs were given by A. Brill(40), (43), (46) and (47). Other proofs were given by Junker(170); by C. Bobek(37); by considerations in hyperspace by C. Segre(307); see § 12; by the enumerative method by H. Schubert(295); by H. G. Zeuthen(419), generalized to include the proper evaluation of multiple coincidences, in (423). This paper also contains a historical résumé of the preceding literature. A proof by F. Lindemann by the aid of abelian integrals is found in (216), and amplified in the Vorlesungen über Geometrie. Another transcendental proof is given by A. Hurwitz(159). A proof using more elementary methods was found by F. Severi(323). A new transcendental proof appears incidentally in connection with the study of Abel's theorem in a paper by O. Chisini(87).

By means of the Cayley-Brill principle F. Severi, in (322), obtained a large number of properties of the characteristics of algebraic curves in S_r.

In the formula $u=a+b+2\gamma p$ the coefficient γ, called the *valence* (German Wertigkeit) is an integer, positive or negative, or zero. When positive or zero it indicates the number of times the point itself must be added to the group of its corresponding points, in order that the enlarged group shall belong to a linear series.

Every positive valence correspondence between the points (x) and (y) on a curve $f(x)=f(y)=0$ can be defined by one equation $\phi(x, y)=0$ in addition to the above equations. For curves with arbitrary coefficients (moduli) only valence correspondences exist, but the valence may be negative. See also G. Scorza(301).

Two or more correspondences (a_1, b_1), (a_2, b_2) ... on the same curve are said to be dependent if integers λ_1, λ_2, ... can be found such that the group of b_1 points taken λ_1 times plus the b_2 points in the second taken λ_2 times ... move in a linear series as the point of departure describes the given curve.

A correspondence and its inverse have the same valence. On any algebraic curve correspondences exist with given valence, either positive, zero or negative.

The sum of two correspondences with valences γ_1, γ_2 is a correspondence with the valence $\gamma_1+\gamma_2$. The product of two correspondences with valences γ_1, γ_2 is a correspondence with valence $-\gamma_1\gamma_2$. For the rational determination of valence, see M. Shibayama(326).

7. *Singular Correspondences.*—The general determination of all possible algebraic correspondence on algebraic curves was obtained by A. Hurwitz(159), (160), (161). Earlier attempts had been made by F. Lindemann(216). The methods of Hurwitz are treated at length in Klein-Fricke(171) and by H. F. Baker(20). See H. Burkhardt(51).

The correspondences with valence are the only ones which exist on curves with general moduli, but the valence can be negative only for correspondences which are rationally composite, and can therefore be completely defined by means of two auxiliary equations. For particular moduli other correspondences, called *singular* correspondences, exist. Examples of such correspondences were pointed out by F. Klein(175). The particular case of the harmonic cubic was found by F. Klein(174). This with that of the equianharmonic cubic was further developed and all possible correspondences on elliptic curves were found by C. Segre(315). See also F. Amodeo(8).

Hurwitz showed that in every case two equations are sufficient to determine any algebraic correspondence completely. $f(x)=f(y)=0$; $\phi_1(x|y)=0$, $\phi_2(x|y)=0$. The details of a singular elliptic correspondence were given by F. Severi(323). The theory is intimately connected with that of complex multiplication. See L. Bertini(24). G. Tafani(350) gives alternate proofs.

The general theory of singular correspondences is given in Chapter 15.

8. *Zeuthen's Formula.*—Given an (a, b) correspondence between two curves C, C' of genera p, p' respectively. Let η be the number of branch points on C, such that for each of them two of the b corresponding points on C' coincide, and η' the number of branch points on C'. Then

$$\eta-\eta'=2a(p'-1)-2b(p-1),$$

H. G. Zeuthen(421). When $a=1$ or $b=1$, the groups of points on C', respectively C, describe an involution. If $a=b=1$ both η and η' are zero and we have a proof that two curves in (1, 1) correspondence must be of the same genus.

The η, η' points were obtained on the hypothesis that for each branch point only two of the corresponding points are coincident. If for a point A, ν' coincide at A', and similarly if for a point B' ν points B coincide on C, then the first member of the preceding equation is to be replaced by $\Sigma(\nu'-\nu)$. This was found by G. Halphen in (133) and (139). Other proofs are given by H. G. Zeuthen(416); by W. Weiss(383) with applications to ruled surfaces; C. Segre(307); E. Bertini(25). By other methods the theorem was found by X. Stouff(339); R. de Paolis(262). A corollary to the first theorem was given by H. Weber(381). If $p=p'$ and $a=1$, then b must also be 1.

9. *Correspondence on a Cubic Curve.*—The early applications of correspondence and residuation were made to cubic curves. J. J. Sylvester(345) considers μ points on C_3; pass a C_v through them, intersecting in $3v-\mu$ residual points. Through these pass a $C_{v'}$.... If $\mu=3i+1$ the final residual consists of a single point, the position of which is independent of the curves C_v, $C_{v'}$, ...

In (346) Sylvester shows how to draw a tangent to C_3 at a point P on it. Let A, B, be any two points of inflexion; let AP meet C_3 in 2, B_2 in 3, P_3 in 4, A_4 in 5, B_5 in Q. The line PQ touches C_3 at P.

In (347) Sylvester showed that the 27 sextatic points of C_3 may be resolved into 3 sets of 9 each; the line joining any two in the same set passes through a point of inflexion.

W. Schur(299) gives a method of generating cubic curves by pencils of conics.

Em. Weyr(395) generalized his treatment of correspondences on a general cubic.

Further properties of complete intersections with a cubic are given by J. J. Sylvester(348). The generation of cubics by symmetric multiple correspondences is given by Em. Weyr(391) and by J. de Vries(373). Sylvester(349) gives another illustration of residuation. See F. Morley(241). A. S. Hart(146) and (145) gives constructions for the ninth point of intersection of a pencil of cubics through 8 given ones. Other constructions were given by A. Cayley(62) and by E. B. Elliott(111). W. P. Milne(233), (234) generates a cubic by apolar pencils of lines(231).

F. G. Taylor(351) applies quadratic inversion to pencils of cubics. See also W. P. Milne and D. G. Taylor(235).

10. *Generation of Cubic Curves.*—Em. Weyr(390) gives properties of tangentials in generating a C_3. C. Küpper(195) considers a cubic as the envelope of a system of conics through two fixed points. The conics which osculate a cubic are discussed by J. Steiner(336); by H. Durege(110).

Em. Weyr(406) and (407) and (391) generates cubic curves by means of symmetric (2, 2) correspondences of tangents to two fixed conics.

The Steiner polygons inscribed in non-singular cubics, that is, one having its odd sides all pass through one fixed point on the curve, and its even sides through another, have the property that if one such polygon exists for a given pair of points P, P', then one exists starting from any point of the curve as vertex. J. Steiner(336); A. Clebsch(92); C. Küpper(192); P. H. Schoute(292), V. Martinetti(227).

11. *Quartic Curves.*—A. Cayley(61) outlined the parametric representation of plane curves. If $p=0$, a point is expressed by rational functions of a parameter λ; if of genus 1, by two parameters λ, μ rationally, connected by $(\lambda^2, \mu^2)=0$; if of genus 2, by λ, μ rationally, connected by $(\lambda^3, \mu^2)=0$, and an outline of further cases. Cayley supposed that the minimum order of a curve of genus p is $p+2$. He(65) discussed an apparent paradox in the parametric representation when radicals are involved.

M. Azzarelli(19) obtains a particular quartic by means of reciprocal radii. Cayley proved that at a point of undulation a curve is touched by its Hessian; U. Masoni(228) proves that the first polar of a point of undulation on C_4 consists of a tangent and of a conic and conversely. G. Humbert(156) expressed the coordinates of a binodal quartic parametrically in terms of elliptic theta-functions.

C. Le Paige in (211) and (209) constructs C_4 through 14 points.

F. Purser and Haughton(378) propose the question of constructing families of C_4 through restricted basis points and finding other basis points.

A. Ameseder(7) considers involutions formed by conics through the nodes of an elliptic quartic and obtains various configurations, including conditions for inscribed polygons.

L. Gegenbauer(128) considers the I_4^3 formed by the planes of space on an elliptic space quartic curve.

W. Wirtinger(414) discusses the relations between the nodal quartic curve and the Kummer surface. Every such quartic can be generated by two projective pencils of conics with one common basis point.

E. Caporali(53); C. Segre(306); E. Ciani(90) give various derivations of the quartic which possesses a series of apolar conics conjugate to the line meeting C_4 in 4 equianharmonic points.

The nodal quartic was discussed by H. E. Timerding(353) as the projection of a C_6 of S_3 of type (2, 3) on a quadric surface.

M. Antscherl(16) continuing the procedure given by C. F. Geiser(129) shows that the general tangent cone from a point on a cubic surface contains as plane sections the general C_4. The 27 pencils of planes through the lines of the surface give twenty-seven systems of contact conics of C_4. The other thirty-six systems of contact conics arise from the pencils of space cubic curves on F_3 through the point of contact.

W. P. Milne(230) shows that any C_4 can be generated by two projective pencils of conics. A conic of one pencil is apolar to all the conics of the second pencil. In (232) the same author generates a C_4 as the apolar locus of two tetrads of points on a conic.

C. M. Hebbert(148) studies certain quartic curves by means of the transformation $z' = \frac{1}{z^2}$ in the complex plane. Isotropic coordinates $z, \bar{z}$ are employed. See also E. Beltrami(22); G. Cesàro(72); Perma(268); Levy(215).

W. V. der Woude(366) obtains a hyperelliptic C_4 having ∞^1 configurations of Desargues $(10_3, 10_3)$.

A. Harnack(144); Servais(316); R. Montessus(240) give a theory of the relation between elliptic space quartics, plane binodal quartics and plane cubic curves by means of elliptic functions.

12. *Contact Conics of Quartics.*—See (191). C. Küpper(201) considers the system of conics through the nodes of an elliptic C_4, and touching it in two other points.

A. Brill(42) studies curves having three point contact with a given curve. H. Krey(186) applied the same method to start the study of those having four-point contact; W. Spottiswoode(335) and (334), (333) extended the theory of four-point contact.

Härtenberger(147) generates C_4 by means of two projective pencils of conics; A. Ameseder(5). Every C_2 through the four points of contact of a contact C_2 of C_4 meets C_4 in another group of four points of contact. O. Hesse(150), A. Ameseder(6) applies the same method to the discussion of various particular cases.

C. Aronhold(18) shows that 7 of the 28 bitangents of a non-singular C_4 can be chosen at will, and that all the others can be expressed as functions of these. The method used is the dual of the Geiser involution.

G. Kohn(183). Given six pairs of bitangents a_i, b_i belonging to the same one-parameter system of contact conics of C_4, every C_2 touching five of them touches the other and is an involution conic of the system.

M. Noether(258) studies the contact conics of C_4 by means of the (1, 2) correspondence between two planes which is associated with the Geiser involution. Previous studies in this direction had been made by A. Clebsch(93); M. Noether(245); R. de Paolis(261).

G. Kohn(182) obtained relations between different systems of contact conics. Two different systems have the same involution conic.

13. *The Brill-Noether Theory. The Fundamental Theorem.*—It was first shown by M. Noether(250) that if $f=0$, $\phi=0$ are the equations of algebraic curves having P_i for r_i-fold and s_i-fold points respectively, then if $F=0$ represents a curve having P_i to multiplicity r_i+s_i-1, A, B, can always be found so that $F=0$ can be put in the form $F=Af+B\phi$ and conversely. For its application to cubic curves see J. J. Sylvester(344) and A. Cayley(64).

M. Noether(249) further discussed the general case, but did not complete the proof. See also (251), (256), (257); G. Halphen(140); A. Voss(370); W. Weiss(384); C. A. Scott(302); F. Severi(319). For reference to numerous other proofs from different standpoints, see (31), E. Löffler(218) and (217) and E. Lasker(208).

14. *Linear Series of Point Groups.*—Given $f=0$ and a linear system of curves $\Sigma\lambda_i\phi_i=0$. The latter cut from the former a linear series g_n^r in which n is the order, or number of variable points cut from $f=0$ by each curve, and r represents the number of dimensions or of arbitrary constants (degrees of freedom) of the system $\Sigma\lambda_i\phi_i=0$, independent of $f=0$. A. Brill and M. Noether(44) and (41). The same idea had been presented transcendentally by B. Riemann(282), by A. Clebsch and P. Gordan(91); and employed further by M. Noether(246) and (247). For other methods see (31).

If the curves $\phi_i=0$ are *adjoints,* having each s-fold point of $f=0$ to multiplicity $s-1$, then either the curves of the system through one point of $f=0$ will not have another variable point in common on $f=0$, or the same property will hold for every point of f and the points describe an *involution,* as one basis point describes the curve.

If $f=0$ is of order n and genus p, then the adjoints $\Sigma\lambda_i\phi_i=0$ of order $n-3$ cut from $f=0$ the *canonical* series g_{2p-2}^{p-1}.

If g_n^r satisfies the relation $r\geqq n-p+1$, the series is called a ***special*** series.

If the points on $f=0$ are arranged in groups of n such that r arbitrary points determine a group, the system is called an *involution* of order n and dimension r. If $p=0$ all series are involutions. If $p>0$, g_n^1 may be *irrational,* thus not contained in a one-dimensional linear series. If the canonical series $g_{2(p-1)}^{p-1}$ is compounded with an involution, the latter must be of order 2. Any curve having a linear g_2^1 is called *hyperelliptic.* G. Castelnuovo(56); F. Amodeo(9); R. Torelli(358); J. D. de Porte(277). A more geometric development of the Brill-Noether theory was given by O. Tognoli(355) and (356). See also A. Brill(48).

C. Weltzien(335) discussed the parametric representation of hyperelliptic curves in the form

$$Px_i = (a_it+b_i)\sqrt{E(t)} + (c_it+d_i)\sqrt{F(t)}$$

in which a_i ... are constants and $E(t)$, $F(t)$ are fixed polynomials of degree $p+1$. The curves are of order $p+3$ and of genus p.

If $r>1$, $g_n{}^r$ is always rational except for series compounded of r groups of an irrational one-dimensional involution. G. Castelnuovo(60); G. Humbert(155) first proved this theorem by means of abelian integrals. Geometric proofs are given by M. De Franchis(103) and (102), by F. Severi(321) and (323) and for $p=1$, by G. Castelnuovo(58). F. Enriques(113) proved that every $\gamma_n{}^r (r>1)$ is either rational or contained in a rational series $g_n{}^s$, $s>r$. See also F. Severi(318); R. Torelli(361).

If the points of any group G_n cut from $f=0$ by an adjoint $\phi=0$ are divided into two parts, each is called the residual of the other.

If G_n and G'_n are residuals as to G_q, and G_n is a residual of G'_q, then G'_n is a residual of G_q. If G_n is fixed, all the coresidual groups $G_qG'_q$ are cut out by the same linear series, and are called *equivalent*. For examples of residuation see E. Study(340).

A series $g_n{}^r$ is called complete if it is not contained in a series $g_n{}^s$, $s>r$. The complete series to which any group or any partial series belongs is unique. The order, dimension, and quality of completeness are all invariant under birational transformation.

E. Bertini(26) showed that the canonical series g^{p-1}_{2p-2} is the only one of its order and dimension. It is simple if the curve does not possess a linear $g_2{}^1$, is complete, and has no basis points. C. Segre(307) gives a systematic development of the Brill-Noether theory by means of projections from hyperspace. See F. S. Macaulay(225). See also C. A. Scott(303); F. Macaulay(226); F. Amodeo(13) shows that if C_m has adjoints of order $m-3-a$ it defines special series $g_n{}^r$, $n \leqq 2p-2-am$, $r=p-1-\left[am-\frac{x}{2}(a+3)\right]+\rho$. F. Hardcastle(142) gives an extensive bibliography covering 1896 and previous years. F. Severi(325) gives a new demonstration of the Brill-Noether theory.

If a group G_n on f determines a complete series $g_n{}^r$ and belongs to $r'+1$ linearly independent adjoints ϕ of order $n-3$, then $r'=p-n+r-1$. This is the *Riemann-Roch* theorem, which may also be expressed in the form:

Every special complete series $g_n{}^r$ has, as residual to the canonical series g^{p-1}_{2p-2}, another special complete series $g^{r'}_{n'}$ wherein $n+n'=2p-2$, $n-n'=2(r-r')$. See A. Brill and M. Noether(44).

15. *Normal Curves.*—Let the canonical series g_{2p-2}^{p-1} be cut from $f=0$ by $\sum_{i=1}^{p} \lambda_i \phi_i = 0$. If we put $\rho y_i = \phi_i(x)$, and regard $y_i, \ldots, y_p$ as the homogeneous coordinates of a point in a linear space S_{p-1} of $p-1$ dimensions, than as (x) describes the curve $f=0$, (y) will describe a curve in S_{p-1} called the *canonical* curve of $f=0$. It is of order $2p-2$ and of genus p. If $f=0$ has no g_2^1, this curve is simple and has no multiple points.

The birational transformations of $f=0$ are now expressed in terms of projective transformations of the canonical curve. This concept originated with B. Riemann(281). See also H. Weber(382); L. Krauss(185); M. Noether(253); F. Klein(176), (172) and (173); and particularly C. Segre(307). By projecting this curve from an S_{p-3} determined by $p-2$ arbitrary points on it, we obtain plane curves birationally equivalent to $f=0$, of order $p+2$, with a point of multiplicity $\frac{1}{2}(p+2)$ and having $\frac{p}{4}(p-4)$ double points if p is even, or of order $p+3$, having one point of multiplicity $\frac{1}{2}(p+3)$ and $\frac{1}{4}(p-1)^2$ double points if p is odd. These were called normal curves by B. Riemann(282).

By means of a net of adjoints through $p-3$ arbitrary points on $f=0$, Clebsch-Gordan(91) obtained a curve of order $p+1$ with $\frac{p}{2}(p-3)$ double points. See also C. A. Scott(304). By projection from a proper S_{p-3} which contains the maximum number of points, the plane curve can be reduced to one of order $p-\pi+2$ having $\frac{1}{2}(p-\pi+1)(p-\pi)-P$ double points, π being the largest multiple of 3 contained in p.

The normal forms for $p=5$, 6, 7 were obtained by M. Noether(255) by means of the quadratic identities among the adjoints.

For $p=2$, 3, 4, 5, 6 see A. Wiman(413). C. E. Pengra(267). By means of the quadratic relations among the ϕ's, see A. Van Benschoten(365) for $p=4$; J. V. McKelvey(229) for $p=5$; V. Snyder(329) for $p=6$.

16. *Reduction to Forms with Double Points.*—It was proved by M. Noether(248) that any plane algebraic curve may, by Cremona transformation of the plane, be reduced to a curve having multiple points with distinct tangents at each. W. K. Clifford(95) studied the curves C_n normal in S_n or S_{n-1}. The former are rational and the latter can be projected into plane cubics. A C_n in S_{n-1} has n^2 points at which the osculating S_{n-2} has n-point contact.

C. Küpper(196), using the same method as M. Noether(251), showed that C_m in S_2 can be the projection of a C_m of S_3 only when it has at least $\frac{(m-1)^2}{4}$, $\frac{m(m-2)}{4}$ double points respectively, according as m

is odd or even. K. Bobek(39) discussed a curve C_m, in S_3, genus p, $m-3>p>1$ having an $(m-2)$-fold secant. It is hyperelliptic and lies on a ruled surface of order $m-p$ on which the multiple secant is of multiplicity $m-p-1$. E. Bertini(28) gave a proof that any algebraic plane curve can be transformed into one having no higher singularities than distinct double points. For further proofs, see the following:

K. Hensel and G. Landsberg(149); G. Landsberg(207); L. W. Thomé (352); G. Halphen(138) and (134); A. R. Forsyth(122); P. Appell and E. Goursat(17); L. Kronecker(190) and (189); G. Simart(327); E. Vessiot(308); H. Poincaré(276); P. del Pezzo(269); M. Pieri(273); E. Vessiot(369); C. Segre(308); B. Levi(214); G. A. Bliss(34); F. Enriques and O. Chisini(112).

C. Küpper(197) showed that if a plane non-singular C_n, $n>3$ is transformed into another C_n birationally, the transformation must be a collineation. Such curves cannot be projections of space curves of the same order. E. Bertini(27) gave another proof by means of (1, 2) plane transformations.

This theorem was generalized by V. Snyder(328) as follows: The largest number of distinct double points which a curve of order n may have without being birationally transformable into another curve of the same order except by collineation is two less than the minimum number of double points which a space curve of the same order can have $(n>7)$. See also V. Snyder(330).

A. Del Re(105) proved that any plane algebraic curve can be transformed birationally into a curve C_m of S_5 having no singularities.

V. Snyder(331) proved that plane algebraic curves of every order n and of any genus p, $0 \leqq p \leqq \frac{(n-1)(n-2)}{2}$ exist, having distinct double points and with distinct tangents at each.

J. V. Nagy(243) discussed the restrictions of the reducibility from the arithmetic stand-point.

F. Severi(320) showed that every general family of curves of order n, genus p, in S_{n-p}, includes limiting curves composed of an $(n-p)$-sided polygon having $n-p-1$ nodes, together with p arbitrary chords. A family of plane C_n with not more than $\frac{(n-1)(n-2)}{2}$ double points includes all possible n sides, and each C_n contains composite ones composed of all possible C_{n-1} and a line. See also F. Severi [(317) Anhang G.].

17. *Particular Curves of Order* $n>4$.—For the general theory see G. Loria(220). Only those curves will be considered which are connected with some definite transformation or involution. The space C_5, $p=1$

was considered by Em. Weyr(396), (397) and (398). It has an I_3^1 of trisecants forming a quintic ruled surface of genus 1 on which the given curve is double. Ruled cubic surfaces through the curve have tangent planes that describe an I_3^2 which is uniquely fixed by one triad. Every I_4^3 on C_5 can be cut by quadrics through two trisecants. D. Montesano(237). C. Segre(310). Em. Weyr(399) and (400) employs similar methods to obtain involutions on the space sextic curve of genus 1.

H. Schröter(294) considered the locus of the eight points of intersection of three conjugate pairs of surfaces given by three involutions of planes. It is a C_7 having the axes for trisecants. See also P. H. Schoute(293).

D. Montesano(238) found all involutorial birational transformations in S_3 such that lines joining pairs of corresponding points belong to a linear complex. The most general one is determined by a C_{10} of genus 11 which has the property that the $\infty^3 I_{10}^3$ cut on it by the planes of space all lie on cubic curves. The curve does not lie on a cubic surface. A particular case gives C_7 of genus 5 which also appears in the study of bilinear complexes of conics. Every C_7 of genus 5 in S_3 has one point O having the property that the six residual points in which any plane through O meets it, lie on a conic; thus defining a rational I_6^2. Each trisecant meets just one other at a point not on C_7, and the plane determined by them passes through O. R. Sturm(340a) and D. Montesano(239). Other properties of particular forms of the curve are given by Th. Reye(280), G. Koenigs(181); G. Humbert(157); D. Montesano(236).

18. *Irrational Involutions.*—Given two curves C, C', in $(k, 1)$ correspondence, such that to a point on C corresponds one point on C', but to a point of C' correspond k points on C. Then C is said to contain an involution γ_k^1 of the genus p' of C'. If $p'>0$, C is said to contain an irrational involution of order k and genus p'. If p is the genus of C it follows from the Zeuthen theorem that $p' \not< p$.

A γ_4^1 on C_6 of genus 5 was given by A. Cayley(63); it is of index 5.

If $p'=0$, the notation g_k^1 will be used. In this case the k points of each group G_k of g_k^1 belong to a linear series. Any curve containing a g_2^1 is hyperelliptic.

L. Fuchs(127) published the theorem that the necessary and sufficient condition that an algebraic curve $f(x, y)=0$ is invariant under an involutorial birational transformation is that it can be transformed birationally into the form $\psi(x)y^2=\phi(x)$, or that it is hyperelliptic, but it was pointed out by A. Hurwitz, that this is incorrect. It is true if the curve does not have a γ_2^1.

C. Küpper and C. Bobek(205) determine the forms of hyperelliptic curves of order $3n$ which belong to the Geiser or the Bertini involutions.

C. Segre(315) determines all possible involutions on general and singular elliptic curves. On the former the only involutorial birational transformations are the central g_2^1 associated with every point of the curve, and the three γ_2^1 which can be expressed as the product of two central projections at any points having a common first tangential. For harmonic curves, cyclic involutions I_4^1 also appear, and for equianharmonic curves cyclic involutions of order 6. This is intimately connected with the theory of complex multiplication of elliptic functions. See G. Greenhill(132); G. Halphen(135); A. Hurwitz(159) and (160); (158) and (161); F. Klein(174); C. Segre(310).

The involutions can be expressed in terms of the parameter u of an elliptic function by $u'=\pm u+C$ in the general case, and by $u'=\pm \omega u+C$, $u'=\pm \omega^2 u+C$, where $\omega^2+\omega+1=0$, in the equianharmonic case, and by $u'=\pm iu+C$ in the harmonic case. The earlier literature contained theorems which were stated incorrectly. See A. Harnack(143) and (144); Em. Weyr(393) and (394); C. Küpper(192).

Associated with every (n, n') correspondence between two curves of genera p, p' is an intermediary curve of genus

$$P=2(p+1)+n'=2(p'-1)+n,$$

n' and n being the number of branch points on C', C, on which exist two involutions, one of order n and genus p', the other of order n', genus p. By means of this theorem V. Snyder and F. R. Sharpe(332) showed how (n, n') correspondences could be constructed, which can be expressed by two auxiliary equations, but not by one.

G. Fontene(121) considered a composite (m, n) correspondence on a given curve by means of a connex.

G. Castelnuovo(56) showed that if a curve of genus p contained an irrational involution γ_ν^1 of genus π and also a rational series $g_n^{\nu-1}$, the second not containing the first, then there exist $y=(n-\nu+1)-p+\nu\pi$ groups of the first which belong to the second. If $n-\nu<p-\nu\pi$, then a group of the former series imposes at most $\nu-1$ conditions on a group of the second in order to be contained in it (generalization of the Riemann-Roch theorem). If on the canonical C_{2p-2} *of* S_{p-1} exists a γ_ν^1 of genus π, through $\eta=2(p-1)-2\nu(\pi-1)$ coincidences passes an $S_{p-\pi+1}$ from which the curve can be projected into $C_{2\pi-2}$ of $S_{\pi-1}$ counted ν-fold. F. Amodeo(9); C. Segre(314); C. Küpper(198), determines a number of conditions on point-groups in the plane such that an C_m through them has an excess on the number of degrees of freedom left. R. Torelli(358) applied these results to hyperelliptic curves.

The theorem of Lüroth(224) that rational curves cannot contain irrational involutions was extended by E. Goursat(131) to apply to curves of genus p as follows: if the coordinates of a point on an algebraic curve C are rational functions of those of C', the genus of C is at most equal to that of C'.

R. Torelli(362) showed that if on C of genus p there exists γ_μ^1 of index ν, then the number z of groups containing $r+1$ points of a g_n^r is given by $z=\binom{\mu-1}{r}\nu n-\frac{1}{2}\binom{\mu-2}{r-1}d$, in which d is the number of double points of γ_μ^1. In (357) R. Torelli showed that if a curve of genus $p=2$ contains a γ_ν^1, it also contains a second irrational involution. If C is of genus 3 and contains a γ_3^1 of $\pi=3$, it also contains an elliptic involution.

G. Castelnuovo(57) obtained the condition that a γ_n^1 of index ν is contained totally in a linear series. The index means the number of groups of γ_n^1 through a given point. Such a series can have at most $2\nu(n+p-1)$ coincidences. The necessary and sufficient condition that the series is contained in a linear series is that this number is actually attained. This was generalized by R. Torelli(360). If a γ_n^1 of index ν is so constituted that the totality of the ν groups G_n of the series through a point P (including P itself counted ν times) describes a linear series as P describes the curve, then the given series is linear. This was first proved by transcendental means by F. Severi(318), see also (324). Also proved by G. Castelnuovo by an application of the theorem just cited.

If a curve C of genus p contains an involution γ_2^1 of genus π C depends on $2p-\pi-1$ moduli. C. Segre(307).

R. Torelli(359) obtains further relations among curves with singular moduli which contain irrational γ_n^1. Some of these are extended to series of higher dimensions by E. S. Allen(2), and (1); J. V. de Porte(277).

Every curve of genus 3 representable on a double curve of genus 2 is hyperelliptic. F. Enriques(114).

O. Chisini(86). Given two plane curves $f_1(x, y)=0$, $f_2(x, y)=0$ in (2, 2) point correspondence. Three cases are possible:

(a) f_1, f_2 are birationally equivalent,
(b) f_1, f_2 have the same γ_2^1,
(c) f_1, f_2 are conjugate of each other.

If the correspondence is not periodic, both f_1 and f_2 are rational. A curve ϕ can be constructed in (1, 2) correspondence with each f_1, f_2. ϕ may be of genus 0, 1, 2, ..., and (2, 2) may be of period 2^r.

19. *Relations to Enumerative Geometry.*—J. Steiner(337) treated the problem of generating plane algebraic curves of order $m+n$ by means

of projective pencils of orders m and n, by a method previously employed to generate cubics and quartics by M. Chasles(76) and (77). That an involution is associated with a (1, 2) correspondence was shown in (78), generalized in (79) by the same author. The concept of an involution of higher order was given by E. de Jonquières(164.1). The number of coincidences of an (a, b) correspondence on a straight line was given by E. de Jonquières(167) and used repeatedly by L. Cremona(97). M. Chasles explicitly gave the formula $u=a+b$ (80) and applied it to numerous problems on conics(81).

E. de Jonquières(166) showed that in a system ∞^1 of curves of order m, index μ, the number ν which touch a given line is $\nu=2(m-1)$. Problems in multiplicity of elements and in recognition of extraneous factors were considered by H. G. Zeuthen(419) and applied to the derivation of Plücker's formulas(424). The later derivation of these formulas by O. Zimmermann(425) is an extension of the same method. The formulas were extended to space curves by H. G. Zeuthen(418); to systems of rational curves(420); to algebraic surfaces in (423); and by R. Sturm(342). Application to the surface of centers of curvature by G. Darboux(101); G. Fouret(124). Other applications by H. Schubert(297) and (298); by H. Krey(187) and (188); by L. Saltel(288) and (289) and (286). The extensive applications made by Em. Weyr were collected and reproduced in his book(386). G. Halphen(136) showed that in a system ∞^1 of conics the characteristic is always of the form $a\mu+b\nu$, μ being the number through a point, ν the number touching a given line.

H. G. Zeuthen(421) proved that in an (a, b) correspondence between two curves of genera p_1, p_2 with η, η^1 coincidences, then

$$\eta-\eta^1=2b(p_1-1)-2a(p_2-1).$$

The first member was replaced by $\Sigma(\eta-\eta^1)$ by G. Halphen(139); H. G. Zeuthen(417), in which higher coincidences are replaced by their equivalents in terms of elementary ones. See also A. Cayley(67); M. Noether(252); G. Halphen(137).

The Cayley-Brill formula has already been given. E. Bertini(30); G. Castelnuovo(59); M. Noether(246) and (254) gave proofs of the theorem that if C of genus p contains g_n^r, then

$$p\leqq(k+1)\left(n-r-\frac{k(r-1)}{2}\right)$$

wherein k is the maximum integer in $\frac{n-r}{r-1}$. For the application of the theorem to possible projections of C_n in S_r, see G. Veronese(367); C. Segre(311) and (312).

If C_m is the projection of another C_m of S_r, the order of the minimum adjoint of C_m is in the interval μ, where

$$m-3>\mu>m-3-\frac{m-r}{r-1}.$$

See C. Küpper(193). G. Schaake(291) determined the effect of actual multiple points on the surface of trisecants of a space curve.

20. *Applications of Binary Forms.*—For cubic curves inscribed in a hexagon, see A. Cayley(68). For concyclic points, see E. Beltrami(21). For problems in elimination see L. Saltel(287) who obtains the order of the eliminant of three ternary polynomials in the general case. F. Morley(241.1) obtained the resultant of three ternary forms of the same order. The general problem in any number of variables is discussed by Morley and A. B. Coble(242). V. Retali(279) employs quadratic inversion and elimination to obtain various theorems regarding conics having multiple contact with the given curve. E. Ciani(88) studies the $(16_6, 16_6)$ configuration when the six coplanar points are in k-fold involution. J. Fairon(115) illustrates properties of binary forms by means of points on a conic. P. Savio(290) obtains the complete system of 18 forms derived from the classification of (2, 2) correspondences. The method is purely algebraic; binary in two sets.

G. Roch(283) showed that if $F(s, z)=0$ defines an algebraic function z of s, then any s', rational in s and z has $m-p+1+q$ arbitrary constants, in which m is the order of s', p is the genus of $F(s, z)=0$ and q is the number of poles of $\frac{\phi}{F'}$.

The logarithmic integrals when $p=2$ and $p=3$ were determined by the same author in (284). An application to the determination of the bitangents of the non-singular quartic was given by the same author(285). The theory of automorphic functions was applied to algebraic curves by G. Humbert(151), (152), (153) and (154). Two new covariants were added by H. S. White(412).

A parametric representation for curves of genus 4 was determined by H. Poincaré(275). The canonical representation of the abelian integrals of all three kinds for the case in which the function is represented by a curve having only linear branches at multiple points was obtained by J. C. Fields(117).

The general problem of rationalization of the branches of an algebraic curve has been exhaustively discussed by P. Koebe(177), (178), (179) and (180). Criteria for adjointness were given by J. C. Fields(118) and relations between branch-points and double points of an algebraic curve were obtained by the same author(119), who develops the whole general theory of algebraic functions of one variable(120). The periodicity of abelian integrals of the second kind was determined in a different form by E. Picard(270).

J. Fairon(116) gives a representation of a binary quartic on a ruled cubic surface and on a cubic curve.

M. Stuyvaert(343) studies various geometric configurations which arise from the condition that two simultaneous binary forms have more than one common root. A. B. Coble(96) generalizes the idea of closure to apply to (k, m) correspondences. See also (96.1). Moreover, using the notation and interpretation of W. F. Meyer(229.1), he obtains a considerable number of new results in the applications of binary forms to involutions. See (96.2), (96.3), (96.4).

21. *Polar Properties.*—E. Waelsch(380), Em. Weyr(392); also C. Le Paige(212), and A. Del Re(104) gave a number of theorems concerning polar properties which are applied to adjoints of a given curve.

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CHAPTER VIII

CREMONA TRANSFORMATIONS IN SPACE AND HYPERSPACE

After preliminary papers by Cayley(21) and Noether(125) in 1870 the main lines of this theory were developed for space S_3 by Cremona and Noether in 1871. The account given here follows in a general way the same outline as Chapter IV. However, the involutorial transformations which have received much attention are discussed in the following chapter. At the end of each section such extensions to hyperspace as have been made are indicated. General accounts of space transformations are given by Sturm[(166); Chapter XI], by Doehlemann[(47); Part II], and by Salmon[(149); Chapter 17]. Loria[(97.1), 250-254] gives a historical account, and Pascal[(138.1), 963-967, 978-985] a short résumé of the subject. Hudson(78.4) gives an admirable treatment of the matters discussed in sections 1, 2, 4, 5.

1. *Definitions and General Properties.*—We set the planes of an $S_3(x)$ into projective correspondence with the surfaces, ϕ, of a homaloidal web of order n in $S_3(y)$ and obtain a Cremona transformation T from a point x of $S_3(x)$, the base point x of a net of planes, to point y of $S_3(y)$, the base-point variable with x of the corresponding net of surfaces ϕ in the homaloidal web. Then planes in $S_3(y)$ are mapped by means of their sections with the web ϕ upon surfaces ϕ' of order n' in $S_3(x)$ which necessarily constitute the homaloidal web in $S_3(x)$ determined by T^{-1}. A line l' in $S_3(x)$ passes by T into a curve λ of order n' in $S_3(y)$; and a line l in $S_3(y)$ passes by T^{-1} into a curve λ' in $S_3(x)$ of order n. We use unprimed letters for occurrences in $S_3(y)$ and the corresponding primed letters for corresponding occurrences in $S_3(x)$. Thus the use of a primed letter in $S_3(y)$ indicates a theorem; e. g., if above the orders of T, T^{-1} for planes are n, n', then the orders of T, T^{-1} for lines are n', n, respectively.

If, in S_r, T similarly defined transforms the S_{r-1}'s, S_{r-2}'s, ..., S_1's of $S_r(x)$ into manifolds of orders $n, n', n'', \ldots, n^{(r-2)}$ in $S_r(y)$, then T^{-1} transforms the S_{r-1}'s, ..., S_1's of $S_r(y)$ into manifolds of orders $n^{(r-2)}, n^{(r-3)}, \ldots, n', n$, respectively in $S_r(x)$.

The first paper which brings out clearly the distinctions between the various kinds of *fundamental* or *F-points* and the various kinds of *principal* or *P-surfaces* and *P-curves* is that of Cremona[(41) (1871)]. The

n^3-1 fixed intersections of three surfaces of the web ϕ are absorbed by a *fundamental basis* or F-basis of the web which consists of certain F-curves $C_{q_1}^{r_1}, \ldots, C_{q_i}^{r_i}, \ldots, C_{q_\rho}^{r_\rho}$ of orders $q_1, \ldots, q_i, \ldots, q_\rho$ and of multiplicity $r_1, \ldots, r_i, \ldots, r_\rho$ for the general surface ϕ; and of certain F-points $f_1, \ldots, f_j, \ldots, f_\sigma$ of multiplicity $k_1, \ldots, k_j, \ldots, k_\sigma$ for the general surface ϕ. At a point of the F-basis not all surfaces ϕ contain a general direction. There is then a net of surfaces ϕ on such a direction and therefore an image x of this direction in $S_3(x)$. At an *isolated* F-point f_j, the directions correspond to points x on a rational P-surface in $S_3(x)$ whose order is the number of times a curve λ passes through f_j [Cremona(38), Noether(126).] This P-surface is a double factor of the jacobian $J(\phi')$ [Cremona(38)]. At an F-point on the F-curve $C_{q_i}^{r_i}$ the directions correspond in general to points x on a rational curve of order r_i called the P-curve of the F-point. In general, also, as the F-point varies on $C_{q_i}^{r_i}$ the P-curve in $S_3(x)$ varies over a P-surface, of genus equal to that of $C_{q_i}^{r_i}$ and of order equal to the number of variable intersections of a curve λ with $C_{q_i}^{r_i}$ [Cremona(38), (41); Noether(126)]. Such a P-surface is a simple factor of $J(\phi')$ [Cremona (38)]. It may happen, however, that the P-curve of an F-point on C_q^r does not vary as the F-point runs over C_q^r. The P-curve is then an F-curve $C'^{r'}_{q'}$ of the web ϕ' and directions at an F-point on either curve correspond to the points of the other F-curve. Such corresponding F-curves in inverse webs are said to be of the *second kind.* Cremona(38), (41) had inferred from particular cases that for two such curves $r'=q$ and $q'=r$ a view which persisted for a long time [cf. Sturm(166), p. 342]. Nevertheless Montesano[(114) (1918); cf. also (115)] by constructing examples shows that transformations in S_3 exist for which two corresponding F-curves of the second kind of orders ν', ν have multiplicities on the webs ϕ', ϕ equal to $k\nu$, $k\nu'$, respectively, where k, ν, ν' are arbitrary positive integers. Marletta(93.1) extends this theorem to transformations in S_n.

The isolated F-point is characterized by the fact that its consecutive points determine a P-surface. It may be a particular point on one or more F-curves. An F-curve is of the first or the second kind according as it is met by a general curve λ in variable points or in fixed points only.

The jacobian $J(\phi)$ has an isolated F-point f_j of multiplicity k_j on the web ϕ as a point of multiplicity $(4k_j-2)$; it has an F-curve of the first kind $C_{q_i}^{r_i}$ as a curve of multiplicity $(4r_i-1)$ [Cremona(38), (41); Noether(126), for special cases (125)]; and it has an F-curve of the second kind C_q^r as a curve of multiplicity $4r$ [Cremona(38)]. Doehlemann(48), Levi(94) and Gerbaldi(65) discuss the jacobian of four surfaces by algebraic methods and reach the same conclusions.

A plane α in $S_3(y)$ is mapped by the system of curves cut out by the web ϕ—a system of order n with q_i base-points of order r_i—upon a surface of the web ϕ'; similarly, a plane α in $S_3(x)$ is mapped upon a surface of the web ϕ. Hence

$$n'=n^2-\sum_{i=1}^{\rho} q_i r_i^2, \qquad n=n'^2-\sum_{i=1}^{\rho'} q'_i r'^2_i$$

[Cayley(31); Cremona(41)]. The general surface ϕ', the transform of α by T^{-1}, meets the P-surface of an isolated F-point only in F-curves; and meets the P-surface of an F-curve $C_{q_i}^{r_i}$ in q_i P-curves [Cremona(41)].

Loria(97) observes that a plane α and a surface ϕ in $S_3(y)$ correspond respectively to a surface ϕ' and a plane α in $S_3(x)$ whence the genus of plane sections of surfaces of either web is called the *genus* of the transformation (cf. § 2). The equal genus of the two sets of plane sections had also been noted by Noether(127) who combined the resulting relation

$$(n-1)(n-2)-\sum_{i=1}^{\rho} q_i r_i(r_i-1)=(n'-1)(n'-2)-\sum_{i=1}^{\rho'} q'_i r'_i(r'_i-1)$$

with the above equations to get

$$4(n-n')=\sum_{i=1}^{\rho} q_i r_i-\sum_{i=1}^{\rho'} q'_i r'_i.$$

Pannelli in seeking an extension of the Clebsch-Cremona theorem that in the plane the number of F-points of inverse transformations is the same finds relations among the characteristic numbers of the F-bases(133), (134), (135). Of these the most noteworthy is the theorem that if as above the web ϕ has σ isolated F-points and ρ F-curves $C_{q_i}^{r_i}$ of genus p_i respectively, then

$$\sigma+\rho+\sum_{i=1}^{\rho} p_i=\sigma'+\rho'+\sum_{i=1}^{\rho'} p'_i.$$

Tummarello(171.4) states that if p, p' are the genera of the F-curves in the two spaces then $\sum p=\sum p'$ and for each p there is an equal p'. This is proved by Hudson[(78.3) and (78.4) p. 277].

Miss Hudson(77) shows that a transformation can be constructed with an F-curve of any genus however great; and that a product $T=T_1T_2$ has F-curves of genera equal to those of the factors T_1, T_2; and that consequently the general space transformation cannot be expressed as a product of a finite number of types of elementary transformations. She proves also [(78.4), p. 358] that a transformation can be constructed which has an arbitrarily given curve as an F-curve of the first kind.

Autonne(7.1) shows that to an isolated F-point which is a higher singularity of the web there corresponds a degenerate P-surface, or merely

F-curves, or both. He introduces(7.8) a very artificial nomenclature for the F-system and P-system.

For Cremona transformation in space $S_k (k>3)$ there may exist F-loci of dimensions $0, 1, \ldots, k-2$ but no exhaustive analysis of the possible behavior of such F-loci with respect to their P-loci has been made.

2. *Determination of the Types.*—Cayley(81) remarked that a type of transformation T would be determined if an F-basis could be selected whose *postulation* (i. e., the number of linear conditions imposed by the prescribed basis) for surfaces of order n is $\frac{1}{6}(n+1)(n+2)(n+3)-4$ and whose *equivalence* (i. e., the number of intersections of three surfaces absorbed by the prescribed basis) for surfaces of order n is n^3-1. The surfaces of such a web would necessarily be rational. He gave for a curve of order m with h apparent nodes the postulation $m(n+1)-\frac{1}{2}(m-1)m+h$ and the equivalence $m(3n-2)-m(m-1)+2h$.

Noether(187) first gives reasonably complete formulas for (a) postulation: the ϕ constitute a web; (b) equivalence: three ϕ's have a single variable intersection; and (c) genus: the general ϕ has arithmetic genus $p_a=0$. Let as before the web ϕ of order n have ρ F-curves $C_{q_i}^{r_i}$ of order q_i and rank (order of developable of tangents) a_i which are r_i-fold on the ϕ's; $C_{q_i}^{r_i}$ have c_{ii} actual double points and meet $C_{q_j}^{r_j}$ in c_{ij} points $(i \neq j)$; let the ϕ's have σ isolated F-points f_l of multiplicity k_l and let the curve $C_{q_i}^{r_i}$ pass through f_l with d_{il} branches, the tangent cones of the ϕ's at f_l having no part fixed for all the ϕ's; and finally let the ϕ have certain points of contact of order $s-1$, i. e., a common $(s-1)$-fold point with common tangent cone of order $(s-1)$. Then the conditions (a), (b), (c) on the ϕ's lead to the following relations in which

$$\Sigma_1 = \sum_{i=1}^{\rho}, \qquad \Sigma_2 = \sum_{i,j=1}^{\rho} (i \leqq j), \qquad \Sigma_3 = \sum_{l=1}^{\sigma}, \qquad \Sigma_4 = \sum_{i=1}^{\rho} \sum_{l=1}^{\sigma}:$$

$$\begin{aligned}
\text{(a)} \quad & \tfrac{1}{6}(n+1)(n+2)(n+3)-4 = \Sigma_1 \tfrac{1}{6} r_i (r_i+1)\{(3n-2r_i+5)q_i \\
& \quad -\tfrac{1}{2}(2r_i+1)a_i\} - \Sigma_2 \tfrac{1}{6} r_i (r_i+1)(3r_j-r_i+1)c_{ij} \\
& \quad + \Sigma_3 \tfrac{1}{6} k_l (k_l+1)(k_l+2) - \Sigma_4 \tfrac{1}{6} r_i (r_i+1)(3k_l-2r_i+2) d_{il} \\
& \quad + \Sigma \tfrac{1}{2} s(s+1);
\end{aligned}$$

$$\begin{aligned}
\text{(b)} \quad & n^3-1 = \Sigma_1 r_i^2 \{(3n-2r_i)q_i - r_i a_i\} - \Sigma_2 r_i^2 (3r_j-r_i) c_{ij} \\
& \quad + \Sigma_3 k_l^3 - \Sigma_4 r_i^2 (3k_l-2r_i) d_{il} + \Sigma s^2;
\end{aligned}$$

$$\begin{aligned}
\text{(c)} \quad & \tfrac{1}{6}(n-1)(n-2)(n-3) = \Sigma_1 r_i (r_i-1)\{(3n-2r_i-5)q_i \\
& \quad -\tfrac{1}{2}(2r_i-1)a_i\} - \Sigma_2 \tfrac{1}{6} r_i (r_i-1)(3r_j-r_i-1)c_{ij} \\
& \quad + \Sigma_3 \tfrac{1}{6} k_l (k_l-1)(k_l-2) - \Sigma_4 \tfrac{1}{6} r_i (r_i-1)(3k_l-2r_i-2) d_{il}.
\end{aligned}$$

From these by forming proper linear combinations there follow:

$$
\begin{aligned}
(1)\quad & \tfrac{1}{6}(4n-1)(4n-2)(4n-3)-1=\textstyle\sum_1\tfrac{1}{6}4r_i(4r_i-1)\{(12n-8r_i-5)q_i\\
&\quad -\tfrac{1}{2}(8r_i-1)a_i\}-\textstyle\sum_2\tfrac{1}{6}4r_i(4r_i-1)(12r_j-4r_i-1)c_{ij}\\
&\quad +\textstyle\sum_3\tfrac{1}{6}4k_l(4k_l-1)(4k_l-2)\\
&\quad -\textstyle\sum_4\tfrac{1}{6}4r_i(4r_i-1)(12k_l-8r_i-2)d_{il}+\sum s(7s-3);\\
(2)\quad & 2(n^2-1)=\textstyle\sum_1 r_i\{(n+r_i)q_i-\tfrac{1}{2}r_ia_i\}-\sum_2 r_ir_jc_{ij}+\sum_3 k_l^2\\
&\quad -\textstyle\sum_4 r_ik_ld_{il}+\sum\tfrac{1}{2}s(s+1);\\
(3)\quad & 11(n-1)=\textstyle\sum_1 r_i(5q_i-\tfrac{1}{2}a_i)-\sum_2 r_ic_{ij}+\sum_3 2k_l\\
&\quad -2\textstyle\sum_4 r_id_{il}+\sum s(s+3).
\end{aligned}
$$

Equation (1) expresses the fact that the jacobian $J(\phi)$ is completely determined by its $(4k_l-2)$-fold points f_l and its $(4r_i-1)$-fold curves $C_{q_i}^{r_i}$. Equation (2) in connection with an equation of Section 1 leads to

$$
\begin{aligned}
(4)\qquad 4n'-4&=\textstyle\sum_1 2r_i\{(n-r_i)q_i-\tfrac{1}{2}r_ia_i\}-\sum_2 2r_ir_jc_{ij}+\sum_3 2k_l^2\\
&\quad -\textstyle\sum_4 2r_ik_ld_{il}+\sum s(s+1)\\
&=\textstyle\sum_1 Q_i+\sum_3 2R_l+S;
\end{aligned}
$$

where

$$
\begin{aligned}
(5)\quad & Q_i=2r_i\{(n-r_i)q_i-\tfrac{1}{2}r_ia_i\}-r^2c_{ii}-\sum_{j=1}^{\rho} r_ir_jc_{ij}-\textstyle\sum_3(2r_ik_l-2r_i^2)d_{il}\\
& R_l=l^2-\textstyle\sum_1 r_i^2d_{il}\\
& S=\textstyle\sum s(s+1)
\end{aligned}
$$

[Noether gives for Q_i the value

$$
\begin{aligned}
Q_i=2r_i\{(n-r_i)q_i-\tfrac{1}{2}r_ia_i\}&-\sum_h r_h^2c_{hi}\\
&-\sum_j(2r_ir_j-r_i^2)c_{ij}-\sum_l(2r_ik_l-2r_i^2)d_{il}
\end{aligned}
$$

where $h<i$ and $j\geqq i$, a value which does not have the requisite symmetry.]

Since $4n'-4$ is the number of variable intersections with the $J(\phi')$ of a line in $S_3(x)$ the term Q_i is the number of variable intersections of a curve λ in $S_3(y)$ with the F-curve $C_{q_i}^{r_i}$ and is the order of its P-surface in $S_3(x)$. The term R_l is the order of the P-surface in $S_3(x)$, which occurs twice in $J(\phi')$, of the isolated F-point f_l. Also, the order of the P-surface in $S_3(x)$ which corresponds to the contact of order $s-1$ in $S_3(y)$, and which occurs $(s+1)$ times in the $J(\phi')$, is s [cf. Cremona(38) Nota 2^a].

Miss Hudson(75) remarks that Noether pointed out that the foregoing formulas for the postulation and equivalence of an isolated F-point A of a homaloidal web require modification when the tangent cone at A degenerates and contains a fixed part due to the passage through A of certain branches of F-curves. She then studies the equivalence and postulation of the condition that the tangent cone at A contain a fixed part irrespective of the passage through A of F-curves. She considers also the condition that certain sheets of the surfaces have contact of higher order at A [cf. also (74), (78)]. Applications are made to *monoidal* transformations for which A is a point of multiplicity $n-1$.

The determination of types by means of the above formulas for postulation and equivalence occurs only incidentally. The first general method for obtaining a wide variety of transformations in space is due to Cremona(38). A surface ϕ of a homaloidal web is cut by the other surfaces of the web in a homaloidal net of curves on ϕ. Since ϕ is rational it can be mapped on a plane π by a net $\sum$ of curves of order r on a certain base B. All surfaces of order n cut ϕ in curves which map from a system $n\sum$ in π of curves of order nr on the base nB. To determine the space transformations in which ϕ can figure as a member of the homaloidal web we select from the system $n\sum$ a net which contains a fixed part L and a variable part which constitutes a plane homaloidal net H on a base B'. In space the surfaces which cut out on ϕ the maps of the homaloidal net LH form with ϕ a homaloidal web whose F-curve is the map of L. The order n' of the inverse transformation is the order of the map on ϕ of the variable curve H on π. Since the types of nets H presumably are known, those types which can be augmented by L to lie in the system $n\sum$ may be counted out. Cremona applies (Nota 1^a) this process to the cases where ϕ is a quadric, a general cubic surface, a nodal cubic surface, and a ruled cubic surface; also (Nota 2^a) where ϕ is a cubic surface with two or three nodes or with a uniplanar point, a Steiner quartic surface, and a quintic surface with a double cubic curve, for all of which the mapping on the plane π is known in advance. De Paolis(136) uses this method to determine monoidal transformations of order n, particularly those for which $n'=n$ [cf. also Bianchi(15)]. This is the first determination of a type with arbitrary n. Loria(96) followed Cremona in the discussion of the case where ϕ is a general cubic surface. Sturm(170) reestablishes the types $n=3$ by a discussion of the homaloidal nets on the cubic surface itself. Aroldi(4) applies Cremona's method to the general quartic surface with a double conic. Tummarello(171) applies the method to the determination of monoidal transformations for which $n'=n^2$. They are divided into two types according as the ϕ's have not or have the same tangent cone of order $n-1$ at O; of the first type there are three classes, of the second one, for each type of plane homaloidal net.

Montesano(104) considers transformations for which the ϕ's are surfaces of order n with an $(n-1)$-fold line. These are of genus zero and are included among the types given by Loria(97) of genus zero: (a) the ϕ's and ϕ''s are Steiner quartics; (b) the ϕ are Steiner quartics, the ϕ''s quadrics; (c) the ϕ's have an $(n-1)$-fold line, the ϕ''s an $(n'-1)$-fold line. Dr. Grazia Nobile(124) refers to lectures at Naples (1918-19) in which Montesano constructed all webs $\phi(n=3)$ which are monoidal or pluri-monoidal; and to unpublished theses of Drs. De Cesare and Berardi in which all webs ϕ consisting of quartics with a double conic, and quintics with a triple line and two skew double lines incident to the triple line, are determined. All these types are of genus one. Dr. Nobile completes the construction of webs of genus one by adding 32 new types.

Segre(154.1) studies an (n, n) transformation with an $(n-1)$-fold F-line obtained by two projections from higher space. Perazzo(188.2) obtains transformations of low degree in S_3 from incidence theorems in S_5. Marletta(98.1) also obtains by projection a new homaloidal web with an $(n-2)$-fold F-line. Tummarello(171.1) obtains new webs of order n with an $(n-3)$-fold line and $(n-3)$ triple points. He(171.2) also gives an $(n, 2n-3)$ transformation for which the web of order n is monoidal and the other consists of surfaces due to DeVries; and later(171.3) discusses monoidal (n, n) transformations with osculation at a vertex.

Loria(96) discusses webs ϕ of general cubic surfaces with an F-curve, certain simple F-points and a number of common contacts of order r with a given plane at a given point. The latter conditions give rise to diophantine equations similar to those in the plane and the geometric solutions are obtained.

Noether(126) first makes systematic use of products to obtain a variety of types of space transformations. Miss Hudson(76) adds to the seven types $(n=3)$ for which ϕ is a non-singular cubic surface those types for which $\phi(n=3)$ has one, two, three, or four nodes or a nodal line. Of the 75 types enumerated, 26 can be obtained as products of two $(n, n'=2, 2)$ transformations; 16 as products of a $(2, 2)$ transformation and others of the list; and 33 are not expressible as products of simpler types. She further(78) lists those types which are products of any two $(2, 2)$ transformations; and also(72) lists 127 types of $(3, 3)$ transformations for which the sextic F-curve degenerates into 6 lines which meet or coincide in various ways and gives for each type the triad of bilinear forms which determine it. Cech[(21.1) cf. also (78.1)] adds to these one type. Montesano(117) takes as elements of a product the cubic transformation determined by three pairs of projectively related pencils of planes whose F-basis in either space consists of a space cubic curve and three bisecants, say C'_3, b'_1, b'_2, b'_3, and C_3, b_1, b_2, b_3. If a number of such transformations $K_1, \ldots, K_t$ are carried out in succession in such a way that K_l, K_{l+1} have $C_3^{(l)}$, $C'_3{}^{(l+1)}$ in coincidence and perhaps also some of the bisecants there arises a transformation of order $2n-1$ which has the final $C_3^{(t)}$ as an $(n-1)$-fold F-curve and a number ρ of bisecants of $C_3^{(t)}$ as F-curves of multiplicity $s_1, \ldots, s_\rho$ where $n, s_1, \ldots, s_\rho$ form a ternary "Cremona group of numbers." In this way an aggregate of types of space transformation simply isomorphic with the aggregate of types of plane transformations is obtained. Tinto(170.2), (170.3) also studies these transformations. In an earlier paper Montesano(116) had considered the case where the C_3's degenerate into a conic and a secant line.

Miss Beloch(13) proves the two following theorems stated by Castelnuovo: (I) For a homaloidal web the adjoint surfaces of all indices h (i. e., the surfaces of order $n-4h$ with multiplicity $i-h$ (or zero) for every i-fold F-curve and with multiplicity $\alpha-2h$ (or zero) at every α-fold isolated F-point of the given web) are lacking. (II) Every web of order $n=4k$ possesses either (a) an F-curve of multiplicity $i \gtrless k+1$ and of order $\lesseqgtr 15$ or (b) an isolated F-point of multiplicity $\alpha \gtrless 2k+1$; with similar results for $n \equiv 1, 2, 3$ (mod. 4). She then considers transformations of type (a) for which the F-curve of greatest multiplicity is (1) a plane curve; (2) a curve on a quadric; (3) a curve on a cubic surface, and determines for them some of the other F-curves and F-points. She shows that there are no transformations of type (b) with only F-points without contacts; that every transformation with only F-points with contacts is monoidal and of type: (1), (2, 4), quadrics on a tetrahedron with a fixed tangent plane at one vertex; or (2), (3, 9), cubics with a double point and simple points with a contact of third order at one; or (3), (4, 16), quartics with a triple point and a contact of the fifth order at a simple point. Of these three types the first two were noted by Cremona while the third is new. Miss Beloch also proves that if in type (a) the F-curve of maximum multiplicity is of genus 3 and order 6 the transformation is the product of a (3, 3) type and of one of order less than n.

Montesano(108) obtains a type for which the ϕ's contain a space quartic of the first kind C_4 as an m-fold F-curve and $2m$ bisecants of C_4 as simple F-curves. The inverse web is of the same kind, $n=n'=2m+1$, and the genus is m. In superposed spaces it may be involutorial. Del Pezzo(141) studies transformations for which the ϕ's are cones. The simplest is a quadric type T and higher types of order n are obtained as products of types T. Berardi(13.2) gives fourteen types of quintic transformations with a triple and two double lines.

Many types such as that of Montesano(108) above have been discussed in connection with involutorial transformations. These are reported more fully in the next chapter. We mention also the determination by Montesano(106) of involutorial types ($n'=n$) for which the ϕ's have an $(n-2)$-fold line. Also, many types are mentioned in connection with the complex Γ of Section 5 and in connection with the regular transformations of Section 3.

In higher space relatively few homaloidal systems have been exhibited. Cayley(21) remarks that as two bilinear relations in S_2, and three in S_3, determine respectively transformations of orders 2 and 3, so r bilinear relation in S_r determine a transformation of order r. The properties of this transformation known in S_3 have been generalized by Godeaux(67) to S_r. The involutorial case in S_4 determined by four polarities is studied by Miss Alderton(1); cf. also Wong(177). Del Pezzo(139) determines 16 homaloidal systems of quadrics in S_4; and also(140) exhibits a homaloidal system of quadric cones in S_4 which is then generalized to S_r. He(142) considers also a quadratic transformation in S_4 whose inverse is of order 8, as well as a number of its special cases; also a quadratic transformation whose inverse is of order 4. Carrone(19) follows the method of Cremona explained above in using the mapping system which arises from stereographic projection of a quadric in S_r upon an S_{r-1}. He obtains thereby various homaloidal systems $(n=2)$ which for $r=4$ are nearly all given by del Pezzo. Carrone also obtains a type from the fact that in S_{r+1} r independent S_{r-1}'s, $\sigma_1, \ldots, \sigma_r$ have the property that on a point P of S_{r+1} there is a single line which meets $\sigma_1, \ldots, \sigma_r$. If then in S_{r+1} two S_r's are given, a point P of the one determines a line which meets the other in a point P'. This transformation from P to P' is discussed. This symmetrical case is extended to the case of fixed spaces $S_{r_1}, S_{r_2}, \ldots, S_{r_{p-r+1}}$ in $S_p (p>r)$ where $r_1+r_2+\ldots+r_{p-r+1}=n$. Then the S_{p-r}'s of S_p which meet the fixed spaces form a system ∞^r, one on each point of S_p and again points P, P' of two given spaces S_r, S'_r are images in a Cremona transformation. Veneroni(174) observes that the manifolds of order r in S_r which contain $r+1$ arbitrary S_{r-2}'s form a homaloidal system. They contain also the variety $V_{r-2}^{(r-1)(r-2)}$ formed by all the lines which meet the $r+1$ given S_{r-2}'s. The inverse system is of the same type and in superposed spaces the transformation has 2^r fixed points. Marletta(98.1) obtains new homaloidal systems in S_r.

3. *Integer Linear Transformations Associated with Regular Groups.*—Coble(27) has defined the *regular* Cremona group in S_r to be that group which is generated by a transformation of the form $x_j x'_j=1$ $(j=1, \ldots, r+1)$ and collineations; but the discussion is confined to those elements for which the isolated F-points are placed in general position. In S_3 the transformations of this group have been described by Kantor(88) as having no F-curves of the first kind. He gives their more striking properties in S_3 and notes the possibility of generalization to S_r. We take up at once the general case. The regular transformations have properties

quite analogous to those in the plane since they are determined to within collineations by their finite number ρ of isolated F-points. This number ρ is the same for the direct and inverse transformations. In S_r they have F-spaces of dimension $r-2, \ldots, 1$ but the existence of these is a necessary consequence of the existence of the isolated F-points. There is but one type of principal *variety*—that whose points correspond to the aggregate of directions at an isolated F-point. Such a P-variety is determined uniquely by its behavior at the isolated F-points.

The order n of such a regular transformation T from $S_r(x)$ to $S_r(y)$ has the form $(r-1)\mu+1$. It transforms an $S_{r-1}(x)$ into a variety in $S_r(y)$ of order n with multiple points of orders $(r-1)_{s1}, \ldots, (r-1)_{s\rho}$ at the isolated F-points in $S_r(y)$. The P-variety which corresponds to the F-points, $F_i (i=1, \ldots, \rho)$, in $S_r(x)$ has the order r_i with multiplicity a_{ij} at the F-point F_j in $S_r(y)$. If a variety $V_{r-1}(x)$ has the order x_0 and the multiplicity x_i at the point F_i this variety is transformed by T into one in $S_r(y)$ with the order y_0 and the multiplicity y_j at F_j where

$$G: \quad \begin{aligned} y_0 &= [(r-1)\mu+1]x_0 - r_1x_1 - \ldots - r_ix_i - \ldots - r_\rho x_\rho, \\ y_1 &= (r-1)s_1x_0 - a_{11}x_1 - \ldots - a_{i1}x_i - \ldots - a_{\rho 1}x_\rho, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_j &= (r-1)s_jx_0 - a_{1j}x_1 - \ldots - a_{ij}x_i - \ldots - a_{\rho i}x_\rho, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_\rho &= (r-1)s_\rho x_0 - a_{1\rho}x_1 - \ldots - a_{i\rho}x_i - \ldots - a_{\rho\rho}x_\rho. \end{aligned}$$

The behavior of $V_{r-1}(x)$ and its transform at pairs of ordinary corresponding points may be indicated by further equations

$$\begin{aligned} y_{\rho+1} &= -(-1)x_{\ +1}, \\ &\cdots\cdots\cdots\cdots \\ y_m &= -(-1)x_m. \end{aligned}$$

Since T is generated by the particular transformation above, the linear transformation G is generated by the particular element,

$$A: \quad \begin{aligned} y_0 &= rx_0 - (x_1 + \ldots + x_{r+1}), \\ y_j &= (r-1)x_0 - (x_1 + \ldots + x_{r+1}) + x_j, \qquad (j=1, \ldots, r+1), \end{aligned}$$

and by permutations of the variables $x_1, \ldots, x_m$. This particular element, and therefore G itself has an invariant linear form, and an invariant quadratic form,

$$\begin{aligned} &(r+1)x_0 - x_1 - x_2 \ldots - x_m, \\ &(r-1)x_0^2 - x_1^2 - x_2^2 \ldots - x_m^2. \end{aligned}$$

The existence of these forms imposes a system of relations on the coefficients μ, r_i, s_j, a_{ij} of the transformation G entirely analogous to the system [Chapter IV (2)] which prevails in the plane [(27) p. 366]. To the inverse Cremona transformation T^{-1} there corresponds the inverse linear transformation G^{-1} which is obtained from G by interchanging the r_i, s_j and the a_{ij}, a_{ji}.

To every type of regular transformation T with prescribed coordination of the two sets of isolated F-points there corresponds a linear transformation G; and the aggregate of such types with $\rho \leqq m$ F-points gives rise to a collineation group $g_{m,r}$ which is infinite and discontinuous for $m > r+3$ except in the cases $(m, r) = (6, 2)(7, 2)(7, 3)(8, 2)(8, 4)$ [(27) p. 362]. For $r=3$ the symmetric types

$$(n, \rho, r_i) = (3, 4, 1)(7, 6, 2)(15, 7, 4)$$

have been given by Kantor[(88) p. 27]; for higher spaces Coble[(27) p. 368-69] adds the types: in $S_4(49, 8, 10)$; in $S_r(r, r+1, 1)$; in S_r(r odd), $(\frac{1}{2}(r-1)(r+3)+1, r+3, \frac{1}{2}(r+1))$. By considering the particular case for which the $\rho \leqq m$ F-points are placed on a normal elliptic curve, Coble[(27) § 6] derives a collineation group $e_{m,r}$ with rational coefficients which is a transform of $g_{m,r}$ for which the invariant linear space becomes a fixed coordinate space.

Kantor pursues the study of the transformation G with particular reference to the determination(87) of periodic cubic transformations in S_3, and(88) of finite groups of regular transformations in S_3 (cf. § 6). Coble pursues a similar study for *congruent sets of points* in S_r. Two ordered sets of m points, P_m^r in $S_r(x)$ with points $p_1, \ldots, p_m$ and Q_m^r with points $q_1, \ldots, q_m^r$ in $S_r(y)$ are congruent under the regular transformation T if $\rho \leqq m$ of the pairs p_i, q_i are the F-points of T, T^{-1} and if the remaining $m-\rho$ pairs are ordinary corresponding pairs. By mapping such a set P_m^r by means of its absolute projective invariants upon a point P of a linear space $\Sigma_{r(m-r-2)}$, the aggregate of sets congruent to it under regular Cremona transformation is mapped upon an aggregate of points P in $\Sigma_{r(m-r-2)}$ which form an aggregate of conjugate points under a Cremona group $G_{m,r}$ in $\Sigma_{r(m-r-2)}$ which is isomorphic with the collineation group $g_{m,r}$. This group is discussed in Sections 7 and 10.

With any set P_m^r there is "associated" projectively a set P_m^{m-r-2} in S_{m-r-2}. For such associated sets the groups $G_{m,r}$ and $G_{m,m-r-2}$ in $\Sigma_{r(m-r-2)}$ coincide. This association of sets of discrete points is studied by Coble(86), (87), (31), and, in a particular case, by Musselman(123). The groups $g_{m,r}$ and $g_{m,m-r-2}$ are simply isomorphic.

If in forming products with the generator $x_j x'_j = 1$ one F-point is held fixed, the types thus generated have a monoidal character and the coefficients of the transformation G for T in S_r may be inferred from those of a transformation G' for a regular transformation T' in S_{r-1} [cf. Coble(30)]. This procedure has been called the dilation of T'. If two points are fixed, the types obtained in S_3 are called *dyoidal* by Kantor(88). When the spaces are superposed and one F-point is held fixed, all lines on this point are transformed among themselves; if two F-points are held fixed, all planes on the line of the two are transformed among themselves, etc.

The set of points P_m^r defines a discontinuous aggregate of P-varieties which consists of the directions at any one of the m points, the S_{r-1}'s on any r of the m points, etc.; in general, of any variety V^k_{r-1} defined by its behavior at the points of P_m^r and of a type which can be obtained by regular transformation from the directions at a point. Under regular transformation the P-varieties of a set P_m^r are transformed into the P-varieties of a congruent set Q_m^r. There is also attached to a set P_m^r an aggregate of discriminant conditions of which the simpler types are the condition that two points of P_m^r coincide in some direction, and the condition that $r+1$ points of P_m^r are on an S_{r-1}. The other types are such as arise from these simpler types by regular transformation. Properties of P_m^r which persist under regular transformation to congruent sets Q_m^r, i. e., invariant properties of P_m^r under regular Cremona transformation are expressed in terms of these discriminant conditions [cf. Coble(27) §8; (28) §3; (30); (32)].

Kantor has developed a theory similar to the above for transformations in S_r which can be generated by collineations and a quadratic transformation with an isolated simple F-point and an F-variety which is a quadric in an S_{r-1}. His main objective has been the periodic types and the finite groups [cf. § 6].

Kantor(90) considers the behavior of the group of all Cremona transformations in S_r from the standpoint of general group-theory. He concludes that it is a group whose elements contain certain continuous parameters $a_1, a_2, \ldots$; and certain discontinuous parameters $\alpha_1, \alpha_2, \ldots$; each set of parameters infinite in number. The α's vary with the type of transformation; for given type the a's vary continuously with the choice of the F-varieties. Subgroups may be characterized by invariant relations on either type of parameter.

4. *Projective Descriptions of Particular Types.*—The simplest Cremona transformation in space, the quadratic $(n=2,\ n'=2)$ transformation, is studied by Schiaparelli(152) and Geiser(62) as a quadric inversion; see also Cayley(21) and Cremona(38), (41). Aschieri(5) constructs this type by means of a correlation between two S_3's and two projectively related bundles of rays, and later(6) studies it in connection with a certain linear complex. The transformation $(n=2,\ n'=3)$ is studied by

Cayley(21) and Cremona(38), (41). The transformation $(n=2, n'=4)$ is discussed in detail by Cremona(40) and by Medugno(99).

The most interesting space transformation $(n=n'=3)$ is that with a sextic F-curve of genus three (or its special cases) in either space for which the P-surfaces are the octavic loci of trisecants of the F-sextics. It is first obtained in 1837 from three correlations by Magnus(98) who also recognizes the existence of the sextic F-curve, C^6. It is discussed from the same point of view by Cremona(37); by Cayley(21) who notes the special case where C^6 consists of four lines and their two transversals; and by Sturm(169) who observes that it is determined by 13 pairs of corresponding points and that it has in superposed spaces 8 fixed points [cf. also Terquem(170.1), Kantor(82), Machovec(97.2), and Baker(12.1)]. The properties of the transformation are developed by Reye[(148); Vol. III].

Of particular cases of the above transformation, von Krieg(92) first gives that which arises by extending to space the Sedydewitz construction for a planar quadratic transformation. It is generated by three pairs of projective pencils of planes and has for F-curve the three base-lines of the pencils and a cubic curve bisecant to each. It is applied by von Krieg to various construction problems. This case is more fully investigated by Ascione(7) particularly for phenomena which appear in superposed spaces; and later is discussed by Doehlemann(49) who observes a case which has a cubic space curve as a locus of fixed points. The particular case where the three base lines of the pencils form a triangle and the transformation is of the form $x_i x'_i = 1$, $(i=1, \ldots, 4)$ was given earlier by Silldorf(159) who observes that it is determined by the two F-tetrahedra and one ordinary corresponding pair; and that the lines on two related F-points are in a quadratic correspondence.

Miss Hudson(71) gives 28 types of the cubic transformation (3, 3) for which the one sextic F-curve consists of six lines with 8 simple intersections or their equivalent in multiple intersections. The cases for which the six lines are the edges of a tetrahedron is discussed by Beltrami(13.1), Amigues(2.1) and Obeslo(128.1); and, in metrical aspects, by Hunyadi(78.5) and Meyer(99.1), (99.2). Eckhardt(49.1) compounds such transformations with two fixed F-points to produce regular transformations.

If the three correlations which define the (3, 3) transformation become polarities, the transformation is an involution with pairs x, x' apolar to a net of quadrics. The geometry associated with this involution is developed by Sturm(167), (168) without particular reference to the general theory. The case in which the net of quadrics is on a cubic space curve is discussed by Cantone(18), by Reye[(148) Vol. 2]; and by Schoute(154); also by Haskell(70) who employs the comitants of a binary cubic.

An extensive account of the quadratic and cubic transformations is given in Sturm's treatise[(166) p. 347-401]; a more elementary account by Doehlemann[(47) Part 2]. The cubic reciprocity defined by three collineations is discussed by Sturm(169); and a metric case, in which the plane determined by a point P is on the feet of the perpendiculars dropped from P to three skew lines, by Guradze(69).

Cremona(38) gives a great variety of particular cases for each of which the bases and the jacobians of the inverse homaloidal webs are described. If the general surface of the web ϕ is a general quadric, the order of the inverse web ψ is 2, 3, or 4; for ϕ a general cubic surface, the ψ are of orders 3, 4, 5 (two types), 6 (two types) to which Sturm(170) adds another); for ϕ a nodal cubic surface, the ψ are of orders 5, 6, 7 (three types), 8, 9; for ϕ a ruled cubic surface, the ψ are of orders 2, 3, 4, 5; for ϕ a binodal cubic surface, the ψ are of orders 3 (two types), 4, 5, 6, 7; for ϕ a trinodal cubic surface, the ψ are of orders 3, 4, 5, 6; for ϕ a cubic surface with a uniplanar point, the ψ are of orders 4, 5, 6, 7, 8, 9; for ϕ a Steiner quartic surface, the ψ are of orders 2, 4; and for ϕ a quintic surface with double space cubic, the ψ are of orders 4, 7. Cremona returns later(41) to the cases ($n=2$, $n'=2, 3, 4$; $n=3$, $n'=3, 4, 5$). Bonicelli(16) investigates the (3, 6) case defined by cubic surfaces on three skew lines, a point, and a point at which there is a fixed tangent plane. Cremona(44) discusses a (6, 4) transformation; as well as(45) a (6, 5) transformation. Aschieri(5.1) treats a (4, 4) transformation, and Godeaux(67.2) discusses transformations which carry a pencil of planes into a pencil of planes, or a linear congruence into a linear congruence.

Saltel(150) defines the Aguesienne transformation in space to be the involution whose pairs x, x' are on a line with a fixed point P and in the involution cut out on this line by a pencil of quadrics [cf. also Deruyts(46)]. The web has a node at P and is on the base curve C^4 of the pencil of quadrics; the P-surface of P is the quadric on C^4 and P, the P-surface of C^4 is the quartic cone on C^4 with vertex at P. Godeaux(66) replaces the bundle of lines on P by the rays of a linear congruence with directrices a_1, a_2; the web consists of sextics with double lines a_1, a_2 and simple C^4. There is a quintic locus of fixed points on a_1, a_2, C^4. Schoute(153) extends the transformations described in Chap. IV, Section 4 to space and obtains for the first involution a cubic transformation with F-tetrahedron and for the second the transform of this by an inversion.

De Paolis(136) gives a geometric and analytic account of transformations whose homaloidal web of order n has an $(n-1)$-fold point; and in particular of those for which $n'=n$.

As noted above, interesting types arise and have been discussed in connection with their involutorial cases. Thus Geiser(63) describes the involution of order 7 defined by pairs x, x' which form with six given points the base points of a net of quadrics. Montesano(108), (110) constructs involutions for which planes pass into surfaces of order $2n+1$ with a space quartic of the first kind as an n-fold curve, and with $2n$ chords of the curve as simple lines. The genus of such a transformation is n. Montesano(113) also discusses involutions determined by bilinear complexes of conics in space.

Conner(35) gives an account of the Kantor involution determined by 7 points in space; and Coble(31) describes certain involutions which leave a symmetroid unaltered.

Snyder(163) describes two types of involutions in S_4 which transform a $V_3{}^3$ in S_4 into itself. The first is the projection of the variety upon itself from a point of the $V_3{}^3$. In the second, each plane on a line π of $V_3{}^3$ meets $V_3{}^3$ in a conic, and the projection of this conic into itself from the pole of π is part of a transformation in S_4.

Further types are mentioned in Section 5 in connection with the complex Γ, and in Chapter IX on Involutions.

5. *Fixed Points and Cyclic Sets. The Complex* Γ.—When the spaces $S_3(x)$ and $S_3(y)$ are superposed, the transformation T carries x into a point $y=x'$ of the same space and, as in the plane, fixed points, or invariant cycles of points, may arise. T then may be periodic or belong to a group in $S_3(x)$. In this and the three following sections the theory thus arising is reported. The line which joins x to its correspondent x' will in general occupy ∞^3 positions and thus will run over a complex Γ. For particular transformations Γ may reduce to a congruence. The complex Γ has been widely used as a means of classifying transformations S_3.

Aschieri(5) finds that the general quadratic transformation has 6 fixed points and finds also the fixed points and loci of fixed points for the involutorial cases. Kantor(81) shows that the number N_k of cyclic sets of k points for the transformation $y_i=1/x_i (i=1, \ldots, 4)$ where $k=e_1{}^{m_1}e_2{}^{m_2}\ldots e_\nu{}^{m_\nu}$ ($e_j=$a prime) is given by

$$N_k=2[3^k-\sum 3^{k/e_j}+\sum 3^{k/e_ie_j}\ldots(-1)^\nu 3^{k/e_1\cdots e_\nu}]/k.$$

He also(81) finds that the number N_k for the transformation $y_i=1/x_i$ $(i=1, \ldots, r+1)$ in S_r is given by

$$N_k=(r-1)![r^k-\sum r^{k/e_j}+\sum r^{k/e_ie_j}\ldots(-1)^\nu r^{k/e_1\cdots e_\nu}]/k.$$

James(80) proves for a Cremona transformation in S_r whose orders are $n, n', \ldots, n^{(j)}, \ldots, n^{(r-2)}$ (the respective orders of the transforms of an $S_{r-1}, S_{r-2}, \ldots, S_{r-j-1}, \ldots, S_1$) that the number of fixed points is $(\sum_{j=0}^{r-2} n^{(j)})+2$, the number of involutorial pairs is $\sum_{j=0}^{r-2}\frac{1}{2}n^{(j)}(n^j-1)$; and gives also a formula for the number N_k of cycles of k points. Hudson and Wren(78.2) discuss the two involutorial pairs of the (2, 2) transformation in S_3.

Pannelli(130) uses the formulas of Noether given in Section 2 to make a detailed study of the complex Γ. De Paolis(137) studies the complex Γ for the involution associated with a (1, 2) transformation in space; also(138) involutorial transformations for which Γ degenerates into a congruence consisting of (a) lines on a point; or (b) bisecants of a cubic space curve; or (c) transversals of a space curve of order μ and a $(\mu-1)$-fold secant. Vogt(175) studies the transformations for which Γ degenerates into a linear congruence. T has orders (n, n) and an F^{n-1} of fixed points. Montesano(101) finds that if Γ is a non-special linear complex and T involutorial an upper limit for n is 11; a special case is treated later(103). Pieri(143) finds that for Γ a special linear complex the order of n is unlimited. Aschieri(6) considers the case where Γ is linear and special and T quadratic. Montesano(105) discusses the case where Γ is tetrahedral for which n in general is 19 though reductions occur in particular cases. Del Re(147.1) constructs transformations of the third and higher orders for which Γ is linear or is a congruence and gives extensions to hyperspace. Pieri(144) considers the case when Γ is a Hirst complex determined by two correlative planes in space. Montesano(107) discusses the transformations T for which the complex Γ is quadratic; and Pieri(145) the particular case where Γ consists of the secant lines of a conic. Montesano(109) defines an involution, for which Γ consists of the tangents of a surface, in terms of a linear (one conic on a point) congruence of conics in space whose planes envelop a surface F of class >1. For given x the plane of the conic on x has a contact with F and the line through x and this contact meets the conic again at x'. Pieri(146) lists cases where Γ consists of the tangents of a surface; and Montesano(112) adds to this list. Montesano(111) studies the type (a) of Loria(97) in which the webs are Steiner quartics and finds that Γ is either a special linear complex or a cubic complex. Miss Caldarera(17) investigates the cases for which Γ consists of the secants of a space cubic curve.

6. *Periodic Transformations and Finite Groups.*—Montesano(100) defines an abelian G_{16} of quadratic involutions by means of a plane π and five points $O_i (i=1, \ldots, 5)$ in space. The quadrics Q_i with O_i, π as pole and polar and the other four points as self-polar tetrahedron meet π in the same conic K. The involution I_i has F-point O_i and F-conic K and $I_iI_j=I_jI_i=I_kI_lI_m$. The G_{16} possesses invariant cubic surfaces on which the group is the map of the plane Cremona group G_{16} determined by five points. Montesano also finds a Cremona G_{32} in space by mapping

the lines of a quadratic complex upon an S_3 and taking the map of the group of collineations and correlations which is defined by the complex. Del Re(147) considers the transformation T, $x_i x'_i = a_i$ $(i=1, \ldots, r+1)$ in S_r with 2^r fixed points. These fixed points and T itself are invariant under the linear group G, $x'_i = \pm x_i$ of order 2^r. Then G and T generate a Cremona group of order 2^{n+1}. An analogous group is obtained by combining $x'_i = \mu_i x_i (\mu_i^m = 1)$ with T.

S. Kantor in a series of papers(83), (84), (85), (86) derives the types of periodic transformations and of finite groups for that particular subgroup of the Cremona group in S_r which is generated by collineations and a quadratic transformation defined by an isolated F-point P, and a general quadric in an S_{r-2} not on P as an $F-M^2_{r-2}$. In forming products of these generators the F-manifolds of the factors are so chosen that an isolated F-point does not lie on the S_{r-2} of an $F-M^2_{r-2}$. This sub-group has an arithmetic character analogous to that of ternary transformations. A cyclic group generated by a periodic transformation, or a finite group has an invariant linear system of varieties V_{r-1} whose order and dimension may be reduced by a properly defined "adjoint" process to the point where the resulting linear system can be transformed into one of a finite number of types, and the associated group into one of a similar set of types.

Kantor carries out a similar process for regular periodic substitutions in S_3(87) and for regular finite groups in S_3(88). A regular transformation is one generated by $x_i x'_i = 1 (i=1, \ldots, 4)$ and collineations. The list obtained in (88) contains (a) a collineation group in S_3; (b) 34 types with an invariant bundle of lines; (c) 10 types with an invariant pencil of planes on a line a, 5 of which leave the line bundles on points a_1, a_2 of a each invariant while the other 5 interchange these bundles; (d) 5 types of groups of cubic transformations; (e) a group G_{720} determined by 5 points; (f) 7 types of groups determined by 6 points; and (g) 14 types of groups determined by 7 points. The groups (f) have an invariant Weddle surface determined by the 6 points and arise from collineation groups on the Kummer surface which is birationally equivalent to the Weddle. The groups (g) have an invariant locus of nodes of quadrics on the 7 points and arise from collineation groups which the plane quartic curve, birationally equivalent to the nodal locus, admits.

E. H. Moore(118) observes that all of the cross-ratios of a binary $(r+3)$-ic are rationally expressible in terms of r of the cross-ratios which he takes for the coordinates of a point P in S_r. When the roots are permuted, P is transformed by a Cremona transformation which in general is of order r. The $G_{(r+3)!}$ thus generated has a collineation subgroup

$G_{(r+2)!}$ and is determined by $r+2$ points. The F-spaces of the elements are linear spaces of various dimensions determined by these points. For $r=3$ this is Kantor's group (e) just above. For $r=2, 3, 2p-1$ Coble(24), (25), (33) has discussed this group in connection with the solution of the equation of degree 5, 6, $2p+2$ (cf. § 10). In (26) he generalized this group in connection with a set of m points in S_r, P_m^r. If the first $r+2$ of the m points be placed at the reference basis and the unit point in S_r and if the last coordinate of the other points be taken as equal then the $(m-r-2)r+1$ coordinates of these points are taken as coordinates of a point P in a linear space $\Sigma_{r(m-r-2)}$. As the points of P_m^r are permuted the point P is transformed under a Cremona $G_{m!}$ in $\Sigma_{r(m-r-2)}$. The set P_m^r in S_r is "associated" with a set P_m^{m-r-2} in S_{m-r-2} whose $G_{m!}$ coincides with that of P_m^r. For $r=1$ or $r=m-3$ this group is Moore's cross-ratio group. One case $m=6$, $r=2$ had been given earlier by Kasner(91) who obtained it from four independent irrational invariants of a double six on the cubic surface mapped from the plane by cubic curves on P_6^2.

Coble(27) obtains a group $G_{m,r}$ of transformations in $\Sigma_{r(m-r-2)}$ by adding to the permutations of the points P_m^r the operation of passing from P_m^r to a set Q_m^r *congruent* to P_m^r under regular Cremona transformations in S_r (cf. § 3 Chap. IV). Again the group $G_{m,r}$ attached to P_m^r coincides with the group $G_{m,m-r-2}$ attached to the associated set P_m^{m-r-2}. This group is in general infinite (cf. §7) but three finite cases $G_{6,2}$, $G_{7,2}=G_{7,3}$ and $G_{8,2}=G_{8,4}$ are of particular interest being isomorphic with the group of respectively, the 27 lines of a cubic surface, the 28 double tangents of the plane quartic, and the 120 tritangent planes of a space sextic of genus four on a quadric cone. Coble(28) uses the $G_{6,2}$ in Σ_4 for the determination of the lines on a cubic surface (cf. § 10).

Coble(33) defines by means of $2p+2$ points in S_{2p-1} an abelian $G_{2^{2p+1}}$. It is generated by transformations of the type $xx'=1$ with isolated F-points at $2p$ of the points which interchange the other two. The group has an invariant variety of dimension p, a generalization of the Weddle surface, which is mapped by the simplest linear system invariant under the group upon the hyperelliptic Kummer manifold of genus p.

7. *Infinite Discontinuous Groups.*—Certain infinite discontinuous groups in S_r which are subgroups of the general Cremona group in S_r are themselves sufficiently general to have the constitution ascribed by Kantor[(90), cf. § 4] to the Cremona group itself. This is true in S_3 of the monoidal group of Cremona transformations with an invariant bundle of lines, and of the dyoidal [cf. Kantor(88)] group with an invariant pencil of planes. The group of regular transformations and the group

generated by a quadratic transformation (cf. § 6) have a similar character. Other examples may be given such as the group in S_3 which has an invariant quadric or the group in S_r which has an invariant elliptic curve C^{r+1}. Such groups are characterized by the fact that their elements contain continuous parameters in addition to the discontinuous parameters. Other infinite groups are properly discontinuous in the sense that an element contains no continuous parameters.

Hutchinson(79) observes that the Kummer quartic surface admits 60 cubic involutions ($x_i x'_i = 1$) with F-points at a Göpel tetrad of nodes, and that the Weddle surface admits 15 similar involutions, which on either surface generate an infinite discontinuous group. Baker(12) notes that an element to which Hutchinson ascribes an infinite period is actually of period two. He adds relations among various products of involutions. Sharpe and Craig(36) use Severi's basis theory to confirm Hutchinson's general result and to add others. Snyder(162) shows that the infinite discontinuous group of birational transformations which occur on certain surfaces of Rosenblatt, which contain a pencil of elliptic curves, can be replaced by a similar group of Cremona transformations. Snyder and Sharpe(157) obtain an infinite discontinuous group of non-involutorial Cremona transformations which leaves invariant a quartic surface containing a space sextic of genus three. They prove also that the known birational group of a quartic surface which contains a space sextic of genus two can be replaced by a group of Cremona transformations; and earlier(156.1) prove a similar theorem for the space sextic of genus three. Snyder(163) combines the two types of involutions which carry a V_3^3 in S_4 into itself [cf. § 4] to form a discontinuous group with continuous parameters.

Coble has defined a doubly infinite array of infinite discontinuous groups, $G_{m,r}$ (cf. § 6), in $\Sigma_{r(m-r-2)}$. To an aggregate of conjugate points P under $G_{m,r}$ in $\Sigma_{r(m-r-2)}$ there corresponds in S_r the aggregate of sets of points P_m^r congruent to one of the sets under regular Cremona transformation in S_r or the associated aggregate of sets of points P_m^{m-r-2} in S_{m-r-2}. However, the transformations of $G_{m,r}$ in Σ are not regular. The group $G_{m,r}$ contains subgroups simply isomorphic with $G_{m'r'}$ if $m' \leqq m$ and $r' \leqq r$ [(30), p. 258]. The infinite case occurs for $m > r+3$ with the exceptions noted in Section 6. For particular points P in Σ the aggregate of conjugate points under $G_{m,r}$ may be only finite. Then the set of points P_m^r in S_r is *special* and is self-congruent under an infinite discontinuous Cremona group in S_r. Coble[(27), pp. 377-8; (30)] gives as examples 9 points on an elliptic plane cubic, or 8 points on an elliptic space quartic, the sum of whose canonical parameters is a sub-multiple of a period;

also the ten nodes of a rational plane sextic, and nine or ten nodes of a symmetroid. Another method of defining infinite discontinuous subgroups of $G_{m,r}$ arises from its isomorphism with the collineation group $g_{m,r}$ (cf. § 3) with integer coefficients. Those elements of $G_{m,r}$ whose corresponding elements in $g_{m,r}$ are congruent to the identity modulo j form a discontinuous infinite subgroup of $G_{m,r}$ whose factor group is finite. For $j=2$ these finite factor groups have been determined by Coble(29) to be known groups which occur in connection with period transformations of theta functions in p variables. Coble(31) also notes that the normal variety V_2^{r-1} in S_r admits an infinite discontinuous group of regular transformations, a property shared by the Veronese V_2^4 in S_5.

8. *Continuous Groups.*—Noether(128) shows that every continuous group of quadratic transformations in S_3 is one of the following 5 types or a subgroup of one: (a) the group (∞^{10}) which transforms the system of quadrics on a proper conic into itself; (b) the group (∞^{11}) of a system of quadrics on two intersecting lines; (c) the group (∞^{13}) of a system of quadric cones which touch along a common generator; (d) the group (∞^{11}) of a system of quadrics on a line and an outside point; and (e) the group (∞^{13}) of a system of quadrics which are on a line and touch at a point of the line.

Enriques and Fano(53) show that every continuous Cremona group in space can be transformed into one of the following 6 types or a subgroup of one: (a) the projective group; (b) the group (∞^{10}) of conform transformations which transform spheres into spheres; (c) a generalized Jonquières group with an invariant bundle of lines; (d) a generalized Jonquières group with an invariant pencil of planes; (e) a group (∞^3) of cubic transformations with an invariant linear system of cubic surfaces on a space quartic of the second kind; and (f) a group (∞^3) of transformations of order 7 with an invariant linear system of dimension 12 of surfaces of order 7, with a common double conic, with a simple line l tangent to the conic at p, with a fixed tangent plane along l, and with a triple point at p.

Fano(59) divides the primitive types (a), (b) above into 10 sub-types, 9 of which had been given by Lie, and a tenth—the ∞^6 conform transformations of a sphere into itself. Fano(56) divides the primitive types (c), (d) into 12 sub-types. Fano(57) shows that the types of continuous groups of Cremona transformations given by Lie are reducible to the types of Enriques-Fano. Fano(55) gives a résumé of the above and states the theorem that any continuous Cremona group in S_r has an invariant linear system of dimension n and can be mapped upon a projective group in S_n which has an invariant rational V_r. Fano(58) gives a complete enumeration of continuous groups not contained in a larger group as well as the infinitesimal transformations of such groups. Fano(60) gives a general report on continuous groups.

Young and Morgan(120) study the continuous group defined by three invariant pencils of planes in space in each of which projectivities are simultaneously defined, a type given by Fano(56). The authors indicate the generalizations to S_r and study particularly the geometry under this group of the system of space cubic curves with the base lines of the three pencils for bisecants.

9. *Geometric Applications.*—One of the first applications of space transformations was to the study of surfaces of higher order as the transforms of known surfaces of lower order. Geiser(64) considers the quartic surface with a double conic as the transform of a cubic surface by a quadric inversion. He(62.1) uses metric cases both of the general and of the tetrahedral (3, 3) transformation in a study of the quadric and cubic surface. Noether(126) employs the quadratic transformation and the (3, 3) transformation to obtain from planes or other rational surfaces a variety of different types of rational surfaces. Cremona(39) uses a similar method to extend this variety. He(37.1) uses the special (2, 2) transformation with *F*-point on the *F*-conic to get the quartic surface with double conic from a cubic surface and a special case from a quadric surface. Cremona(42) uses a quadratic transformation to transform a certain cubic surface into a quartic surface with a cuspidal conic, and a cubic transformation to transform a quadric into a quintic surface with a cuspidal space quartic. Eckhardt(49.1) uses the tetrahedral transformation to study the Steiner quartic surface. Saltel(150) proceeds from surfaces already constructible to others by the Arguesienne transformation; and(151) studies the behavior of transforms at acquired singular points. Cremona(43) transforms a cubic surface into a quartic surface with a self-contact by a quadratic transformation. Caporali(18.1) employs a (3, 5) transformation in connection with the quintic surface with double quintic curve. Montesano(111.2), (111.3) obtains by transformation a large variety of quintic surfaces. Pieri(142.1) uses an involution whose complex Γ is quadratic and special to study the sextic surface with a four-fold line and 10 nodes. Nobile(124.1) uses monoidal transformations in the study of rational surfaces of order n with an $(n-\nu)$-fold line and ν-fold points. Wren(177.1) uses Cayley's (2, 3) transformation to study curves on a surface of order n with an $(n-1)$- or $(n-2)$-fold line. Von Krieg(92) applies the (3, 3) transformation determined by three pairs of projective pencils of planes to the construction of a conic in space, a cubic space curve, a rational space quartic, a quadric surface, and a cubic surface. Chizzoni(22), (23) uses a particular involution of order 3 in space to study certain families of conics and rational surfaces (including

cyclides) connected with them; and also gives new constructions for the space quartic on 8 points and the quadric on 9 points. Morley and Conner(84) define the Weddle surface to be the aggregate of pairs x, x' apolar to the net of quadrics on a cubic space curve and also apolar to another quadric. Snyder(161) uses the involution determined by a web of quadrics on 6 points in the study of the Kummer and Weddle surfaces. Van Veen(172) employs an involutorial transformation from plane to plane to establish theorems concerning space quartic curves of the first kind which have 3 or 4 intersections with a fixed space cubic curve. Miss Castelli(20) applies a particular (3, 3) transformation noted by Cremona (defined by pairs x, y whose line is in a linear congruence and which satisfy a correlation) to the study of quartic and quintic surfaces with particular singularities.

The study of Cremona null-systems was begun by Ameseder(2) who developed synthetically the properties of the general quadratic null-reciprocity. Lazzeri(93) first discusses the Cremona reciprocity and finds that in a transformation from point to plane of orders (n, n') for incident point and corresponding plane, the locus of the point is a surface of order $n'+1$, and the envelope of the plane is a surface of class $n+1$ whose relations to the F-bases are described. The reciprocities are polar only in the case of the polar system of a quadric $(n, n'=1, 1)$; of pole and polar plane as to a tetrahedron $(n, n'=3, 3)$; and one case $(n, n'=2, 2)$. He finds only two null-systems for which $n, n'=1, 1$ and 3, 3. However, Montesano(102) obtains infinitely many types by combining an involution whose pairs are on lines of a linear complex Γ with the linear null-system defined by Γ and proves that all Cremona null-systems arise in this way. In connection with other transformations Francois(61) employs the reciprocity which relates a point P to the plane of the feet of the perpendiculars from P on three coordinate axes.

On linear systems of surfaces in S_3 which relate to Cremona transformations and on the reduction of such systems by Cremona transformation we find papers by Guccia and Enriques. Guccia(68) defines p_0 to be the dimension of a linear system of surfaces determined by an ordinary base L consisting of curves and points; p_1 to be the geometric genus of (F); p_2 to be the genus of the variable intersection M of F, F'; and p_3 to be the number of variable intersections of F, F', F''. He proves that if (F) is determined by L and $p_1=0$ and if on a generic F the linear system M contains a smaller system of genus 0 or a smaller system of genus 1 and dimension >1 then $p_0+p_2-p_3=2$. This is extended to the case where $p_1>0$ and F contains a smaller system F' with $p'_1=0$ and properties like the above, in which case $p_0-p_1+p_2-p_3=2$. From these he deduces that any linear system with $p_3=1$ is determined by its base; also that $p_3=1$ is the necessary and sufficient condition for a homaloidal web.

Enriques(52) considers linear systems of surfaces which are *simple* (i. e., surfaces on x do not necessarily pass through an x') and such that the variable intersection of two surfaces is a curve which is (a) rational; (b) elliptic; or (c) hyperelliptic. He shows that such systems can be transformed by Cremona transformation into the following types: (a) into (1) quadrics on a conic; or (2) quadrics tangent at a point to a plane; or (3) surfaces of order n with an $(n-1)$-fold base line and perhaps other base elements; (b) [cf. also Enriques(51)] if the grade (number of variable intersections of three surfaces) >3 into one of the following systems of grade n and dimension $n+1$ or into a system contained in one of them; (1) for $n=4$, the system of cubic surfaces determined by a quintic partial intersection with a quadric (which may degenerate); (2) for $n=5$, 6, 7, 8, 9 the system ∞^{n+1} of cubic surfaces with a double base point P, $9-n$ base lines on P, and the same tangent quadric cone at P; *as well as* (3) for $n=5$ the system ∞^6 of cubic surfaces determined by a quartic of the second kind (which may degenerate); (4) for $n=6$ the system ∞^7 of cubic surfaces on three skew lines; (5) the system ∞^7 of cubic surfaces with a double point P and a space cubic curve (which may degenerate) on P; (6) the system ∞^7 of cubic surfaces with a biplanar base point P with a fixed osculating plane at P, and a plane cubic curve with a node at P; (7) for $n=7$, 8 the system ∞^8 or ∞^9 of quadrics with 1 or no base points; and (8) for $n=8$ the system of quartics with a triple point P, two double base lines on P and the same tangent cone at P; (c) [cf. also Enriques(50)] into a system of surfaces of order n with an $(n-2)$-fold base line, and a base curve which cuts a plane on the base line in two points outside the line, and perhaps other base elements.

Pannelli(132) shows that a twisted curve in space with any singularities whatever can be transformed by Cremona transformations of the type $x_i x'_i=1$ into a curve with only ordinary multiple points. Levi(95) shows that any space curve can be transformed by Cremona transformation into a space curve without multiple points.

Of isolated geometric applications we note the following. Hesse(70.1) uses the involution defined by a net of quadrics in the study of the plane quartic curve. Montesano(106.1) applies an (8, 13) transformation in a study of the analytic dependence of two complex variables. He also(111.1) uses a space involution of order 7 in connection with the problem of projectivity for a group of linear complexes or congruences. Montesano(106.2) employs a (4, 5) transformation to convert a linear system of conics on a point into a line congruence; and Godeaux(67.1) a similar transformation to convert a congruence of twisted cubics into one of lines. Montesano(110.1) uses many types of transformation in the study of linear congruences of conics. Doehlemann(46.1) uses a monoidal transformation in discussing the surface of incidence of corresponding rays of two sheafs.

Pannelli(131) and Wimmer(176) put the points of a space Σ' in projective correspondence in two ways with the web of planes in a space Σ and also with a web of quadrics in Σ. Thus P' in Σ' determines a pair x, x' of points in Σ which describe a quartic surface as P' describes a plane. The properties of this involution and of the quartic surface are developed. Autonne(8), (9), (10), (11) extends to space and to $N-1$ dimensions his theory of birational transformations (cremoniennes) of forms in two series of contragredient variables, which leave invariant the condition of contact. Kantor(89) remarks that the Plücker formulas which express the class, the number of double tangents and the number of flexes $(=x, y, z)$ in terms of the order, the number of double points, and the number of cusps $(=x', y', z')$ constitute a Cremona involution in S_3 which he discusses. This idea is extended to the Cayley formulas for space curves. Berry(14) uses quadratic and cubic transformations in the discussion of integrals of the first kind of total differentials on certain quintic surfaces. Aprile(3) discusses systems of twisted quartics of the second kind. Ota(129) develops the transformation $x_i x'_i = a_i$ $(i=1, \ldots, 4)$ from the geometric side; [cf. also Vegas(173)].

Snyder(160) employs the involution determined by a net of quadrics and other transformations in the investigation of line congruences contained in the complex of generators of the net. Sharpe and Morgan(119) find Cremona transformations which leave a quartic surface with two nodes invariant and which are periodic on the surface. Snyder and Sharpe(158) discuss transformations of the Jacobian of a web of quartic surfaces into itself which may be effected by Cremona transformations. They express the transformations in terms of a Severi base on the surface which consists of a plane section, a sextic of genus three, and nine of the ten lines.

10. *Algebraic and Other Applications.*—The solution of the quintic equation described in Chap. IV § 12 in which the algebraic part is simplified by the use of an auxiliary form-problem defined by a finite Cremona group has been extended by Coble to equations of higher degree. For the sextic case the cross-ratio group of Moore, a g_6 in S_3, has for simplest invariant linear system the system of quadrics on five points which suggests a natural resolvent sextic developed earlier by Brioschi and Maschke(25). Related to these is the discussion of the form-problems of the groups $G_{5!}$, $G_{6!}$ attached to the sets P_5^2, P_6^2 (cf. § 6) of 5, 6 points in S_2 given in [(26) §§ 8, 10]. Coble(33) also outlines a program for a

development, along similar lines, of a solution of the general equation of even degree.

A similar procedure may be applied to the equation for the 27 lines on a cubic surface(28). The group is the extended group $G_{6,2}$ (cf. § 7) defined by P_6^2, a group in space Σ_4. The simplest invariant linear system divides under the operations of the invariant subgroup g_{25920} of $G_{6,2}$ of index two into two linear systems of dimension four contragredient to each other under a collineation group of Maschke which is in immediate relation with hyperelliptic modular functions of grade 3 and genus 2.

The complete system of invariants of the $G_{6!}$ of P_6^2 is given in [(26) § 5]; systems of irrational invariants of $G_{6,2}$ are given in [(25) § 3]; and of $G_{7,2}$ in [(27) § 8]. These are connected respectively with the cubic surface and the planar quartic curve. The latter system is used by Morley(122) in the determination of the condition that the curve be of the Lüroth type. Other examples of invariants of the groups $G_{m,r}$ appear in [(27), pp. 380-381] and [(31), § 4].

Applications of this theory to abelian modular functions are given by Coble[(32) § 14].

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CHAPTER IX

INVOLUTIONS AND (1, 2) CORRESPONDENCES IN S_3, S'_3

1. *Reciprocal Radii.*—The early history and the properties of reciprocal radii will be found in Chapter I. B. Tortolini(124) showed that the element of surface and of volume, when transformed by reciprocal radii, become

$$dS'=\left(\frac{k}{r}\right)^4 dS, \qquad dV'=\left(\frac{k}{r}\right)^6 dV,$$

wherein

$$\frac{x'}{x}=\frac{y'}{y}=\frac{z'}{z}=\frac{k^2}{r^2}.$$

Envelopes of spheres touching three given spheres were studied by O. Bonnet(9), M. Chasles(15), J. A. Serret(101). The latter showed that a sphere cutting a cyclide along a line of curvature meets it always at the same angle. Joachimstahl(43) proved that any sphere having double contact with a cyclide meets it in two circles. A. Mannheim(49) and (50) showed that the three fixed spheres could be transformed by reciprocal radii into others having their centers on a straight line. The Dupin cyclide can be generated in two ways as the envelope of spheres touching three given ones. A. Mannheim(51) defined as principal pole the center of a sphere of inversion under which a given surface remains invariant. It can always be transformed into a surface symmetric about a plane. If it has a curve (circle or line) of principal poles, the surface can be transformed into one of revolution. J. Moutard(71) considered a surface of order m, having the pole O to multiplicity p, and containing the circle at infinity to multiplicity q. By reciprocal radii it is transformed into another of order m', etc., such that

$$m'=2m-p-2q; \qquad p'=m-2q; \qquad q'=m-p-q.$$

It may be invariant if $p+2q=m$. Such a surface is called anallagmatic; it is the envelope of a sphere orthogonal to the sphere of inversion and having its center describe a fixed surface. Quartic surfaces for which $q=2$ are anallagmatic as to five poles. The five spheres of inversion are mutually orthogonal and the surfaces of centers are confocal quadrics. The line joining any two poles is perpendicular to the plane of the other three. The tetrahedron having any four poles for vertices has the fifth for orthocenter. Through any point of the surface pass ten circular sections. See also J. C. Maxwell(55). G. Fouret(29) showed that the Dupin cyclide is the only surface invariant under a continuous system of inversions. The lines of curvature of the cyclide are used as parameters by E. Colot(16).

S. L. Ravier(88) associated transformation by reciprocal radii with stereographic projection to generate quadrics. Given a circle C with center O and in plane π. Draw through O any line not in π, and on it take any two points A, A'. Join A to any point M in π and A' to M', inverse of M as to C. Let AM, $A'M'$, intersect in μ. As M describes π, μ will describe a quadric surface.

A somewhat similar transformation was devised by d'Ocagne(72) and (73). Two points, P, P' are called conjugate if they are diametrical opposites as to a sphere passing through a fixed circle. By linear transformation it can be reduced to reciprocal radii. R. Bricard(10) gave an alternate proof of the theorem that lines of curvatures are preserved under transformation by reciprocal radii.

E. Meyer(56) enumerated more minutely the possible conformal point transformations. They are perspective reciprocal radii with a real fixed sphere, or this transformation and (a) reflection as to the center, or (b) as to a plane through the center or (c) as to a line through the center. Also proved by A. Leveque(46). Earlier proofs had been given by E. Goursat(34) and by G. Darboux(18).

G. Valiron(125) gives a geometric proof that lines of curvature are invariant under transformation by reciprocal radii.

For the literature and general theory of transformation by reciprocal radii in space see J. K. Doehlemann(20), R. Sturm(121), V. Snyder and C. H. Sisam(108), F. S. Woods(139).

2. *Involutorial Transformations of Space.*—The first generalization of involutorial transformations to those defined by surfaces of order n was made by R. de Paolis(77); the method was similar to that employed by E. de Jonquières(44), for the plane involutions which bear his name, but which was not then published. Given a surface of order n with an $(n-2)$-fold point O. The transform of a point P is on OP and is the harmonic conjugate of P as to the residual points of intersection with the surface. The conjugate of a plane is a surface of order n, having O to multiplicity $n-1$, and passing through the curve of contact of the tangent cone to the fixed surface from O. This is known as the perspective monoidal transformation. If $O=(0, 0, 0, 1)$, and the fixed surface has the equation

$$u_{n-2}x_4^2+2u_{n-1}x_4+u_n=0,$$

u_i being ternary in x_1, x_2, x_3 of degree i, the equations of the involution have the form

$$x'_i = x_i(u_{n-2}x_4 + u_{n-1}), \qquad x'_4 = -(u_{n-1}x_4 + u_n).$$
$$i = 1, 2, 3.$$

If u_{n-2} is identically zero the fixed surface is itself a monoid, and P' is defined as the harmonic conjugate of P as to O and the residual intersection of OP with the fixed monoid. In this case all the surfaces of the web conjugate to the planes of space have the same tangent cone at O, and the basis curve consists of $n(n-1)$ of its generators.

V. Martinetti(54) obtained all the monoidal involutions compounded of the perspective monoidal form and any harmonic homology which leaves the bundle of lines through the vertex invariant. In the plane all the non-perspective involutions of type JH can be reduced to homology. The corresponding question in S_3 has not been answered. Eleven types were obtained by Martinetti as follows, in terms of the invariant elements:

1. n odd. C_n point by point and of C_1 not in its plane.
2. n even. C_n point by point, and two points on C_1.
3. n arbitrary. C_n point by point. No other point.
4. n arbitrary. C_{n-r} point by point and C_1 (n-odd), two points (n-even).
5. n arbitrary. C_{n-r} point by point only.
6. n even. Plane π point by point and C_1.
7. n odd. Plane π point by point and two points on C_1.
8. n arbitrary. Plane π point by point only.
9. n odd. π fundamental. C_1 point by point.
10. n even. Two fixed points on C_1.
11. n arbitrary. No fixed elements.

It was shown by D. Montesano(60) that any birational involution which leaves a bundle of lines invariant is either the perspective monoidal, ***M***, or this compounded with a ternary involution H, G, B or J which leaves the bundle at the vertex invariant. Thus in addition to the types found by Martinetti, the MG gives $s_1 \sim s_{3n} : \sum h_i^n C_{n(2n-3)}$ and a curve of invariant points of order $4n$, in which h_i is a generator of the cone. The invariant cone K_6 of G may or may not be composed of invariant points. In ***MB***, $s_1 \sim s_{3n} : \sum_{i=1}^{\infty} h_i C_{n(n-3)}$ and a curve of invariant points of order $3n$. The invariant cone K_9 of B may or may not be composed of invariant points.

In ***MJ***, $s_1 \sim s_n O^{n-1} k^{n-2m} (l_1 \ldots l_{2(p+1)})^m C\gamma$,

$$\gamma = (4m-1)n - 2(p+3)m^2, \qquad n = p + v + 2,$$
$$m = \beta + 1, \qquad b + p + 1 = n - 2m.$$

Not all the forms of possible invariant configurations have been determined.

A number of other types have been considered by this same method. Thus D. Montesano(61) considers the involutions of genus 0 in which $s_1 \sim s_n : c_1^{n-1}$; the pencil of planes through c_1 remains invariant. The two double planes of the involutorial pencil are point by point invariant.

Systems of involutions forming finite groups are obtained by D. Montesano(69). The first, of order 8, can be defined as the plane projection of the harmonic homologies which leave each quadric of a web with common self-polar tetrahedron invariant. All are quadratic, four perspective and three non-perspective. Similarly, from quadratic forms in S_4 an analogous group of order 16 in S_3, a second one of order 16, and one of order 32 in S_2. The last is the plane representation of Klein's G_{32} of collineations and correlations which leave the Kummer surface invariant.

Another type found by D. Montesano(66)

$$s_1 \sim s_n : c_1^{n-2} c_{3n-4}, \qquad [c_1, c_{3n-4}] = 3n-7,$$

wherein $[c_i, c_k]$ denotes the number of intersections of c_i, c_k.

Planes of the pencil c_1 are either invariant or form an involution. For $n=3$ this type is included in the preceding one. Both of these types probably include irrational involutions. D. Montesano(68) considers two non-monoidal transformations which leave every conic of a linear congruence invariant. The congruence is defined by a bundle of planes and a projective net of quadrics. A monoidal transformation of order 4, vertex at that of the bundle of planes, leaves each conic of the congruence fixed. A rational plane curve of order n, having a point of multiplicity $n-1$ at the vertex meets the plane of each conic in one point, center of an involution which interchanges the points of the conic. A third involutorial transformation results when the locus of the center in each plane is a surface in (1, 1) correspondence with the planes of the bundle. Lines joining conjugate points form a complex of order μ.

$$s_1 \sim s_{6\mu+4} : C_7^{2\mu+1} C_\nu^{2} \gamma_1^{2s_1} \ldots \gamma_h^{2s_h} t_1 \ldots t_{5(2\mu+1)},$$

Another type for n general is $s_1 \sim s_{2n+1} : c_4^{n} 2n g_i$, c_4 of genus 1, $[g_i, c_4] = 2$. It leaves the congruence of lines formed by the bisecants of C_4 invariant. D. Montesano(59). Two particular involutions of order 4 and genus 0 were given by D. Montesano(63) which were not included in the earlier types. One can be reduced to central harmonic homology and the other to axial harmonic homology.

F. London(47) provides a linear construction for involutions determined by conjugate points as to one or more quadric surfaces.

O. Tognoli(122) includes certain restricted I_2 of S_3 as particular cases of much more general correspondences discussed in Chapter X.

F. Romano(93) gives a detailed treatment of the involution leaving a web of cubic surfaces through three skew lines and four simple points invariant. This and seven other forms were defined by C. Moffa(57). Of these, four types had been found previously, and four were new.

J. R. Conner(17), following S. Kantor(44.1), constructed an I_2 of order 15 by means of a group $[a]$ of associated points. Let $|C_4|$ be the linear system of quartic surfaces having 7 of these points as double points, and $|Q_2|$ the net of quadrics through all eight. A space quartic curve through $[a]$ is uniquely determined by one point P_1, and it meets those C_4 through P_1 in a second point P_2.

$$s_1 \sim s_{15} : 7a_i^8 l_{ij} r^3_{ij}$$

in which l_{ij} is the (parasitic) line through a_i, a_j and r_{ij} is the (parasitic) cubic curve through all but a_i, a_j. The surface of invariant points is $K_6 : \sum a_i^3 l_{ij} r_{ij}$.

A metrical interpretation of oriented pairs of points was found by H. Rothe(97).

V. Snyder(111) supplements D. Montesano(60) by giving the equations of all nonperspective monoidal types and more details concerning properties of perspective monoidal types which satisfy restricted conditions. An example of the latter type was later developed by L. Godeaux(33a); it had been already employed by M. M. Torrey(123), but derived in a different way. Every monoidal involution of the form MJ belongs to a special linear line complex, the axis of which passes through the vertex of the monoid.

F. R. Morris(70) classifies the involutions determined by conjugate points as to a net of quadrics according to the number of composite quadrics contained in the net. The conjugate points as to one or more linear complexes is not included. Fourteen types are enumerated and their equations derived. Other methods of derivation of the general cubic involution are found in the following papers: Castelli(14.1), Pelzner(82.1), Schoute(99.1), (99.2), Cantone(12.2), Chizzoni(15.1), Del Re(19.1), Fiedler (27.2). For the books and bibliography of this section, see R. Sturm(121), J. K. Doehlemann(20), V. Snyder(114).

For groups generated by this transformation and collineations, see Chapter VIII.

3. (1, 2) *Correspondences Between Two Spaces.*—Given a web of surfaces $|\phi|$, all of the same order, such that any three surfaces of the web intersect in two variable points. If we put $x'_i = \phi_i$, we obtain a

(1, 2) correspondence between the points of (x') and of (x). Let P' be a point of (x'), P_1, P_2 its two images in (x). The lines joining P_1P_2 are called principal lines. The locus of $P_1=P_2$ is the surface K of coincidences and its image in (x') is the surface of branch points L'. The points P_1, P_2 are conjugate in an involution I_2, associated with the correspondence.

Particular systems were found by Em. Weyr(137).

The web consisting of quadrics through 6 points was considered by T. Reye(91). K_4 is the Weddle surface, locus of the vertex of the cones of the web.

$s_1 \sim s'_4 : 3c_1^2 \cdot P'^3$
$c_1 \sim c'_2$
$s'_1 \sim s_2 : 6P$
$c'_1 \sim c_4 : 6P$
$K_4 \sim L'_4 : 16P'^2$

The 15 lines joining the basis points by pairs are transformed into 15 double points of L'_4, and the cubic curve through all six has for image the other double point of L'_4.

The surface of branch points is the Kummer quartic surface with 16 double points. The six bundles of lines with vertex at a basis point are transformed into bitangents of L'_4. All the principal lines are bisecants of the cubic curve γ_3 through the six basis points. Their images in (x') are lines of a bundle, vertex a double point of L'.

P. H. Schoute(99) studied properties of particular associated I_2, without developing systematically the idea of the double space.

A particular (1, 2) correspondence, in which to the planes of (x') correspond quadric surfaces was obtained by C. F. Aschieri(6), following F. Schur(100). Given a regulus a^2, designate by (a^2) the net of linear line congruences through it, and let Σ be the system of ∞^3 linear complexes through two given skew lines. If (a^2) is a net in Σ, the polar planes of a point A_1 as to (a^2) form a bundle (A_1) of planes and of lines with vertex at A_1, projective with (a^2). Let (b^2) be another net in Σ, with a bundle (B_1). Make these two bundles respectively projective with bundles (A) and (B). A point M determines a regulus M^2 of Σ. When M describes a line g, m^2 describes a congruence. Given any complex θ not in Σ, any regulus of Σ has two lines in common with θ, hence to any point of S_3 correspond two lines of θ; to a line of S_3, a regulus, and to a plane, a linear congruence. To a plane correspond the two directrices of its image congruence; to a pencil of planes, a regulus, and to a bundle, a linear congruence. The pairs of directrices, images of the planes of S_3, generate a linear complex θ_1 in involution with θ. If θ is now mapped birationally on S'_3, a (2, 1) correspondence is obtained.

The surfaces K, L' are both quadrics, and I_2 is quadratic. A sphere may be associated with each linear complex in involution with a given one, furnishing a (1, 2) correspondence between a system of non-euclidean spheres and ordinary ones.

By means of the properties developed in (91) Th. Reye(92) showed that the $(16_6, 16_6)$ Kummer configuration can be constructed linearly when six points are given arbitrarily.

Every involution in S_r, the points of each group of which can be mapped rationally on the points of another space S'_r is called rational. For $r=1$ and $r=2$ all involutions I_n are rational. R. de Paolis(93) first presents the general problem of (1, 2) correspondences between S'_3 and S_3 and raises the question whether all I_2 in S_3 are rational or not, but does not attempt to answer it.

When P_1 describes a locus, P_2 may describe the same or a different locus. In the former case the locus is invariant under I_2, and in the latter the loci are conjugate. Every curve invariant under I_2, image of a rational curve in (x'), is hyperelliptic, elliptic, or rational.

If any curve C meets K at P, but is not tangent to P_1P_2, its conjugate passes through the same point with a tangent different from either. The image C' in (x') of both the given C curve and of its conjugate touches L' at the corresponding point. If C touches P_1P_2 at P, so does its conjugate, and C' has a cusp at P' on L'. A self-conjugate C meets K along P_1P_2 and C' cuts L' at C'. Th. Reye(91). If C' touches L' at every common point, C has a double point at every intersection with K, and is composite. A surface and its conjugate pass through the same curve on K. Their residual intersection is invariant under I_2. The image surface touches L' at every common point, and has the image of the residual intersection of two conjugate surfaces for double curve.

Let the lines c'_1 of (x') be transformed into $|R_{n'}|$ of (x).

A fundamental point of (x) is called a fundamental point of the first class if all the $R_{n'}$ of the ∞^4 system pass through it; otherwise of the second class. Let ρ be the number of fundamental points f_i of the first class, of multiplicity r_i on $|\phi_n|$ and of multiplicity θ_i on $|R_{n'}|$. Let there be r fundamental curves γ_i of order m_i, multiple to order i_i on $|\phi|$, and having s_i points on $|R_{n'}|$. Then

$$n'=n^2-\sum i_r^2 m_r, \qquad nn'-\sum_{i=1}^{\rho} r_i\theta_i-\sum_{i=1}^{r} i_i s_i=2.$$

The genus of a general $R_{n'}$ is called the genus of the transformation. Let K be of order v, and denote by N the order of the surface conjugate to a general plane in I_2.

4. *Fundamental Elements in the Two Spaces.*—A fundamental point of (x) of the first class may also be fundamental in I_2. The $R_{n'}$ through it may have any direction, in which case its conjugate in I_2 is a surface, or it may be fixed in direction. In this case the point lies on a fundamental curve of I_2. If not fundamental in I_2, its conjugate has the same property in the (1, 2) correspondence, or finally it may be self-conjugate in I_2. These four types of possible fundamental points in the (1, 2) correspondence are known as the first, second, third and fourth kinds respectively.

If a point f in (x) is fundamental of the first kind, its image surface in (x') is rational. Such a surface, of order θ, is a fundamental surface of the first kind. If f is of the second kind, the image surface is met by the images of the planes of (x) in variable curves, but not all the R'_n, images of the lines of (x) meet it in variable points. If the point is of the third or fourth kind, its image surface in (x') is not met in variable points, by the images of the planes of (x).

To a fundamental point of the second class in (x), of multiplicity r on $|\phi|$, corresponds a curve of order r in (x').

A fundamental curve γ of (x) is of the first class when every $R_{n'}$ meets it in variable points; otherwise it is of the second class. A multiple curve γ_m of multiplicity i on $|\phi|$ may be fundamental in I_2. If the ∞^4 curves R_l, conjugates of lines in I_2, meet it in σ variable points, each of its points is conjugate to a curve and these generate a surface of order σ, image of γ in I_2. Such a curve is of the first class and the first kind. If the $|R_l|$ do not meet γ, each point of γ is conjugate to the same fixed curve. If the conjugate is different from γ, then γ is of the second kind, if it coincides with γ, of the third kind. If γ is fundamental in the (1, 2) correspondence but not in I_2, but is conjugate to another curve, also fundamental in (1, 2), it is of the fourth kind. If conjugate to itself, but not point for point, it is of the fifth kind, and if each point is invariant in I_2, of the sixth kind. A similar diversity of fundamental elements exists in (x').

A curve a in (x) which does not intersect $|\phi|$ in variable points is called parasitic. To any point of a corresponds the whole curve in I_2. A parasitic curve of order r corresponds to a fundamental point of (x') of second class and second kind; it is of multiplicity r on all the surfaces of the quadratic system, images of the planes of (x). It is fundamental in I_2 and lies on K.

The jacobian of the web $|\phi|$ consists of K, of the fundamental surfaces, images of fundamental curves of the first class of (x'), and of the fundamental surfaces, images of the fundamental points of the first class in (x'), the latter each counted twice.

$$\gamma = \Sigma\sigma + 2\Sigma\theta' = 4(n-1).$$

L' is of order $2(p+1)$, p being the genus of $|R_{n'}|$. An i-fold fundamental curve of the first class on $|\phi|$ is of multiplicity $4i-1$ on $J(\phi)$; an i-fold fundamental curve of the second class is of multiplicity $4i$ on $J(\phi)$.

5. *Various Particular Cases.*—A case in which the web $|\phi|$ is composed of cubics with basis elements in restricted position was given by M. Pieri(87). Given an elliptic space C_4 and four coplanar points A_i not on C_4. These and the points of C_4 in their plane determine a pencil of cubic curves with a ninth basis point A_5. Through C_4, $5A_i$ pass a web of cubic surfaces $|\phi|$. Two meet in a space quintic curve of genus 2 and any three meet in two variable points. The C_4 may be composite. The surface L'_6 has a four-fold line and 10 double points. $K_7: C_4^3 5A_i$. The associated involution is of order 11. Through each point A_i pass two parasitic lines. They lie on K_7, and their images are the double points of L'_6.

Another particular case was given by G. de Longchamps(48) and by Elgé(22). Given a fixed point O. As M describes a given locus, find M', M'' on OM such

$$OM.MM' = OM' \cdot MM'' = k^2, \qquad k \text{ being given.}$$

The (1, 2) correspondence associated with the web of quadrics through six points was treated exhaustively by R. de Paolis(80).

M. Pieri(85) mapped the involutions belonging to a special linear complex with axis p on the points of S'.

$$s'_1 \sim s_n : p^{n-3} 0_{5n-11}, \qquad [p,\, 0_{5n-11}] = 5n-18.$$
$$C'_1 \sim C_{n+2}, \text{ genus } n-1, \qquad [C_{n+2},\, p] = n, \qquad [C_{n+2},\, 0_{5n-11}] = 5n-4.$$
$$L'_{2n} \sim K_{3n-4}.$$

Within the web $|s_n|$, is a linear system ∞^1 consisting of a fixed plane through p, and a pencil s_{n-1}. The image in (x') of the plane is a line d'; any point of d' is the image of a cubic of the pencil PP' and the seven basis points in the plane. The s_{n-1} having p, PP' to multiplicity $n-3$ is also singular; its image in (x') is another line e'. The residual s_1 is any plane through p. The points of e' have for images the plane sections of s_{n-1} by the planes through PP'. d', e' intersect—their point has for images in (x) the exceptional conjugate points P, P'.

The involutorial correspondences found by M. Pieri(84) give rise to a Hirst quadratic complex. T. A. Hirst(35) and (36). Given two planes in (1, 1) point-line correspondence. The lines joining the points of each to all the points of the associated line in the other generate a Hirst complex.

A (1, 2) correspondence defined by a web of quadrics through a fixed conic was investigated in great detail by J. Cardinaal(^{14}) with application to quartic surfaces with a double conic, image of quadrics generated by two projective pencils of planes. The transformation has $p=0$. L is a quadric and K is a quadric. The same method was applied by J. Cardinaal(^{13}) to determine and classify the quartic surfaces having a double line. The correspondence is now (1, 4) but is made $(1, 4-k)$ by the appearance of k isolated double points.

The (1, 2) correspondence between the Kummer and Weddle surfaces was developed from a different standpoint by R. W. H. T. Hudson(^{87}) in his book, and still more in detail by V. Snyder(^{117}), wherein a number of infinite discontinuous groups are derived, all generated by involutions.

R. Bricard(^{11}) again proved that the lines P_1P_2 in (x) are bisecants of a space C_3.

C. Rosati(^{94}) obtains the asymptotic lines of the Kummer surface.

Another (1, 2) correspondence defined by ∞^1 quadric surfaces was given by E. Cairo(^{12}).

A systematic treatment of (1, 2) correspondences which includes all rational involutions was begun by F. R. Sharpe and V. Snyder(^{105}) by means of three equations in two sets of coordinates (x) and (x'), together with an enumeration of the cases associated with webs of non-singular surfaces of orders 2-5, having independent basis elements. For a regular system, the genus of the transformation is one greater than the arithmetic genus of a general surface of the web $|\phi|$ of invariant surfaces. The types there enumerated include the rational types previously known, some twenty new isolated types, and seven families of types in terms of an arbitrary n. The problem of reducibility is not considered, except in a few cases. The same process was applied to obtain those types which leave a web $|\phi|$ of sextic surfaces each invariant, and having certain basis elements, by J. O. Osborn(^{74}). It is not complete as restrictions were put on the basis elements. That connected with the non-monoidal cubic involution was found by F. R. Sharpe and V. Snyder(^{106}). It is expressed by the table:

$$\begin{array}{llll}
s'_1\sim s_{12} & :\gamma_6{}^3+7l+21m; & c'_1\sim c_{27} & :P_8{}^6+7P_i{}^4. \\
s_1\sim s'_{27} & :\gamma 6'^{16}+7l^{8'}; & c_1\sim c'_{12} & :[c'_{12}, l']=4. \\
\gamma_6\sim\Gamma'_{54} & :\gamma'_6{}^{12}+7l'^{16}; & \gamma_6'\sim\Gamma_{16} & :\gamma_6{}^4+P_8{}^8+7P_i{}^6+21m^2.
\end{array}$$

The particular case $x_i=\dfrac{1}{x_i}$ is considered in detail. A further study of restricted basis elements for a web of cubics $|\phi|$, mentioned by M.

Pieri(87) was made by V. Snyder(110). All three of the types can be reduced to those discussed by M. Pieri. The basis elements may be

(a) c_2, c_1 skew to it, 5 pts. $A_i : A_5$ on $F_2 : c_2c_1 2A_i$, then A_6 also on F_2.
(b) c_1, $\bar{c}_1$ skew P^2 on c_1.
(c) plane c_3, c_1 $[c_1, c_3]=1$.

In all these cases $p=2$, $p_a=0$.

Generalizing the scheme of C. Moffa(57), V. Snyder(112) obtained all the types associated with a web of invariant monoids with a non-composite basis curve.

If there are seven isolated basis points, the involution is monoidal; that with six does not exist. Those with 5, ..., 0 all exist. For every value of n, L' is of order 4. If there are no basis points on $|\phi|$, L' has no singular plane, and has eleven double points.

Two of these forms for $n=3$ had been found by C. Moffa. Another category of monoidal involutions, namely those in which the monoidal conjugates of the planes of space all have a common tangent cone at the vertex have been determined by the preceding methods by M. M. Torrey(123).

A. Emch(28.1) considers the I in S_n defined by $x'_i=\phi_i(x)$ and employs $p_{ik}=x_i\phi_k-x_k\phi_i$ which changes sign under I, and $g_{ik}=x_i\phi_k+x_k\phi_i$ which is invariant. Functions of the p, g are used to investigate contact loci. The lines joining pairs of conjugate points are mapped on certain varieties, illustrated by known cases. G. M. Caldarera(18.1) discusses an I of order 27 in which lines PP' meet a fixed cubic curve. The correspondence on the bisecants of C_3 is isomorphic with the Geiser G in the plane. L. Godeaux(33.2) generates an I of order 5 by means of two skew directrix lines and a pencil of quadrics. $s_1 \sim s_5 : d^2d'^2 12l$. F. Aschieri(6.1) constructs an I_4 by means of a quadric polarity, a null system, and a projectivity. $s_1 \sim s_4 : d^2c_3c_2 2c_l$. In (33.3) an I is discussed which leaves a congruence of lines invariant. Montesano(64.2) defines an I_7 by means of 5 arbitrary linear complexes, and the null polarity as to the complex k in involution with each of them. Lines PP' belong to k and each such line contains three pairs of conjugate points. In (64.1) the same author develops the theory of an I_6 contained in a linear complex. The cubic surfaces of a web are invariant.

6. (1, 2) *Correspondence between S_4 and S'_4.*—A (1, 2) correspondence between the points of two linear S_4 was obtained by G. Aprile(2) by means of ∞^4 quadric varieties having for basis elements a c_3 of S_3 and three simple points, the plane of which does not meet c_3. These quadrics are made projective with the planes of (x').

The lines of (x') are transformed into c_6, $p=1$, and the lines of (x) go into conics of (x').

Each basis point $T_i \sim S'_3$. The basis $c_6 \sim \Gamma'_6$, enveloped by S'_3. The complete image in (x) of a tangent S'_3 of Γ'_6 is a point on c_3 and a conical variety of order 4 with vertex on c_3. The variety of branch points is rational, of order 4.

In the associated I_2 in (x), linear spaces of three dimensions $\sim V_8 : 3T^5c^4$. Any S_3 through the isolated basis points meets c_3 in three points P_i. The k_3 through the six points T_i, P_i is parasitic. $\infty^1 k_3 = K_3$ on the quartic variety of coincidences, and triple on the conjugate of S_3.

H. C. Shaub(107.1) derived the equations of the (1, 2) correspondences which leave a ∞^4 system of quadric varieties in S_4 invariant.

7. *Involutions of Lines in* S_3.—The problems associated with lines in S_3 can be interpreted as point problems in S_4, but they are sufficiently numerous and have such direct applications to ruled surfaces, congruences, and complexes, that they shall be treated in terms of lines.

F. Aschieri(4) determined an I_2 of lines of a particular cubic complex as follows: Given a pencil of quadric surfaces. The polar planes of a point as to the quadrics of the pencil form a pencil, the axis of which is associated with the given point. The lines form a tetrahedral complex, the fundamental tetrahedron being the self-polar tetrahedron of the pencil of quadrics. Lines meeting pairs of quadrics of the pencil harmonically generate a quadratic complex of Battaglini.

When p is a line of T, its polar complex as to all possible complexes B from pairs of quadrics having the fundamental tetrahedron of T for self-conjugate tetrahedron have a line p' in common also in T. The $p \sim p'$ becomes $P \sim P'$ expressible by $x_i x'_i = 1$. The lines PP' generate a cubic complex.

Another I_2 was given by F. Aschieri(3). Given a linear line complex θ_1 and a fixed cone γ_2. Given any point P. Its polar plane as to θ_1 is π, and as to γ_2 is π_1. The lines π, $\pi_1 = l$ generate the quadratic complex θ_2. Similar dual construction. The line l is the polar line of the given point P as to θ_2.

To every line l there is a pole X as to θ_2, and a conjugate l' as to θ_1, which has a pole X'. The points X, X' are conjugate in a cubic I_2 of points, image of that defined by the lines l, l'.

An I_p of tangents on a rational C_n of S_3 gives rise to a complex made up of the (l_2) pairs of associated tangents and the linear congruences of which they are the directrices. If $n=3$, $p=2$, a cubic complex results.

A further study of Cremona transformations among lines of S_3 was made by F. Aschieri(5).

A particular involution of lines derived from a generalization of Pascal's theorem to S_3 was given by F. Klein(45).

J. de Vries(130) generates a line complex by means of an I_p on a rational space curve C_n. Draw the tangents to C_n at all the p points of a group of I_p and construct the linear congruences having each pair of tangents of a group for directrices. As I_p describes C_n, the congruences describe a complex of order $(2n-3)(p-1)$. The complex curve in each plane has $(n-2)(2n-5)(p-1)^2$ double tangents, which form a congruence of this order and class. Detailed study of $n=3$, $p=2$.

A. Del Re(19) gives a representation of line complexes and line congruences in descriptive geometry, and applies it to graphical statics. J. de Vries(130) constructs a particular line complex as follows: Given two space cubic curves C_3, K_3. From a point P can be drawn one bisecant of each, determining a plane, but in each plane are 9 such points. Construct R, conjugate of P as to cubic involutorial transformation defined by C_3. Similarly for the point Q on K_3. Consider the complex of lines R, Q. It is of order 18.

Similarly, any line t determines the C_3 of the system of which it is a bisecant, meeting C_3 in P and Q. Find P', Q' in the point involution. $t'=P'Q'$ is the conjugate of t. A line through a basis point $\sim$ quadric cone; a line in plane $A_i a_k \sim$ plane pencil. The complex of coincident lines is the cubic PP'. A bundle of lines $\sim$ congruence (4, 6) and a plane field $\sim$ congruence (6, 6).

Another example is given by two harmonic homologies, A, a in plane α and B, b in plane β. A line t of S_3 meets α in P, β in Q. $t'=P'Q'$.

The congruence having a, b for directrices is composed of invariant lines. The planes α, β are singular.

A plane pencil $\sim$ quadric regulus.

A bundle $\sim$ linear congruence.

A special linear complex $\sim$ a quadratic complex.

See J. de Vries(134) and (135). The same author constructs a different I_2 by means of a plane homology A, a in α and a congruence of space cubic curves $|C_3|$ through 5 basis points. A line t fixes the C_3 of the congruence of which it is a bisecant and meets α in P.

Through P' draw the bisecant t' to the same C_3. The bundle A and the special linear complex with axis a are all invariant lines. Basis points are all singular, the bundles $\sim$ complexes of order 4.

A plane pencil $\sim$ ruled surface of order 9.

Plane field $\sim$ congruence (8, 5).

Bundle $\sim$ congruence (7, 8).

Linear congruence $\sim$ congruence (15, 13).

Special linear complex $\sim$ complex of order 9.

A slight variation is furnished by that I_2 defined by the $\infty^2|C_4|$ through 8 associated basis points. [J. de Vries(136).] A line t fixes the C_4 of the system, of which it is bisecant. t meets a fixed plane in P, through which can be drawn the other bisecant lines. This I_2 belongs to the complex of generators of the quadrics of the net through the 8 points.

G. Schaake(98.1) considers the I_2 in line space in which a pencil is conjugate to a regulus. Given a linear complex Ψ, a line m not in Ψ, and a line n in Ψ. Construct the quadric determined by l, m, and n. It has a second line 0 in Ψ. The line l', harmonic conjugate of l as to m and 0 is a line of the regulus to which l, m, n, 0 belong. This defines I_2. When l describes a pencil, l' describes a regulus through n, and if l describes a plane, l' describes a linear congruence containing n. A linear complex is transformed into quadratic complex having n as double line.

Given r, q any two space quartic curves. There are ∞^4 space quartics of the first kind which meet r in eight points and q in eight points. Any line t of S_3 is bisecant of a single curve of the system. Now by means of a plane x the I_2 is extended to all the lines of S_3.

Pencil $\sim$ ruled surface of order 9.

Bundle $\sim$ congruence (22, 24).

Plane field $\sim$ congruence (24, 23).

An involution I_{10} of lines of S_3 may be constructed as follows: Given the bilinear congruence of $|C_3|$ through 5 fixed points in S_3. A curve is uniquely fixed by having a given line for bisecant. Given two such congruences A and B. A line t determines a curve of each, and these have 10 common bisecants, hence the lines of S_3 are arranged in an involution of order 10. All the basis points are singular; a line through one is transformed into a ruled surface of order six. A general pencil of lines $\sim$ ruled surface of order 36, on which all 10 basis points are 12-fold. [J. de Vries(132).]

An I_4^4 of points determined by four points on each line of S_3 can be constructed by a general pencil of cubic surfaces; the surface of trisecants of the basis C_9 is composed of singular lines. It contains the 27 lines of each surface of the pencil. C. H. van Os(126).

Given four pencils of lines in four different planes. A line p of space pierces each plane, hence fixes a line of each pencil. These four lines have a second transversal p'. The relation between p, p' was obtained by J. de Vries(131). A plane pencil is transformed into a ruled surface of order 7. In (133), the two pencils are replaced by a quadric surface. Similarly, given a bilinear congruence of space cubic curves through two fixed points A_1, A_2 and having three fixed bisecants a_i. A point P determines the C_3 of the system passing through it. On C_3 find the harmonic conjugate P' of P as to A_1, A_2. This defines a cubic point involution. A line p is bisecant of a single C_3 of the system, meeting it in P_1, P_2. Construct P'_1, P'_2 by the point involution. $P'_1P'_2=p'$ is the conjugate of the given line in an involution of lines. C. H. van Os(127).

G. Schaake(98) shows that the ∞^3 systems of quadrics through a line of each of three of the defining pencils in the above scheme is not linear. Through three arbitrary points pass 5 quadrics of the system. The involution I_2 found previously contains 6 congruences (1, 1) of singular lines. A general bundle of lines goes into a congruence (5, 6).

8. *Line Complexes and Congruences Associated with I_2.*—Lines joining pairs of conjugate points in I_2 generate either a complex or a congruence of lines. When the line joining one pair P_1P_2 of conjugate points contains no further pairs, the involution is said to belong to the complex.

R. de Paolis(81) obtained the order of the complex. The bundles having vertices in fundamental points are singular. The lines determined by two conjugate fundamental points are simple lines of the complex. Moreover, every bisecant of a parasitic curve is a line of the complex. Now follow detailed analyses of number and kind of fundamental and singular lines in terms of the elements of the (1, 2) point transformation. A scheme is outlined by means of which all rational (1, 2) point transformations can be obtained. In a second memoir R. de Paolis(79) determines those involutions in which the lines joining conjugate points belong to a congruence instead of a complex. There are three types:

(a) The lines form a bundle;
(b) The lines are bisecants of a space cubic curve;
(c) The lines are secants of a rational C_m and of a line which meets it in $m-1$ points.

A limiting case determined by an (1, m) correspondence between the points of a line and planes through the line is not considered there. The first is the monoidal involution; the second is a particular case of the cubic involution expressed by three bilinear equations—a polar system as to a net of quadric surfaces through the cubic curve. The third case becomes axial harmonic homology when $m=1$. This case was considered further by W. Vogt(128) in connection with a wider problem of a Cremona transformation.

V. Eberhard(21) showed that the involutorial transformation of order 7 associated with the web of quadric surfaces through 6 basis points belongs to type (b), but not in the particular way defined by the bilinear equations. Also proved by G. Fontene(28).

D. Montesano(60) showed that cases (b) and (c) of R. de Paolis can be reduced to monoidal type, hence if the lines P_1P_2 of any I_2 belong to a congruence of rational curves of odd order the involution can be reduced to a monoidal form. The I_2 for which the lines P_1P_2 generate a linear complex were obtained by D. Montesano(65). Given any line r of S_3. In any plane through it draw the line P_1P_2 belonging to the complex Γ. Since the line meets r, it meets r', its conjugate as to Γ. On lines of Γ meeting r, r' points P_1P_2 describe a quartic surface F_4. A pencil of linear congruences in Γ have F_2 in common, through r_1, r_2. $[F_2, F_4]=C$, $p=3$ that is hyperelliptic. Construct another F_4 with second pair of lines r, r' but same C_6. Residual is C_{10} of $p=11$, basis curve of $\infty^4 F_4$. Any F_4 of system through P also passes through P'. I_2 is of order 11, $s_1 \sim s_{11}: C^8{}_{10}a_1 \ldots a_{20}$—quadrisecants of C_{10}. Any plane meets C_{10} in 10 points of C_3, but C_{10} does not lie on F_3. C_{10} may be composite. $s_1 \sim s_{11-\mu}: L_0{}^3 \ldots L_3{}^0 a_1 \ldots a_{20}$.

If the complex is special, I_2 for every $n \geqq 3$ exist. [M. Pieri(85).] It is determined and generated by a pencil of invariant surfaces F_{n-1}, each having a variable $(n-3)$-fold point on the axis p, containing the line to multiplicity $n-4$ all having the same tangents through p'. There is a basis curve of order $5n-11$, having $5n-18$ points on p. Every plane through p is invariant, and in it the involution is the Geiser plane involution.

$$s_1 \sim s_{4n-5}: p^{4n-8}c_{5n-11}: c_{5n-11} \sim \Omega_{12n-18}: c^8{}_{5n-11}p^{12n-39}.$$

$$p \sim \pi_{4n-6}: p^{4n-14}c^8{}_{5n-11}: K_{3n-4}: p^{3n-10}c^2{}_{5n-11}.$$

The I_2 associated with the web $|\phi_3|: C_5 2P$ is of this type. It is a generalization of the procedure used by C. Moffa(57), including the case in which C_5 has a triple point at O, and all the surfaces of the web have a double point there. V. Snyder(112).

The types contained in a tetrahedral complex were also determined by D. Montesano(67).

$$s_1 \sim s_{19}: c^5{}_{11}4p^2.35u_4K_{12}: c^8{}_{11}4p35u_4.$$

The same form was studied by J. Wimmer(138) and by F. R. Sharpe and V. Snyder(105), the last named by means of two bilinear equations and one quadratic in (x), linear in (x'). The type defined by a web $|\phi_3|: C_4 3P$ belongs in this category.

Montesano(68) considered the involutions which leave every conic of ∞^2 system invariant. Lines joining corresponding points form a complex Γ_μ with a singular bundle O. On each conic the conjugate points form an involution. Three cases arise. The center of the involution (a) is always fixed at O, or (b) describes a curve $C_\mu: O^{\mu-1}$, in which case every bisecant OG is associated with all the conics of the system through OG, or (c) the center describes a surface F_m.

In (a) it is the perspective monoidal of R. de Paolis(78) and is of order 4. In (b), C_μ is fundamental, and also the C_7, $p=5$ residual of a conic, intersection of two cubic surfaces. The $5(2\mu+1)$ trisecants of C_7 which meet C_μ not at O are all parasitic, t.

$$s_1 \sim s_{6\mu+4}: C_7{}^{2\mu+1}C\ {}^2C_2{}^{2\mu}\Sigma t_i.$$

The jacobian is composed of

$$\Gamma_{8(2\mu+1)}: C_7{}^{2\mu+1}c_\mu C_2{}^{2\mu}\Sigma t_i,$$
$$\Gamma_{9(2\mu+1)}C_7{}^{6\mu+2}C_\mu{}^6C_2{}^{6\mu}\Sigma t_i,$$

conjugates of C_7, C_μ. The surface of invariant points is

$$K_{3\mu+4}: C_7{}^{\mu+1}C_\mu C_2{}^{\mu}\Sigma t_i.$$

In (c), the complex Γ is composed of ∞^2 pencils, point on the surface F_m as vertex and containing a line of the bundle O.

$$s_1 \sim s_{6\mu+4}: C_7{}^{2\mu+1}C^2{}_{2m+\mu}\Sigma\gamma_s{}^{2s}\Sigma t.$$
$$K: C_7{}^{\mu+4}C_{2m+\mu}\Sigma\gamma_s{}^{2s}\Sigma t.$$

No I_2 can belong to the general quadratic complex. [D. Montesano(82).] If it contains a bundle of lines, it is included in those just mentioned. If I_2 contains a system of linear congruences, then

$$s_1 \sim s_{13}: C_4{}^5C^2{}_{10}t_1 \ldots t_{16}$$

in which $K_8: C_4{}^3C_{10}t_1 \ldots t_{16}$.

That contained in the tetrahedral complex has been considered.

A particular quadratic complex was considered by M. Pieri(84), namely that of characteristic [(11) (11) 2] generated by two correlative planes. [T. A. Hirst(85).] The associated involutions are:

$$s_1 \sim s_{10}: C_2{}^2C'_2{}^2 0_2{}^5C_4{}^2Vh_1 \ldots h_8, \qquad V \sim 0_2,\ C_2 \sim \Gamma_8,\ C'_2 \sim \Gamma'_8.$$
$$s_1 \sim s_{13}: C'_3{}^3C_5{}^3 0_2{}^3 0_4{}^5h_1 \ldots h_{16}, \qquad C'_3 \sim \Gamma_{12},\ C_5 \sim \Gamma_{12}.$$
$$s_1 \sim s_{16}: 0_2{}^2 0'_2{}^2C_6{}^2C_7{}^5h_1 \ldots h_{25}, \qquad C_6 \sim \Gamma_{15},\ 0_7 \sim \Gamma_{15}: 0_2{}^6 0'_2{}^6C_6{}^7h_1{}^3 \ldots.$$

All the lines h_i are parasitic.

They can be generated by webs of invariant surfaces of orders 2, 3, 4 respectively. None is contained in a tetrahedral complex. A second particular quadratic complex is that composed of the secants of a fixed conic. M. Pieri(86) determines the involution belonging to this complex. The lines P_1P_2 from each point of C_2 generate a surface F_6 with C_2 and the vertex of the bundle for triple point. These ∞^1 F_6 have C_{13}, $p=14$ in common, $[C_{13}, C_2]=13$.

$$\begin{aligned} &s_1 \sim s_{28} : C_2^{14} C^5_{13} h_1 \ldots h_{39} \qquad \text{all } h_i \text{ parasitic,} \\ &C_2 \sim P_{27} : C_2^{13} C^5_{13} h_1 \ldots h_{39}, \\ &C_{13} \sim \Omega_{78} : C_2^{13} C_{13}^{39} h_1 \ldots h_{39}, \\ &K_{18} : C_2^{9} C^3_{13} h_1 \ldots h_{39}. \end{aligned}$$

P_{27} and K_{18} touch along all 9 sheets of the latter at every point of C_2; Ω_{78}, K_{18} touch along all 3 sheets of the latter at every point of C_{13}. Those involutions contained in a special complex, i. e., one composed of the tangents to a surface were derived in part by M. Pieri(83) and completed by D. Montesano(64). The ∞^3 tangents can be transformed into the points of ∞^2 conics forming a linear system. Each conic is invariant in the involution and lies in a tangent plane to the given surface. Instead of conics, there may be a linear congruence of invariant lines, reducible to monoidal types. The former may be formulated thus: Given in S_3 a linear congruence of $|C_2|$ situated in the tangent planes of a surface. If the two points of each conic collinear with the point of contact of its plane are regarded as conjugate, the correspondence thus established is a birational and involutorial transformation, belonging to the complex of tangents to the surface. The latter are contained in the two categories which follow: Given two linear line congruences γ, γ', with d as common directrix, and d', d'', as second respective directrix. If two lines are homologous which meet d at the same point, and d', d'' in projective ranges, this homology must be involutorial, or if a linear system Σ of homologies be established between d', d'', this system may be referred projectively to the points of d, homologous lines meeting on d, and meeting d', d'' in the pair of points associated with the homology belonging to the point on d. This system Σ must be made up of involutions.

The complexes (or congruence) to which those involutions belong which leave a web of cubic surfaces invariant were determined by C. Moffa(57). The cubic complex associated with the cubic involution was found by Th. Reye(90) and its equation derived by F. R. Sharpe and V. Snyder(106). The I_2 of order 15 found by S. Kantor(44.1) and J. R. Conner(17) belongs to this same complex.

Another involution belonging to Γ was obtained by D. Montesano(58). Let $B_1 \ldots B_8$ be the basis points. Consider the fixed quadric cone $B_6^2 B_1 \ldots B_5$. A point P fixes the C_4 of the net which passes through it. This C_4 meets K_2 in one variable point Q. The line PQ is in Γ. Conversely, given any line p of Γ. It meets K_2 in two points Q_1, Q_2, each of which determines a C_4 passing through it. Each curve meets p in one pair of points P_1, P_2. Thus between the lines of Γ and the points of S_3 there is a (1, 2) correspondence, and the associated involution is that of P_1, P_2

$$\begin{aligned}
&s_1 \sim s_9 : d^8{}_{78} c^5{}_{78} c_{16} \ldots c_{56} g_1 \ldots g_5,\\
&d_{78} \sim D_7 : d^4{}_{78} c^2{}_{78} c_{16} \ldots c_{56} g_1 \ldots g_5,\\
&c_{78} \sim c_{78},\\
&c_{16} \sim s_1 : B_i d_{78}.
\end{aligned}$$

The lines g_i are also parasitic. Each g_i lies on the quadric through $c_{78} c_{6i} d_{78} d_{6i}$, and is a generator of the other system through B_i.

9. *Infinite Discontinuous Birational Groups.*—Compare Section 7 of Chapter VIII. As nearly all the infinite discontinuous groups of Cremona transformations thus far known can be generated by involutorial transformations, they should be treated in this connection, although the transformations themselves are not involutorial. G. Humbert(38), (39) and (40) gave the first illustration in connection with a particular form of the Kummer surface. The method employed is transcendental, but it can be expressed as the product of two projections of the surface upon itself from two nodes—thus two monoidal involutions. Another illustration was furnished by P. Painlevé(76), which involved those elliptic functions which admit complex multiplication. J. I. Hutchinson(41) and (42) showed that the methods previously employed could be applied to the general Kummer surface, and to certain other surfaces birationally equivalent to it. G. Fano(25) showed that any quartic surface through a hyperelliptic sextic curve of genus 2 has two infinite series of such curves, is invariant under the characteristic $g_2{}^1$ of each, and that their product is not periodic. This quartic surface does not contain a system of elliptic curves; the cases previously mentioned all depend essentially on the non-central involutions on a non-singular cubic curve. F. Enriques(23) proved that every surface invariant under an infinite discontinuous group of birational transformations has a pencil of elliptic curves, except when $p_a = P_1 = 1$.

In these papers the authors were concerned only with points on the surfaces, hence did not raise the question whether the transformations were contained in a Cremona group or not.

Numerous examples of the first category are given by V. Snyder(115). Quartic surfaces with $m(1 < m \leqq 16)$ nodes exist under such a group. Another form was given by Snyder(109) by means of congruence of lines which can be arranged in conjugates on quadric surfaces. Interchanging the points of contact with the focal surface on each generator is an involution, and the two conjugate systems generate a non-periodic transformation. Further development by H. B. Owens(75).

A. Rosenblatt(95) gave examples of irregular surfaces with a system of elliptic curves; in (96) these results were considerably extended. Particular examples of quartic surfaces were given by Garnier(30) and by Remy(89). Further developed by V. Snyder(116).

F. Severi(102) expressed the transformations of Fano by means of the base, obtaining all the linear transformations which leave a certain quadratic form invariant. V. Snyder(117) gave a number of further infinite groups belonging to the Kummer surface. The question raised by H. F. Baker(7) is answered and the result expressed in terms of the Severi base by F. R. Sharpe and C. F. Craig(103). Later Snyder(118) generalized the result of Rosenblatt to surfaces of any given index of irregularity, and showed that all the transformations employed are Cremonian.

L. Godeaux(31) proved that the most general quartic through a conic is not invariant under any birational transformation. His proof is not complete. For certain other rational curves on quartics, see Godeaux(32). F. R. Sharpe and V. Snyder(105) proved a similar theorem for quartics through a straight line, a conic. For a single space cubic one involution exists; for two space cubics the product of these involutions is not periodic.

A résumé is given in the presidential address of H. F. Baker(8).

L. Godeaux(33) showed that the quartic through a general space sextic curve of genus 3 is invariant under an infinite discontinuous group of birational transformations. V. Snyder and F. R. Sharpe(119) showed that this and all other known transformations of infinite order under which a surface remains invariant are contained in a Cremona transformation. This one does not contain involutions nor does the surface contain infinite systems of hyperelliptic curves, thus being more general than the Fano surface. The quartic through an unrestricted C_8, $p=2$ is invariant under two Cremona involutions, which generate an infinite group. F. R. Sharpe and V. Snyder(107). This and two other forms were considered by G. Fano(27) and (28). Thus, a quartic surface with two skew lines has three operations which generate an infinite group. The first is the involution of points on the transversals of the two lines, and the second and third are expressed by an even projection of each cubic curve through one line or the other into itself. The symbolic form of every transformation of the group is expressed in terms of the Severi base.

This is more general than the particular case $n=1$ of C_1 and a rational C_m meeting it in $m-1$ points treated by Sharpe and Snyder(104), since the parts played by the two skew lines may be interchanged. The second case is that of a quartic with a double point and a simple line through it, and the third is an F_4 with C_m, $p=2$. If $m=6$, the Fano surface results; $m=8$ gives that discussed by F. R. Sharpe and V. Snyder. An infinite group is associated with every even value of $m>4$, and for certain odd values.

10. *Involutions Belonging to M_3 in S_4.*—Although all involutions I_n of order n in S_1 or S_2 are rational, it was proved by F. Enriques(24), that this is not true in S_3. Groups of n points exist such that any point uniquely determines the group to which it belongs, that cannot be mapped rationally on the points of S'_3. Examples with smaller value of n were given by G. Aprile(1). I_2 belonging to varieties M_3 in S_4 were obtained by G. Marletta(52) and (53). One new theorem is: Every M_n of S_4 having an S_2 to multiplicity $n-3$ and in it an S_1 to multiplicity $n-2$ is representable by the pairs of an I_2 in S_3.

It is not known whether this M_n is rational or not. It has been shown by G. Fano(27a) that irrational varieties in S_4 exist which have all invariants zero.

It was shown by V. Snyder(120) that the projection of the cubic variety $V_3{}^3$ of S_4 into itself from points on it cannot form a continuous group. Nor can the transformation defined in any plane through a line p on $V_3{}^3$ by means of the harmonic homology having p for axis and its pole as to the residual conic in the plane as center. Nor can the combination of the two generate a continuous group.

The details of mapping the general cubic variety of S_4 on a double S_3 were given by V. Snyder(113). The lines joining pairs of conjugate points of I_2 in S_3 form a special linear complex, but each line contains two pairs of points. The I_2 is of order 6 and is monoidal of type *MJ*.

It is not known whether certain of the involutions I_2 in S_3 are rational or not, in particular

$$s_1 \sim s_n : c_1{}^{n-2} c_{3n-4},$$

given by D. Montesano(66) and the types given by him in (69).

Numerous I_2 belonging to the cubic V_3 of S_4 having 10 double points were given by E. Ascione(6a), but these are all rational.

A more general form was given by G. Marletta(53). Given in S_3 a point A and a C_8 of $p=7$, the residual intersection of two cubic surfaces through a common line. An S_2 through A meets C_8 in 8 points, through which can be drawn C_4 having A^3. The $\infty^2|C_4|$ describe a rational congruence of order 2. The $|F_5|:A^4C_8$ is a pencil.

Each C_4 is on one F_5; residual is S_1 through A, not meeting C_8. Any S_2 through this S_1 meets F_5 in a C_4 of the congruence. Each F_5 contains two such S_1 through A, forming I_2. Their S_2 meets $|F_5|$ in basis C_3. In S_4 construct $M_n : S_2{}^{n-2}$ with $S_1{}^{n-2}$ in the multiple S_2, with curve a meeting any S_3 through S_2 in 1 point. Take also $M'_3 : S_1$ and Σ_3. Take a point P on M_n, and construct $\Sigma'_3 = PS_2$. It meets M_n in F_3 and meets M' in F'_3, which have S_1 in common, hence residual is C_8. The point $A = (\Sigma', a)$ is triple on C_4 meeting C_8 in 8 points. Residual is 1_1, 1_2 meeting Σ, S_2. Hence there is a (1, 2) correspondence between M_n and Σ_3.

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CHAPTER X

REDUCTION OF SINGULARITIES OF SPACE CURVES AND SURFACES

1. In this very short chapter we shall not attempt to do anything further than to bring up to date the two existing and readily accessible accounts by Castelnuovo-Enriques[(5) p. 639], and by Berzolari[(3) Ch. 31 § 5, 36 § 4] which cover the literature down to about 1915. A critical account of singularities of surfaces still remains to be written. Due to the doubts surrounding many of the papers on the subject, to the difficulties inherent in the question, the writing of a report of that nature would be an arduous task indeed. It is one which the committee hopes may be accomplished in a not too distant future.

2. There is relatively little to say concerning singularities of space curves. The only recent writings are Enriques(9), Enriques-Chisini[(11) p. 551], Albanese(2). Enriques-Chisini give a very complete treatment of consecutive singularities. They analyze those of C of S_3 by considering those of its plane projection, or else directly by means of quadratic transformations of the first kind (with fundamental point and conic). They show that the two methods give the same result. They also show that the transformations in question enable one to reduce the singularities of C to multiple points with distinct tangents. They prove that the reduction to a curve without any singularities whatever can be obtained by means of quadratic transformations of the second kind (with three fundamental points and a fundamental straight line). The same authors have also given a good historical account [p. 575].

Albanese(2) gave a new and very rapid reduction birational on C but not on its carrying space.

3. We pass now to singularities of surfaces. Enriques(9), (11), Enriques-Chisini[(12) p. 577], Chisini(7), (8) have investigated at length the behavior of a surface in the vicinity of a singular point. Enriques-Chisini examine also many special cases and give valuable historical data(12) [p. 654]. They consider the relations between multiple points and multiple curves, consecutive singularities and the behavior of the first polars at the singular points. They also examine the behavior of $z(xy)$ and the permutation of its values when (x, y) is in the vicinity of (x_0, y_0) where (x_0, y_0, z_0) is a singular point of F. Of particular interest is the case when the point is an intersection of two or more branches of the

total multiple curve, especially in regard to the local parametric representations by means of power series, Chisini(7), (8). Enriques in his papers, also Enriques-Chisini[(12), p. 634] examine the character of the singularity in regard to the number of distinct locally irreducible branches of F through it; the approximation of F by means of distinct rational surfaces to an assigned infinitesimal degree; the relation of all this to the local parametric representation of F by sets of holomorphic or meromorphic functions of two variables.

Related investigations: Schmeidler(16), H. W. E. Jung(13) first five chapters. Jung in particular gives in Ch. 5 valuable contributions to the theory of the consecutive curves to an assigned point on F. His book is a compendium on his investigations on algebraic surfaces. Its cardinal feature is the extension to surfaces of the notion of *divisor* of the so-called arithmetical theory of functions of one variable, associated with the names of Kronecker, Hensel, Landsberg and presented in a recent book(14) by Jung himself, also by Bliss(4). In the case of surfaces, complications arise due to the fact that *divisors* naturally ascribed to curves, must sometimes be assigned to certain point-neighborhoods or consecutive curves. It is this question which Jung studies in very complete fashion in (13).

4. Chisini(6) has reduced the singularities of a surface assumed in S_3, by birational transformations of S_3 itself. He removes the isolated singularities and exceptional singularities of the multiple curve by a sequence of quadratic transformations whose fundamental lines pass through the points. That the process is finite he shows by means of a certain number, the *rank* ρ of a singular point P which is a non-increasing integer under the operations considered. Let L be the complete intersection of the first polar surface of an arbitrary point O of S_3 with F, each component of L being counted only once in the curve. L is what Chisini calls the *simplified* polar curve. On the projection L' of L from O on an arbitrary plane the projection of P produces a reduction of the class of L' by an amount say v. Let also r be the multiplicity of L at P and c the number of distinct branches of the multiple curve issuing from P. Then

$$\rho=4v-5c-r(r-1).$$

Thus at nodes of the multiple curve of F, $\rho=-4$, at cusps of that curve $\rho=1$. The rank ρ is invariant under a birational transformation regular at P. Under the transformations that he uses Chisini shows that the rank is never increasing. Utilizing this fact he reduces F to a surface whose singular points of the type considered are of rank 1, -4, and are merely double points with distinct tangents for the simplified polar L. F will then have no other exceptional singularities than points where two distinct multiple curves cross one another without contact, or base points of the variable part of L where it is not tangent to the multiple curve. The elimination of sets of consecutive multiple curves is accomplished by Chisini by means of monoidal transformations, his results in (6), (7) playing an important part in this last step. The reduced surface has no other singularities than multiple curves with distinct tangent planes at arbitrary points and with no other exceptional points than those considered above.

5. Albanese gives in (1) a reduction birational on F but not on its carrying space. He first constructs on F a linear system of curves $\sum_k^h$, where $h>6$ is the dimension and k the number of variable intersections of any two of its curves at finite distance from the base points. Furthermore the system is simple, i. e., its curves that are made to pass through an arbitrary point of F, do not necessarily pass through any other. Finally it has the property that any similar subsystem $\sum_{k-j}^{h-i}$, $i \leqq 4$, is irreducible and such that $j<2i+1$. By means of the system F is reducible to F' having only the following singularities: A finite number of isolated double points that may be consecutive to one another; double lines that do not intersect and carry a finite number of simple cuspidal points.

In regard to the singularities of surfaces see also Severi(17). For those of algebraic varieties see Schmeidler(16). Lefschetz[(15) p. 368] established that any algebraic variety of rank one is birationally equivalent to a non singular variety in some space. See also Ch. 17, No. 8.

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CHAPTER XI

MULTIPLE CORRESPONDENCE AND MAPPING IN SPACE AND HYPERSPACE

1. (1, k) *Correspondence between* S_3, S'_3, $K>2$.—A (1, 8) correspondence between the points of (x') and (x) is obtained by associating the quadric surfaces of a web $|\phi|$ with the planes of (x'), $x'_i=\phi_i(x)$. A line in (x') has an elliptic c_4 for image, and on it lie all eight associated points, image of a variable point on the given line. The line joining a pair of associated points P_1P_2 is a component of $\infty^1|c_4|$ of the system through them. It is called a principal line. Th. Reye(114). The surface of coincidences is K_4, jacobian of the web. Its image in (x'), with which it is in (1, 1) correspondence is L'_{16} of class 4. The images of the principal lines are straight lines in (x'), bitangents of L'_{16}, the surface is the complete focal surface of the congruence. An arbitrary line of (x) is transformed into a conic c'_2 which touches L'_{16} in four points.

A plane of (x) has for image a Steiner surface of order 4, having three concurrent double lines. The web of quadrics contains 10 composite ones—pairs of planes; the 10 lines lie on K_4.

If $|\phi|$ has r basis points, there are r bundles of lines composed of principal lines, making r systems of bitangents to L'_{16-2r}. Each congruence is of class 2.

The case of $r=6$ has been considered in Art. 8 of Chapter X. If $|\phi|$ represents a web of surfaces of order n without basis elements, $x'_i=\phi_i(x)$ defines a (1, n^3) correspondence between (x') and (x). The jacobian $J(\phi)$ is the locus of coincidences: L' is of order $4n^2(n-1)$, and of class $4(n-1)^3$. A locus in (x) which is not generated by all n^3 associated points has for image in (x) a locus which touches L' in every image point of those on K. Th. Reye(115).

$c_1(x)\sim c'_n(x)$, touching L' in $4n-4$ points.
$s_1(x)\sim s'_{n^2}$, touching L' along a curve of order $4n(n-1)$.

The case $n\geqq 2$ was further employed by Th. Reye(116) to obtain systems of surfaces containing series of conics. The (1, 8) transformation defined by $x'_i=x_i^2$ which sends the lines of a Battaglini quadratic complex into the ∞^3 conics which touch the faces of a tetrahedron was considered by C. Segre(133).

Many of the properties derived by D. Montesano(89) as applying to I_8 are at once obtainable from the results of Segre.

H. E. Timerding(159) treated a particular case of $n=2$ by putting $x'_i=x_i^2$. The web of quadrics has a common self-polar tetrahedron. Both K and L' are composite, each consisting of the faces of the tetrahedron of reference. A quadric inscribed in L' is transformed into a tetrahedroid in (x)—a special Kummer surface in which the six nodes in any singular plane are in involution. The image in (x') of a general plane of (x) is the general Steiner surface, as in any web $|\phi|$ of quadrics. J. de Vries(167) discusses a (1, 8) correspondence between the Kummer quartic surface and a plane. V. Martinetti(88) gives further properties of the $(16_6, 16_6)$ configuration of Kummer.

J. de Vries(169) obtains further properties of the (1, 8) correspondence. K. W. Rutgers(119) gives alternate proofs for properties of the surface L'.

R. Scaccionoce(121) gave an analytical representation of a surface generated by two planes and a bundle of surfaces of class p having a $(p-1)$ fold line, all birationally related.

Further properties of 8 associated points were found by H. Newmann(91).

The $(1, 8-r)$ correspondence determined by a web of quadrics with $r=0\ldots5$ was considered by V. Snyder and F. R. Sharpe(149). L'_{16} has a double curve of order 60 and a cuspidal curve of order 36. When $P_1=P_2$ describes K_4, the other six points describe a surface R_{24} of order 24, having a cuspidal curve of order 90 and a double curve of order 120. Similar properties when $|\phi|$ has 1, ..., 5 basis points.

J. R. Musselman(90) showed that K_4 contains two distinct ∞^3 systems of sextics of $p=3$. See also J. R. Conner(29.1). It can be transformed birationally into the symmetroid. The ∞^3 self-polar triangles as to a given conic were mapped on the points of S_3 by G. Schaake(122) by means of a (1, 1) correspondence between the points of the plane of the conic and the bisecants of a rational space quartic curve. The vertices of self-polar triangles correspond to three concurrent bisecants. In the plane of the conic is a c_6 of $p=6$ containing ∞^1 configurations $(10_3, 10_3)$ of Desargues.

A. Del Re(32) obtains various multiple correspondences by means of linear systems of monoids with a common vertex.

H. de Vries(166) defines a (1, 9) correspondence between the planes and points of S_3 by means of two general space C_3.

J. de Vries(168) considers an involution of tangents I_p of a rational space curve R_n. The lines in each group are divided into couples, and the linear congruences constructed having each pair for directrices. As the group describes I_p this congruence describes a complex, the order and properties of which are determined. A particular case is developed in detail, leading to a quadratic system of quadrics and its dual which defines a Kummer configuration $(16_6, 16_6)$.

2. *Lines of Complexes Mapped on Points of S_3.*—The lines of a linear complex can be mapped birationally on the points of space. F. Klein(60); M. Noether(92); F. Klein and S. Lie(59); F. Klein(61); S. Lie(73).

The quadratic identity $p_{12}p_{34}+p_{13}p_{42}+p_{14}p_{23}=0$ and the equation of the complex define a three-dimensional quadratic variety M_2 in S_4, which can be projected stereographically upon the points of space. (Noether, Klein). F. Klein(62). S. Lie considered two equations $f_1(x_1, x_2, x_3, x_4; x'_1, x'_2, x'_3, x'_4)=0$, $f_2=0$ homogeneous in two sets of coordinates, making the points of either space correspond to an ∞^3 system of curves in the other. When both equations are bilinear, the curves are lines of a complex.

Given two bilinear equations $\Sigma a_{ik}x_ix'_k=0$, $\Sigma b_{ik}x_ix'_k=0$ between the point coordinates of S_3 and S'_3, if the points (x) of one of them are images of the lines of a linear complex of the other (x), then the points of (x) are images of the lines of a special quadratic complex of the other, consisting of the lines which meet a fixed conic, or the tangents of a quadric surface. S. Lie(72).

For the early history see S. Lie(68). See also S. Lie(74) and the statement made by S. Lie and F. Engel(69).

In particular

$$x_1x'_4-(x'+ix'_2)x_4+x_3x'_3=0,$$
$$(x'_1-ix'_2)x_3-x_2x'_4+x'_3x_4=0,$$

map the lines of the complex $p_{12}-p_{34}=0$ on the points of (x'), and the points of (x) on the complex of secants of $x_1^2+x_2'^2+x_3'^2=0$, $x'_4=0$. When the point describes a line in (x), the image line in (x') describes a regulus through the conic. The lines of the conjugate regulus are images of the points of the conjugate of the given line as to the complex $p_{12}-p_{34}=0$. If the conic is the imaginary circle at infinity, the line-sphere transformation results. S. Lie and G. Scheffers(70). C. M. Jessop(55). V. Snyder(148) has given the details, nothing new.

R. Bricard(20) mapped the points of S_3 on the tangent lines to a quadric surface birationally. When the point describes a line the image line describes a quadric surface circumscribed about the given one. The line-sphere transformation of Lie appears as a particular case. Further cases have been given by E. Study(150), H. Beck(10), M. Fouche(46), L. Berwald(18). See the criticism by S. Kantor(58). E. O. Lovett(78), H. Liebmann(75), have generalized the method to apply to much wider cases.

P. F. Smith(147) mapped the points of S_3 on the spheres of a linear spherical complex, employing a direct generalization of the transformation by reciprocal radii.

P. del Pezzo(105) maps the complex on the points of S_3 by means of the quadrics through two skew lines.

Special linear complexes were mapped on S_3 by A. Wiman(181) by means of the bilinear equations

$$x'_1x_1+x'_2x_2=0, \qquad \Sigma a_{ii}x_ix'_i=0,$$

and applied to the determination of ruled surfaces having a directrix line. The same method is also employed to obtain ruled surfaces in a general linear complex, and those which contain a conic. Further details are given by R. Sturm(152), R. Schuhmacher(127), T. Reye(111), F. Schur(128), L. Cremona(31).

The tetrahedral complex—consisting of the ∞^3 lines which meet the faces of a fixed tetrahedron in points having a fixed cross-ratio—was mapped on the points of S_3 by A. Weiler(178) and (179).

If the complex is $p_2p_5+kp_3p_6=0$, we may put

$$\begin{aligned} p_1&=x_1x_2, & p_4&=(k-1)x_3x_4,\\ p_2&=x_1x_3, & p_5&=-kx_4x_2,\\ p_3&=x_1x_4, & p_6&=x_2x_3. \end{aligned}$$

See also E. Bertini(11). The mapping also follows directly from its method of generation by means of two projective S_3. T. Reye(110). For further details and consideration of special cases, see J. Cardinaal(23); C. Servais(141); St. Jolles(57).

Linear complexes are mapped on the points and circles of a plane by L. Eckhart(34). The complex $\Sigma a_ip_i=0$ is associated with the circle $a_3(x^2+y^2)+(a_2+a_5)x-(a_1+a_4)y-a_6=0$ and the point $(-a_5, a_4, a_3)$.

The general quadratic complex was mapped on the points of S_3 by F. Klein; see M. Noether(92). Using $x_1, \ldots, x_6$ for line coordinates, the equation may be put in the form

$$\phi(x_1, x_2, x_3, x_4)+Ax_5+Bx_6=0,$$

in which A, B contain only $x_1, \ldots, x_4$.

$$\begin{aligned} \rho x_i&=\xi_i(A\xi_3-B\xi_2), & i&=1, \ldots, 4.\\ \rho x_5&=\xi_1\xi_4B-\xi_3\phi, & \rho x_6&=-\xi_1\xi_4A+\xi_2\phi. \end{aligned}$$

G. Darboux(33) shows that the system of quadric surfaces through 4 points gives a map of the general quadratic line complex on the points of S_3. A fundamental curve of order 5 appears in (ξ). See also S. Lie(71), A. Voss(165), and F. Aschieri(3).

The tetrahedral complex is also the projection in S_3 of the lines which meet three planes in S_4, C. Segre(135) and (138); and also the intersection with S_3 of the ∞^3 planes which meet a normal C_4 in S_4 in three points. C. Carrone(24).

Another method was employed by J. de Vries(171). Given a pencil of quadric surfaces $|F_2|$. The polar planes of a point P form a pencil, axis p. As P describes S_3, p generates the tetrahedral complex having the self-polar tetrahedron of the pencil of quadrics for fundamental tetrahedron.

The quadrics circumscribing the tetrahedron go into (2, 2) congruences contained in the complex. A general surface F_n of order n is transformed into a $(n, 3n)$ congruence. G. Castelnuovo(27).

K. Ogura(93) and (94) gives a different method of mapping the tetrahedral complex on the points of S_3, and applies a similar method to map various cubic complexes on S_3.

E. Caporali(21) and (22) investigated in detail the (2, 2) congruences contained in a given quadratic complex by the method of Klein. The same results were obtained by E. Bertini by projection from an S_1 in S_5, see C. Rosati(118). A (1, 2) representation was given by R. Sturm(155) previously proposed and partially carried out by F. Aschieri(6) and (7). A particular form of quadratic complex, generated by two projective pencils of linear complexes was given by F. Aschieri(5). More generally D. Montesano(84) mapped the same quadratic complex on the points of S_3. R. Sturm(156). The method was proposed by C. Segre(139).

Special cases of the complex of Battaglini were mapped on S_3 by D. Montesano(87).

H. Beck(10) maps the points of space on the tangents to a quadric, but each tangent corresponds to two points, conjugate in an axial harmonic homology. The points of the two directrices are transformed into the two reguli of the quadric. The Lie line-sphere transformation appears as one interpretation of a limiting cases when a parameter vanishes.

The cubic complex Σ_3 of the generators of the quadrics in a net has the 8 basis points as vertices of singular bundles. R. Sturm(157) and (158). See also (151). F. Aschieri(8). T. Reye(113) and (112). See also J. C. Kluyver(64). D. Montesano(85) mapped Σ_3 birationally on the points of S_3 as follows. Let π be the plane of any three basis points. A point P determines a pencil in the net, basis curve C_4, which meets π in one variable point P'. PP' is a line of Σ_3, unique when P is given. Conversely, given a line p of Σ_3. It meets π in P' and fixes the C_4 passing through it; p meets C_4 in a second point P. There are 56 such mappings, and the product of any two defines a (1, 1) point transformation in S_3.

These are of three different kinds, according to the number of common basis points in π, π'.

Montesano also defines a (1, 2) correspondence between the lines of Σ_3 and the points of S_3. Let K_2 be the quadric cone through the basis points $B_1 \ldots B_5$ and having its vertex at B_6. A point P fixes the C_4 of the net passing through it, and C_4 meets K_2 in Q. PQ is a line of Σ_3. Conversely, given a line p of Σ_3. It meets K_2 in two points Q_1, Q_2, each of which determines a C_4, meeting p in P_1, P_2. In the associated involution of P_1, P_2 each line of Σ_3 joins one pair of conjugate points. If C_{ik} is the cubic through all basis points except B_i, B_k and d_{ik} is the line joining B_iB_k and g_i is a generator of the quadric through d_{78} and d_{8i}, through B_i, we may write

$$\begin{aligned} s_1 &\sim s_9 : C_{78}{}^3 d_{78}{}^5 \cdot C_{16} \ldots C_{56} \cdot g_1 \ldots g_5, \\ C_{78} &\sim \gamma_7 : C_{78}{}^2 d_{78}{}^4 \cdot C_{16} \ldots C_{56} \cdot g_1 \ldots g_5, \\ C_{i6} &\sim s_1 : B_i d_{78}; \end{aligned}$$

each g_i is parasitic. The lines Σ_3 contain conjugate points of 168 such involutions.

Another method of mapping Σ_3 is obtained by associating each line p of Σ_3 with the point P on it such that the tangent plane to the quadric of the net through p at P shall pass through a given basis point B_8. Conversely, given P; the line PB_8 determines a quadric of the net, and the other generator through P is p. There are 8 such mappings, and the product of any two defines a birational transformation, in which the lines PP' belong to Σ_3. F. R. Sharpe and V. Snyder(145).

It was shown by G. Fano(43) that the general complex of order higher than 2 cannot be mapped birationally on the points of S_3. The ∞^3 circles of a plane were mapped on the points of S_3 by P. H. Schoute(134). Given a triangle ABC, $U=0$ its circumscribed circle, and P any point. Project A, B, C from P on opposite sides, a is length of side $B'C'$, etc., any circle then has the form

$$U + \lambda a^2 + \mu b^2 + \nu c^2 = 0.$$

Let λ, μ, ν be rectangular coordinates of a point in S_3. The condition for orthogonality is then expressed as polarity as to a fixed quadric surface. A variation was employed by K. W. Walstra(177) the center being the x_1, x_2 of the image point, and the third coordinate of the point being the power of the origin as to the given circle. Further developed by J. de Vries(170). Another form was employed by G. Bordiga(16) who considered a complex of circles in S_4 and mapped them on the points of S_3.

M. Pieri(108), D. Montesano(86) and (88) mapped the complex of tangents to an algebraic surface on the points of S_3. The necessary and sufficient condition that the correspondence is (1, 1) is that the surface is rational. A summary of complexes associated with (1, 1) correspondences is given by G. Loria(76).

The lines of S_3 were mapped on the pairs of lines and points of a plane by B. Mayor(83). Given two projective bundles of planes; map the planes of one on the lines of an auxiliary plane, and those of the other on the points of the same plane. A line not through the vertex of either bundle uniquely determines a line and a point, and conversely. The locus of united elements is the image of a linear complex. The exceptional elements are a special linear complex, and the image of its axis. Further applications were made by G. Marletta(79) including the theory of algebraic congruences. A more extensive study of defining lines by an ordered pair of points in the plane was made by J. Grünwald(51) with applications to kinematics. Two different methods are developed, one linear, and the other by means of the Plücker quadratic identity. The Lie line-sphere transformation is included in the latter.

L. Eckhart(34.1) generalized the various methods employed in descriptive geometry and gives in analytic form the principal schemes of mapping the lines of space or of hyperspace on groups of points in the plane.

A similar study of pairs of points of S_2 mapped on a quartic variety of S_5 was made by G. Schaake(122.1). The pairs of conjugate points as to a fixed conic of the plane are mapped on the points of S_3. This was carried further by the same author in (122.2), by means of a fixed space cubic curve.

3. *Lines of S_3 Mapped on Points of S_4.*—The quadratic identity $\pi=0$ among the six homogeneous line coordinates defines a four-dimensional variety in S_5 which can be projected stereographically into S_4. F. Klein(62). The linear transformations under which π remains invariant can be interpreted in S_4 in terms of motion, similarity and transformation by reciprocal radii. A similar interpretation can be made concerning an $n-1$ quadratic variety in S_n. A. Giacomini(47).

If $\pi=0$ be written in the form $x_1^2+x_2^2+x_3^2+x_4^2-x_5x_6=0$, then

$$\begin{aligned}&\rho x_i = x'_i x'_5, \qquad i=1, \ldots, 4.\\ &\rho x_5 = x_5'^2.\\ &\rho x_6 = x_1'^2+x_2'^2+x_3'^2+x_4'^2.\end{aligned}$$

The exceptional elements in S_4 are the hyperplane $x'_5=0$ and the quadric hypercone $x_1'^2+x_2'^2+x_3'^2+x_4'^2=0$, and in line space, the directrix of the special complex $x_5=0$.

Other methods of mapping the lines of S_3 on the points of S_4 were given by F. Aschieri(8); F. Chizzoni(29); G. Loria(77). Let Σ_3, S_4 be two linear spaces. In S_4 given two points, vertices of projective systems of S_1, S_2, S_3. To a line r of Σ_3 associate a perspective S_2 of the first hyperbundle, and construct its projective image S'_2. Let S_2, S'_2 intersect in P, image of r, and conversely. This is a direct generalization of F. Aschieri. The method of Chizzoni was also generalized by G. Loria. Given Σ_3 and in it two planes α, β. In S_4 given two lines a, b. Make points of α projective with planes of S_4 through a; points of β projective with the planes of S_4 through b. A line r of Σ_3 meets α, β each in a point. The image planes of these points meet in P, image of r, and conversely.

G. Lazzeri(65) mapped the lines of S_3 on a plane point line connex.

G. Scorza(129) gave an interpretation of the complete ternary field in terms of the lines of space.

The lines of S_n can be mapped birationally on the points of a rational variety of dimensions $2(n-1)$ in $S_{\binom{n+1}{2}-1}$ by the method of H. Grassmann(48) and (49). Let (x), (y) be any two points of S_n. The coordinates of the line joining them are the two rowed determinants of the matrix

$$\begin{vmatrix} x_1 & x_2 & \cdots & x_{n+1} \\ y_1 & y_2 & \cdots & y_{n+1} \end{vmatrix}$$

These determinants are regarded as point coordinates of $S_{\binom{n+1}{2}-1}$, and the identities among them define the variety $M_{2(n-1)}$. E. d'Ovidio(95) and (96). H. Grassmann(50). F. Engel(38). The order of $M_{2(n-1)}$ was found by H. Schubert(126) to be $\frac{1}{2n-1}\binom{2n-1}{n-1}$. The case of $n=4$ was derived again by G. Castelnuovo(28) and $n=3$ by C. Segre(140). This method was employed by B. Levi(67) to determine the system of bisecants of an algebraic curve in S_n. The resulting forms will not be multiple, since not all bisecants of C_m in S_n can be trisecants. See E. Bertini(12) and the note by G. Castelnuovo.

A more general proposition is given by P. del Pezzo(108). For reduction of singularities by projection and by Cremona transformations, see P. del Pezzo(106); M. Pannelli(99); C. Segre(132); H. G. Zeuthen(183); and B. Levi(66).

R. Schumacher(127) considers two fixed skew lines g_1, g_2 in S_4 and two fixed planes π_1, π_2 in line space(77). A point P of S_4 determines the planes $g_1{}^P$, $g_2{}^P$ and a line g determines two points P_1 in π_1, P_2 in π_2. Now establish a projectivity between the ∞^2 planes through g_1, and the ∞^2 points of π_1. Similarly for g_2, π_2. This establishes a projectivity between g and P. M. Pannelli(101); M. Pieri(107); E. Veneroni(164). Compare R. Sturm(153); C. M. Jessop(60).

4. *Linear Line Complexes Mapped on Other Elements.*—The system of ∞^5 linear complexes can be mapped on a part of the plane line-point connexes. G. Lazzeri(65). Let π, π', π_1, π_2 be any four planes of S_3, Associate the points of π and of π_1 projectively; and the points of π_2 with the lines of π' projectively.

A line meets π in P, and π_2 in P_2, hence is uniquely associated with the point P_1 of π_1 and the line p' of π'. In this way a linear line complex is represented by a lineo-linear connex with a fixed pencil. Or on the conics of a plane [C. Segre(137); see also R. Sturm(154)] or on a point and a circle of a plane, L. Eckhart(34), or the totality of quadric surfaces through four fixed basis points, J. Kloboncek(63). In the last memoir the special linear complexes are mapped on the quadratic system of conics previously considered by L. Cremona(30) and by F. Aschieri(4). In the latter the lines of space (axes of special complexes) which are images of composite conics form a cubic complex.

Another method was given by J. de Vries(172). Given a pencil of lines (A, a) not in the complex, and a point M not in a. Let the bundle A' be made projective with the bundle A. Given a line r of the complex. Call the plane $(A, r) = \rho$, and let ρ' be its associated plane of A'. If r meets a in R, then the point ρ', $MR = R'$ is the image of r.

Conversely, if R' is given, draw $A'R'$; it is the axis of a pencil of planes of A', one plane of which passes through the point a, $MR' = R$. The image line r is the intersection of this plane and the polar plane of R. A pencil in the complex $\sim$ conic through M.

A linear congruence $\sim$ cubic surface with node at M.

A regulus $\sim$ space curve of order 4 with node at M.

The singular elements are all derived.

Another method is given by the same author(176), by means of an axial involution, the axes of which are not in the complex nor conjugate with regard to it, and an arbitrary point C. Given a line r of the complex. Let its image in the involution be r'. The transversal of r and r' through C meets r in its image point R. In this representation each line of the complex passes through its image point. Conversely, given any point R, let ρ be its polar plane, and L, λ its conjugate in the involution. Then R is the image of that line of ρ which meets CR and its conjugate in L, λ.[*]

A pencil $\sim$ conic meeting the invariant lines a, b of the involution.

A linear congruence $\sim$ cubic surface through a, b and the directrices of the congruence.

A [2, 2] congruence $\sim$ plane.

A cubic ruled surface $\sim$ line.

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[*] **The author is indebted to Dr. Marian M. Torrey for detailed abstracts of a number of recent papers to which he did not have access.**

M. N. van der Bÿl(20a) systematizes the latter method by obtaining all the mappings in which each line of the complex passes through its image point; in particular, those in which a pencil of the complex is mapped by a conic in its plane and passing through its vertex, and a linear congruence is mapped on a cubic surface. The singular elements are obtained in the general case and in various particular cases.

J. de Vries(173) generalizes the method employed in (172) to map a quadratic complex on the points of S_3.

A quadric cone in the complex $\sim$ space C_4 with a fixed double point M.

A pencil cone in the complex $\sim$ conic through M.

A conic cone in the complex $\sim$ plane C_4 with singularity at M.

A congruence (4, 4) in the complex $\sim$ plane.

5. *Various Correspondences in Hyperspace.*—The theorem of Chasles concerning the number N_1 of coincidences in (a_0, a'_0) correspondences on two superposed S_1's, $N=a_0+a'_0$ was generalized by G. Salmon(120). If there is no locus of ∞^1 invariant points, then if a_1 is the order of the locus in (x') image of a straight line in (x), then

$$N_2=a_0+a'_0+a_1(a_1=a'_1).$$

This was extended by H. G. Zeuthen(184).

$$N_3=a_0+a'_0+a_1+a'_1(a_1\neq a'_1).$$

See also H. Schubert(125).

M. Pieri(109) generalized this by induction, with the following results: If S_k, S'_k are in (a_0, a'_0) point correspondence such that an S_i in S_k has a locus of order a_i in S_k and there is no ∞^1 locus of invariant points, then the number of coincidences N_k has the two forms

$$N_{2m}=\sum_{i-1}^{m} a_i+\sum_{i-1}^{m} a'_i \qquad \text{with } a_m=a'_m$$

$$N_{2m+1}=\sum_{i-1}^{m} a_i+\sum_{i-1}^{m} a'_i, \qquad a_m\neq a'_m \text{ in general.}$$

F. Palatini(97) established a correspondence between the points (x, y) of a plane, and the real points (a, b, c, d) of S_4, wherein $x=a+ib$, $y=c+id$. Another representation was given by A. Henschel(52). [C. Segre(134). G. Scheffers(123)].

G. Castelnuovo(26), (27), and (28), constructs a (3, 6) congruence of lines in S_4 by means of three projective ∞^2 systems of S_3 through three skew lines as vertices. The congruences are projected into S_3 and mapped on the points of a plane. A configuration of 15 planes and 10 points in S_4 is found; through each point pass 6 planes and in each plane are 4 points.

G. Bordiga(14) and (15) proposes a generalization of the linear line complex of S_n, and the generation of normal curves by its means; also surfaces by projective systems of S_4 with S_3 as basis in S_6, the resulting surface can be mapped on a plane. A complex is defined as the totality of ∞^3 lines in S_4 which satisfy three given conditions, in particular, those meeting three given surfaces, or the bisecants of one which meet another or finally the trisecants of one surface. A birational transformation between the points of two S_3 is established, in which the planes of each are transformed into the web of quintic surfaces through a space cubic doubly and a plane quartic simply. Numerous errors are contained in the papers. E. Ascione(9) considers the complex of trisecants of certain F_6, F_5, F_4 in S_4 and cuts the latter by S_3, the section being the general Steiner quartic surface with three concurrent double lines. F. Palatini(98) determined the characteristics of certain linear systems of linear line complexes in space of 5 dimensions by means of the methods of enumerative geometry. By means of projection from a singular element on a surface or a variety of hyperspace, P. del Pezzo(102), (103), and (104) obtained various theorems concerning tangent hypercones, and the condition under which spatial sections are normal curves.

J. Fairson(44) obtained a correspondence between S_n and S_{2n-1} by means of invariants of systems of binary forms.

C. H. van Os(163) considers q linear systems of order m_i, ∞^p in S_n, projectively related. The locus generated by them is of dimension $n+p=q$ and of order Σm_i.

F. Klein(62) proved that every general line complex could be represented as the complete intersection of M_2 and M_n in S_5.

C. H. Sisam(146) obtained a complete representation of lines of hyperspace, S_1 in S_r by means of connexes; and this was generalized to complete representation of the totality of S_k in S_r by F. Severi(142) by means of Grassmann matrix coordinates of order $k+1$.

A. Brambilla(17) employs a (1, 16) correspondence in S_4 by means of $x'_i=x_i^2$ to study surfaces of order 8, sections by a fixed S_3 of the images of $|S_3|$. This is a generalization to S_4 of the method employed by C. Segre(133) and by H. E. Timerding(159). Other generalizations had previously been made by A. Brambilla(18) and (19).

G. Marletta(80) and (81) showed that the quartic variety with a double plane in S_4 is rational and mapped it on S_3.

C. Segre(136) studied the varieties which represent pairs of points of two S_2 and of two S_3; more generally of S_p and S_q. The resulting variety is of dimension $n=p+q$ in space of $(pq+p+q)$ dimensions, and of order $\frac{n!}{p!q!}$. It can be mapped on the points of S_n.

G. Aprile(1) showed that the variety generated by the tangent lines to a normal ruled cubic variety of S_4 which meet a generating plane of it is rational, and mapped it on the points of S_3.

J. Eiesland(35), (36) and (37) generalized the line-sphere transformation of Lie to apply to hyperplanes and hyperspheres.

F. Severi(143) employs the methods of enumerative geometry to obtain the number of coincidences in a $\infty^{(k+1)(r-k)}$ series of pairs of S_k in S_r. The bibliography of the known particular cases is included.

F. Severi(144) obtains for surfaces and varieties in S_r, $p_q = \dfrac{\pi(\pi-1)}{2}$, $p_a = p_q - \pi$, $p^{(1)} = (\pi-2)(4\pi-5)$ of the surface in (1, 1) correspondence with the non-ordered pairs of points of a curve of genus π. The points are in (1, 1) correspondence with the congruence of bisecants of the given curve.

G. Scorza(130) and (131) considers those varieties whose curve sections by S_k are elliptic, extending to varieties the theorems of G. Castelnuovo(25) and F. Enriques(40) and (41). The variety consists of an ∞^1 elliptic system of kS_{k-1} or, if $n>8$, is rational and is representable on S_k by means of a linear system of cubic forms. If $n=4$, $k>3$, it is the quartic base of a pencil of quadric varieties in S_{k+2}. If $n=5$ it can be reduced either to V_4^5 of S_7, or V_5^5 of S_8 or V_6^5 of S_9. If $n=6$, V_6^6 of S_8.

The case of V_4^4 had been done previously by C. Rosati(118).

The V_6^5 is the $\infty^6|C_1|$ in S_4, and the other cases of $n=5$ can be represented by the lines of one, or two, or three linear complexes of lines in S_4, respectively.

F. Enriques(39) considered the parametric representation of a variety of order n in S_4 with an $n-2$-fold line. It can be represented by an involution in S_3 if it contains a rational non-composite surface of conics through C_1. If V_n of S_r has a linear congruence of rational curves it can be represented by an involution of S_3 if it contains a surface not made up of curves of the congruence.

This idea was extended by G. Fano(42) to apply to three dimensional varieties with rational surface sections. Apart from V_3^3 of S_4, all such varieties are rational.

J. de Vries(174) maps the lines of S_3 on the conics of a ∞^4 system in the plane. Given a space cubic r_3 and its plane image conic r_2. A pencil of planes through an axis l cut r_3 in the groups of an $I_3(A_1A_2A_3)$. Let (B) be its image.

Draw the tangents to r_2 at the points B_k, forming a triangle with vertices C_x. As the plane describes a pencil, the points C_k describe a conic k_2, involution conic of r_2. If l is a secant of r_3, I_3 has one fixed point and an I_2. The congruence of bisecants of r_3 is mapped by the pairs of tangents of r_2.

When l describes a pencil, the conic k_2 describes a pencil.

When l describes a plane the conic k_2 describes a net with 3 basis points, vertices of a circumscribed triangle to r_2.

A regulus is mapped by a system of index two.

A bundle becomes a net with a fixed basis point, pole of the chord, image of the bisecant through its vertex.

If r_2 lies in an osculating plane of r_3, k_2 and r_2 touch each other.

The congruence (3, 1) of axes of osculation is mapped by the k_2 having double contact with r_2.

The pencils of lines in S_3 can be mapped birationally on the conics of a plane; J. Wolff(182). Given four non-coplanar points A_i and S_3, and a plane α which contains none of them. Let the line A_iA_k meet α in B_{ik}, and the plane $A_iA_jA_k = \alpha_i$ meet α in the line b_i. Take a general point A in α. Denote the ∞^5 quadric surfaces through the points A_i by 0. Let (P, π) designate a general pencil. Associate with it that quadric of 0 for which P is pole of α and π polar plane of A. This quadric is cut by α in a conic which is image of the pencil (P, π). Conversely, given a conic in α, there is one quadric of the system which contains it. The other steps can then be retraced. The singular elements are obtained in both systems.

The lines of S_3 are mapped on the pairs of points of a plane by J. de Vries(175) as follows:

Given two fixed planes α, α' and two fixed points A, A' not in either. An arbitrary line l meets α in L, α' in L'. Let AL meet a fixed plane π in P, $A'L'$ meet π in P'. The P, P' are the images of l. The steps taken in inverse order will define a line l. A pencil of lines is mapped by two projective point ranges; a regulus by two projective conics; a bundle by two projective plane fields. A linear congruence determines a quadratic correspondence between pairs of points in the plane.

6. *Compound Involutions.*—Given two multiple correspondences $(1, m)$, $(1, n)$, the first between (x') and (x), the second between (ξ', ξ), and a (1, 1) correspondence between (x') and (ξ'). By this means an (m, n) correspondence is established between the points of (x) and of (ξ), with the property that m associated points in (x) have the same n points for images in (ξ), and conversely. Such a multiple correspondence is called a compound involution.

O. Tognoli(160) applies to S_3 the method employed by C. Wiener(180) for S_2. Given three pencils of surfaces of orders p, q, r, with m common basis points. They determine an involution of order $n = pqr - m$. In (x') given p', p', r', m', n' projective with respective pencils in (x). Particular attention is given to the case $n = n'$.

Similar constructions for involutions were given by J. S. Vanecek(161). Given a fixed quadric F_2, a space curve C_m, and a surface F_p. Let π be the polar plane of a variable point P as to F_2, meeting C_m in m points. The polar plane π_m of one of them, M, meets π in a line which pierces F_p in p points X. The polar plane π_p of any of them meets the line π, π_m in a point P'. The four points P, M, X, P' are vertices of a self-polar tetrahedron as to F_2. When P describes a curve or surface of order μ, P describes a curve or surface of order $4\mu mp$. The dual construction was given by J. S. Vanecek(162). The method of O. Tognoli(160) was applied by M. Pannelli(100) to determine fundamental elements, K, K', L, L' and includes a discussion of systems having basis curves.

A particular I_8 and the configuration 8_4 was given by G. Fontené(45).

A particular (2, 2) involution was obtained by H. Rothe(117). Given a fixed plane and a number $m \neq -1$. Let P_1, P_2 be a pair of ordered points in S_3. Construct P_0 such that $P_1P_0 : P_0P_2 = m$. Pass through P_0 a plane parallel to the fixed one. The points P'_1, P'_2 symmetric as to this plane are the images of P_1, P_2. The pairs can be mapped birationally on the points of S_6.

A different (2, 2) compound involution is that obtained from two arbitrary triangles ABC, $A'B'C'$ in S_3. Given any point P, find P' such that

$$PA = P'A', \qquad PB = P'B', \qquad PC + P'C'.$$

The images of planes are quadric surfaces. The transformation was suggested by A. Jacobi(53) and (54).

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CHAPTER XII

THE MAPPING OF A RATIONAL SURFACE ON A PLANE

1. *The Quadric Surface.*—The stereographic projection of the sphere was known in ancient times; see E. Kötter(104). For the quadric of revolution see A. J. Fresnel and Deviel(78); J. N. P. Hachette(86). For the mapping of the elliptic paraboloid see C. Ciamberlini(40). For the mapping of the general quadric on a plane see M. Chasles(33), (34), (35), (36), (37) and (38). J. Steiner(132). G. P. Dandelin(64). J. Plücker(126).

A. Cayley(30). M. Chasles(38) and (39). If the quadric is projected from a point O of the quadric upon a plane which meets the generators through O in A and B, the image of O is the line AB and the images of the generators OA, OB are A and B respectively. A plane section goes into a conic through A and B; in general a curve of order $p+q$ meeting OA in p points and OB in q points goes into a curve of order $p+q$ having A for p-fold and B for q-fold point. The generators of the systems OB, OA go into lines through A and B respectively. The papers of Plücker and Cayley express the projection analytically. In the case of a cone the points A and B' coincide and the images of the plane sections are conics having a common tangent at a point A.

2. *The Cubic Surface.*—The mapping of a cubic surface F_3 on a plane was first made by A. Clebsch(41) by means of the generation of F_3 by three projective bundles of planes, used by H. Schröter(128) in discussing the 27 lines on the surface and previously given by J. Steiner(134) and H. Grassmann(81) and (82). If (x) is a point on a plane and (y) on the surface there are three relations of the form $\Sigma a_{ij}x_iy_j=0$. These were first given by L. J. Magnus(107). The particular case in which $a_{ij}=a_{ji}$ was further developed by O. Hesse(87). Solving for (y) it is seen that a plane section goes into a cubic curve through 6 fixed points. Each of these points corresponds to a line a_i on F_3. Six other lines b_i correspond to the conics through five of the six points and the remaining 15 lines c_{ij} to the lines joining two of the six points, L. Cremona(57). A line in (x) goes into a space cubic on F_3 meeting each b line in 2 points, each c line in one point and not meeting each a line. A curve of order n in (x) goes into a curve of order $3n-\Sigma m_i$ where m_i is the multiplicity of the curve at the point a_i. A cubic surface with a double point is depicted by the special case in which 6 points lie on a conic or 3 points lie on a line or two are consecutive.

A mapping which is geometrically simpler was given by A. Clebsch(51) by means of the transversals of 2 skew lines on F_3, the skew projection of J. Steiner(131) and (132). The plane sections go into quartic curves with 2 common double points and 5 simple basis points. This system of quartics is transformable into cubics through 6 points so that the two mappings are equivalent. Similarly the mappings given by F. Dimont(73) by curves or orders 4, 5, or 6 are all transformable into cubics through six points. The case of sextics with 3 triple points and 6 simple points corresponds to the mapping by means of the bisecants of a space cubic on F_3. The same remark applies to other mappings given by L. Zängli(145). The mapping given by G. Majcen(108) obtained by defining the cubic surface by the intersection of the transversals of 2 skew lines and a projective bundle of planes is included in the skew projection of Clebsch.

The modification in the mapping of a cubic surface which has singular points was treated by J. Diekmann(72). The 6 basis points of the cubics which represent the plane sections have in each case some special position. The cubic surface with 4 double points is discussed by E. Laguerre(106). The plane sections are depicted as cubics through the 6 vertices of a complete quadrilateral. W. P. Milne(109) applies the mapping of a cubic surface to the consideration of triads of points apolar to a pencil of plane cubics.

3. *Quartic Surface with a Double Line.*—A. Clebsch(46) showed that the quartic surface F_4 with a double line contains 64 plane sections composed of two conics each meeting the double line in the same point. The depiction on a plane is then obtained analytically by forming the equation of a quartic surface with the double line and passing through one of these conics. Geometrically the surface is mapped by drawing the lines which meet the double line and conic. The plane sections are represented by quartic curves with one common double point and 8 simple basis points. The cubic through the 9 points is the image of the double line. W. Frahm(77) considers the case of additional double points. The plane sections are depicted as quintics with one triple point, one double point, and 8 simple points, transformable into quartics with one double point and 8 simple points. J. Cardinaal(19) considers the surface as the locus of the intersection of 2 projective pencils of quadrics through the line.

4. F_4 *with a Double Conic.*—In the first part of a paper by A. Clebsch(43) the author starts with the parametric representation of a quartic surface $y_i=f_i(x_1, x_2, x_3)$ $i=1$ to 4, where $\Sigma a_i f_i=0$, $i=1$ to 5, is a system of cubics through 5 fixed points. He shows that there are 16 lines on the surface represented by the 5 points, the 10 connecting lines and the conic through the points, also the systems of conics repre-

sented by the lines through one point and the conics through 4 points. He also shows that there the surface has a double conic represented by a cubic of the system $\Sigma a_i f_i = 0$. In the second part commencing with the equation of a quartic surface with a double conic $\phi = 0$, $p = 0$, namely, $\phi^2 - 4p^2 q = 0$, he shows how to find the depiction of the surface with which he began in the first part.

This analytical method was continued by G. Korndörfer(100) and (101) when the surface had additional double points and for the case of 2 intersecting double lines in (102) and of 2 intersecting coincident lines in (103). The case of a cuspidal conic was discussed by L. Cremona(59), the image of a cubic surface with two double points by a special (2, 2) quadric transformation. See also Tötössy Béla(137).

C. F. Geiser(79) and L. Cremona(63) showed that an F_4 with a double conic can be found by applying a (2, 2) quadratic transformation to a cubic surface through a conic and a point and also to a quadric. See also J. Cardinaal(19) and (20), and G. Darboux(65.1).

A. Del Re(66) considers the surface $(x^2 + y^2 + z^2) z^2 - h t^2 w^2 = 0$ which has two double lines in the plane $z = 0$. The plane sections are depicted as cubics through 3 fixed points and a 4th point with a fixed tangent.

5. *The Steiner Surface.*—The Steiner surface which has 3 double lines meeting in a point which is a triple point on the surface was mapped on a plane by K. Weierstrass(143), the plane sections going into conics, so that $y_i = f_i(x_1, x_2, x_3)$, $i = 1$ to 4, f_i being a quadratic ternary form.

The equation of the surface was found by eliminating x_1, x_2, x_3 by A. Brill(16). A. Clebsch derived the normal form of mapping $y_1 = 2x_2 x_3$, $y_2 = 2x_3 x_1$, $y_3 = 2x_1 x_2$, $y_4 = x_1^2 + x_3^2$ from the equation of the surface in the form $y_2^2 y_3^2 + y_3^2 y_1^2 + y_1^2 y_2^2 = 2y_1 y_2 y_3 y_4$. He also showed that the general form of Weierstrass could be reduced to the normal form by a suitable choice of coordinates, the fundamental triangle in (x) representing the 3 double lines and the triple point. See also L. Cremona(62), where the cases of 2 or 3 coincident double lines are discussed. For various parametric representations see G. Koenigs(99).

6. *Rational Quartics with No Double Curve.*—The quartic with a triple point is mappable by projection from the triple point. The rational quartics with any other type of singular point can be mapped on a double plane by projection from the double point. Since this double plane is rational, the sextic curve of branch points must belong to one of three types: (1) quartic; (2) a sextic with a 4-fold point; (3) a sextic with two consecutive triple points. The first type of quartic surface was found by L. Cremona(53). The plane sections can be mapped on a plane by sextics with $7P^2$ and $4P^1$ all lying on a cubic which is the image of the

singular point. The other two types were found by M. Noether(120). The second type is mapped on a plane by $C_7(a^3b_1{}^2b_2{}^2\ldots b_9{}^2)$ and the third by $C_9(a_1{}^3, a_2{}^3, \ldots, a_8{}^3b_1{}^2b)$ the 10 points each lying on a cubic as in the first case. See G. Jung(91) and D. Montesano(111).

7. *Quintics with 2 Double Skew Lines.*—A. Clebsch(46) determined analytically the parametric equations of the surface, the plane sections being mapped as sextics with 2 triple points A, B and 13 simple basis points. The 2 skew lines are depicted as quintics with a triple point at A (or B) and a double point at B (or A) and through the 13 simple points. The 13 points cannot be taken arbitrarily but are intersections of two such quintics. A pencil of lines through B (or A) and a quintic A^3B^2 (or A^2B^3) give the web of sextics. There are 13 lines on F_5 images of the 13 basis points. The line AB is a fundamental line. The system of sextics can be transformed into quintics A^2B^2 with 12 simple basis points, the line AB is now the image of the 13th line on F_5 and the 12 points are the intersection of 2 quartics A^2B and AB^2. See A. Cayley(22). See also T. Castelli(20.1).

8. *Quintics with a Double Cubic.*—A. Clebsch(46) found analytically the equation and the parametric equations of the quintic with a double cubic. The plane sections are depicted as quartics through 11 fixed points, the double curve as C_7 with 11 double points.

F_5 with 3 double lines meeting in a triple point. E. Caporali(17) gives the depiction of this surface by sextics with $7P^2$ and $3P^1$.

9. *Quintics with Double C_4, $p=1$.*—The equation of the surface is of the form

$$y_1H_1{}^2+y_2H_1H_2+y_3H_2{}^2=0,$$

where $H_1=0$, $H_2=0$ are quadrics. The surface contains the conics

$$H_2=\lambda H_1, \qquad y_1+\lambda y_2+\lambda^2y_3=0.$$

A straight line through $O\equiv(0, 0, 1)$ meets the surface in 2 pairs of points PQ, $P'Q'$, corresponding to the two values of λ given by $y_1+\lambda y_2+\lambda^2y_3=0$. The locus of the harmonic conjugate of OPQ, $OP'Q'$ is a ruled cubic R, in (1, 2) correspondence with F_5. If R_3 is depicted on a plane, then F_5 can be depicted on a double plane, with a sextic curve of branch points with a P^4. The plane sections correspond to quartics with a P^2 and 7 simple basis points, see A. Clebsch(49) and (44). M. Noether(118) mapped the quintic surface on a second quartic with a triple line and hence upon a plane.

L. Cremona(55) discusses a quintic surface with a quartic cuspidal curve of genus one and a double line which is the reciprocal of a cubic surface with a uniplanar point. A. Cayley(31) gives a summary of Clebsch's work on F_4 and F_5.

10. *Quintics with Double Quintic with a Triple Point.*—A. Clebsch(49), footnote, mentions this surface as being mappable on a double curve with a quartic curve of branch points, the depiction of plane sections on a simple plane being cubics through 4 points. There are 10 lines corresponding to the 4 points and the 6 connecting lines. For a special case see G. Darboux(65). L. Cremona(61) obtained the surface as the image of a quadric under a (3, 3) birational transformation in which the sextic basis curve was composed of a space cubic curve, a line, and conic meeting the cubic in 3 points and the line in 2. The other sextic curve consists of the quintic with a triple point and a line meeting it in 2 points. E. Caporali(18) considers a (3, 5) transformation in which the cubic surfaces have in common a rational quartic and a point with a common tangent plane. The quintic surfaces have in common a double quintic curve with a triple point and a conic meeting it in 4 points. The plane depiction of the quintic surface follows immediately.

11. *Quintics with Singular Points.*—D. Montesano(111), (112), and (113) has investigated various types of quintics having either no multiple curve or multiple curves insufficient for the rationality of the surfaces. The method is that of Cremona, namely, to apply a birational transformation to a known rational surface to obtain a new quintic surface. E. Pensa(122) has developed a general process of determining the multiple elements of a surface so that $p_a = P_2 = 0$, (G. Castelnuovo(28)) which ensures the rationality of the surface and has applied the method to find all the combinations which give rationality to the quintic surface.

12. *Rational* F_n $n > 5$.—For $n > 5$ the rational surfaces have not been completely enumerated. L. Cremona(60) and (61) and J. E. Hill(88) and (89) have found various rational surfaces by considering (3, 3) birational transformations in which the sextic basis curves are composite. M. Noether(119) used a repetition of (2, 3) and (3, 3) transformations. E. Caporali(17) gives 8 kinds of sextic surfaces obtained from the theory of triply infinite linear systems of plane curves. L. Cremona(53) discusses an F_{12} whose plane sections are depicted as sextics with six double points.

M. Noether(118) discusses an F_6 with a double cubic and a non-intersecting double line; a plane section is mapped by a C_7 with $P^3 6P^2$ and 10 simple points.

G. Bordiga(8) applied the known cubic involution of order 2 in which a point P and its conjugate P' are conjugate with respect to a quadric F and lie on the same secant of 2 skew lines a, b conjugate to the quadric. The image of a quadric surface is of the 6th order and has four-fold points where *a, b* meet *F* and has for double lines the edges of the tetrahedron determined by these 4 points.

M. Bonicelli(8) discusses sextic surfaces with 3 double lines, a triple conic and a fundamental cubic, the inverse of a cubic web with fundamental elements, 3 lines, a point with a fixed tangent plane, mappable by cubics through 6 points. P. del Pezzo(124) discusses a sextic surface with 9 double lines and a 4-fold point. A. Brambilla(12), (11), (13), (14), and (15) considers $x_i = \theta_i^n$ $(i=1$ to $4)$, $\Sigma\theta_i = 0$ and its generalization.

D. Montesano(114) considers a rational surface of order n such that to any point P of the surface corresponds a plane of the bundle through a fixed point O which passes through P.

13. *F_n with an $n-2$-fold Line.*—M. Noether(118) showed that there exists on the surface a rational curve of order $n-2$ meeting the multiple line $x_3 = x_4 = 0$ in $n-3$ points and therefore each conic on the plane of the pencil $\dfrac{x_3}{x_4} = \lambda$ in one point. The equation of the surface can be written in the form $AB = CD$ when $ABCD$ are linear in x_1 and x_2. The equations $x_3 = \lambda x_4$, $A = \mu C$, $D = \mu B$ map the surface on the plane λ, μ. The plane sections are depicted as curves of order n with a $n-2$-fold point and $3n-4$ simple basis points. The multiple line is depicted as a curve of order $n-1$ with a $n-3$-fold point and passing through the $3n-4$ basis points.

14. *F_n with an $n-3$-fold Line and $n-3$ Triple Points.*—A. Tummarello(138) obtains these surfaces as the images of cubic, or known quartic and quintic surfaces by a transformation of order n with an $n-1$-fold line and $3(n-1)$ simple basis points.

15. *Rational Ruled Surfaces.*—L. Cremona(54) depicted a rational ruled surface on a plane so that the system of generators is depicted as a pencil of lines and therefore the plane sections as curves of order μ with a $\mu-1$-fold point at the vertex of the pencil. He considers the case of a R_{m+n} with an m-fold and an n-fold directrix $m \geqq n$ and $(m-1)(n-1)$ double generators. The plane sections are depicted as curves of order m with an $m-1$-fold point having $m-n-1$ fixed tangents. If $m=n$ the least value of μ is $m+1$.

A. Armenante(1) found the analytic form of the equations of C_μ by the method of H. A. Schwarz(129) and gave a partial discussion of the representation of the double curve of R_n; A. Clebsch(42) discusses the mapping of the cubic ruled surface on a plane by means of conics through a fixed point which is the image of the double line, the simple directrix being represented by a line. See also A. Clebsch(52).

16. *R_4 with a Double Cubic.*—A. Clebsch(48) depicts the plane sections of the ruled quartic with a double cubic by means of cubics with a double

point and a simple basis point. He finds the equation of the surface and the depiction of the double curve as a quartic with double points at the two basis points. This depiction follows from the definition of the generators as the lines joining corresponding points on 2 conics (in different planes) whose points are in (1, 1) correspondence. See L. Cremona(56).

A. Clebsch(50) showed that by considering 2 plane sections of R_n and the generators joining corresponding points, the plane sections are depicted as curves of order $n+1$ with an n-fold point. By quadratic transformations it is possible to transform these curves to curves of lower order having either (1) one other simple basis point, (2) no simple basis point, or (3) basis points, consecutive to the multiple point so that there are $1, 2, \dots, n-1$ fixed tangents.

17. *R_n with an $n-1$-fold Line.*—M. Noether(118) discussed the mapping of R_n with an $n-1$-fold line. By projection from a point on the multiple line the plane sections are depicted as curves of order n with an $n-1$-fold point and $n-1$ simple basis points corresponding to the $n-1$ generators through the center of projection. The curves can be transformed into curves of lower order by successive quadratic transformations, the number of simple basis points being reduced by 2 each time until there is either one or no simple basis point.

The case $n=5$ is discussed by B. Kalicum(95) and (96).

18. *General Theory of Mapping a Rational F_N on a Plane.*—A. Clebsch(46). If the coordinates of a point (y) on the surface are expressible in terms of the coordinates of a point (x) on a plane by equations of the form

$$y_i = f_i(x_1, x_2, x_3) \qquad i = 1 \text{ to } 4$$

where f_i is a homogeneous polynomial of order n, the section of F_N by the plane $\Sigma a_i y_i = 0$ is represented by the curve $\Sigma a_i f_i = 0$. If this system of curves has a_i i-fold basis points the order N of the surface is given by the equation

$$N = n^2 - \Sigma i^2 a_i.$$

The genus of the plane sections and the image curves is the same. The image of an i-fold basis point is a rational curve, of order i and that of a curve of order m having a j-fold point at an i-fold basis point is a curve of order $mn - \Sigma ij$.

The image of the double curve of F_N is a curve of order $(N-4)n+3$ having a $(N-4)i+1$-fold point at an i-fold basis point which is in (2, 1) correspondence with the double curve. See also M. Noether(117).

E. Caporali(17) considers a triply infinite system of plane curves ϕ of order n genus p and grade N with s basis points, images of the plane sections of genus p of a surface F_N of order N. The linear system of jacobian curves of nets of curves of the system ϕ corresponds to the curves of contact ψ' of the tangent cones of F_N from points outside and are of order $(N-4)n+3$ and multiplicity $(N-4)r+1$ at an r-fold basis point. Two nets of curves of ϕ have in common a pencil. The number of curves of this pencil with an extra double point is

$$3(n-1)^2-\Sigma(r-1)(3r+1)=N+4p+s-1.$$

Hence the class of F_N is $N+4p+s-1$. Two jacobian curves meet in $6(n-1)^2-2\Sigma(3r^2-2r+1)=2(N+4p-s-1)=k$ other points which are common to all the jacobian curves and correspond to pinch points on the double curve Δ' of F_N and to coincidences in the (1, 2) correspondence between the points of Δ' and its image Δ. The curves Δ and ψ and therefore Δ' and ψ' meet in $\alpha=(N-1)(2N-7)+2p(N-11)+3s$ other points. By considering the intersection of Δ' and the second polar of a point not on F_N we have

$$3t+\alpha=(N-2)(\tfrac{1}{2}(N-1)(N-2)-p).$$

The curve Δ has 3 double points for each triple point of Δ'. Hence its genus P can be found.

The genus P' of Δ' can then be found by applying Zeuthen's formula namely from

$$K=2(P-1)-4(P'-1).$$

G. Humbert(90) shows by considering the double integrals belonging to the rational surface F_n that the adjoint surfaces of order $n+q-4$ meet the surface in curves whose images on a plane are curves of order $hq-3$, the images of the plane sections of genus p being of order h, and are adjoint to the curves of order hq, images of the curves of intersection of F_n with surfaces of order q. The dimension of the system of curves is $\frac{1}{2}q(q-1)n+q(p-1)+\frac{1}{6}(q-1)(q-2)(q-3)$.

19. *Surfaces in S_3 Derived by Projection from a Higher Space.*—P. del Pezzo(123) showed that surfaces of order n in S_n are ruled for $n>9$ and for $3\leq n\leq 9$ are mappable on a plane by means of cubics, $p=1$. Their projection on S_3 is a F_n with a double curve of order $\dfrac{n(n-3)}{2}$ with $\dfrac{(n-2)(n-3)(n-4)}{6}$ triple points and genus $\dfrac{(n-4)(n-5)}{2}$.

G. Veronese(139) found F_4 with a double conic as the projection of the intersection of two quadric varieties in S_4. He also(140) found the Steiner surface as the projection of F_4 in S_5, which can be mapped on the plane by conics.

G. Bordiga(4) obtained an F_6 with a double C_7, $p=1$ and one P_3 by projecting an F_6 in S_4 which is the locus of the intersection of 4 projective forms of the second kind and is mapped on a plane by quartics through 10 points. In (7) he considered 4 forms of the 2d kind in S_5 in which the S_3 of one corresponds to the S_4 of the others. The locus of the intersection of corresponding elements is a rational F_7. G. Bordiga(5) discusses the plane representation of ruled normal surfaces F_{n-1} in S_n defined as the locus of the intersection of 2 systems of planes with straight line axes, projectively related. The surface F_{n-1} contains ∞^1 lines and a S_{n-1} meets it in a C_{n-1} depicted on a plane by a S_{n-2} through $n-2$ points on F_{n-1} as a C_{n-1} with $n-2$ fixed simple points and a P_{n-2}.

G. Bordiga(6) derives F_5 and F_6 from S_6, the surfaces being generated by 3 collinear bundles out of S_4 with the centers of 3 dimensions "im Auge Gefast." The sections by S_5 are sextic curves $p=1$ represented on a plane by cubics.

20. *Miscellaneous Methods.*—G. Jung(93) describes a general method of considering a rational surface as the locus of the intersection of a plane of a bundle and the corresponding line of a congruence when the planes and lines are connected by a Cremona transformation, and applies the method to the case of a monoid.

P. Visalli(141) applies a similar method using the lines connecting corresponding points on two planes between which there is a Cremona correspondence. See also G. Jung(94) and A. Del Re(70).

J. de Vries(142) considers a surface derived from the lines of a rational congruence and having the directrices of the congruence for multiple lines so that a line of the congruence meets the surface in only one point not on a directrix. See also R. Sturm(135), G. B. Guccia(85), C. Minea(110).

T. Reye(127) derives rational surfaces from the correspondence between 2 spaces (x') and (x) in which to a plane (x') corresponds a quadric in (x). A plane in (x) goes into a Steiner surface in (x'). A quadric in (x) goes into a rational surface of order 8. If the quadrics which define the correspondence have basis points, a quadric in (x) through r basis points goes into a rational surface of order $8-r$.

S. Kantor(98). The lines joining a point P on a quadric to 3 fixed points meet the quadric again in 3 points which determine a plane whose envelope is a surface of the sixth class. Its image in a certain point-plane transformation is the quadric. Hence the surface is rational. The image of a line in (x) is a conic in (x'). Hence the rational surfaces contain 2 systems of conics corresponding to the generators of the quadric in (x).

A. Del Re(67), (68), (69) and (71) discusses various quintic surfaces and the F_4 with a double conic obtained as the fundamental surface of a (1, 2) connex.

21. *Rational Surfaces and Linear Systems of Curves.*—G. Jung(92) states that if all the linear systems of least order with genus p are supposed known then all the rational surfaces with plane sections of genus p may be deduced and conversely, if all the rational surfaces with plane sections of genus p are supposed known then the linear systems which are the images of the plane sections may be deduced. See also C. Segre(130) and G. Jung(91). If we study rational surfaces and determine the general types we may omit particular cases and so fail to determine all linear systems and we are certain to omit the linear system not representable by rational surfaces. Two linear systems of least order which are not birationally equivalent may correspond to two surfaces one of which is a special case of the other. Any linear system of least order with consecutive basis points may be considered a special case of a system with non-consecutive basis points but not of least order.

22. *Rational Surfaces Mappable on a Multiple Plane.*—A. Clebsch(49) considered various rational surfaces which can be mapped on a double plane, namely the quadric by projection from an external point, the cubic surface by projection from a point on the surface, the quartic with a double conic by projection from a point on the conic and the quintic with a double quartic of $p=1$. The last surface contains a system of conics lying on planes through a point π. The locus of the polar lines of π with respect to the conics is a cubic surface in (1, 2) correspondence with the quintic surface. By mapping the cubic surface on a plane the quintic surface can be mapped on a double plane.

M. Noether(116) shows that a double plane with a curve of branch points of order 2_n and a_i $2i$ or $2i+1$-fold points is rational if

$$\tfrac{1}{2}i(i-1)a_i=\tfrac{1}{2}(m-1)(m-2).$$

M. Noether(121) stated that rational double planes can be transformed birationally into one of 3 types with a curve of branch points either,

(1) C_{2n} with P_{2n-2}
(2) General C_4
(3) C_6 with 2 consecutive P_3

and refers to his paper(116) in which he finds the condition that a double plane shall be rational; he also refers to E. Bertini(8), where the latter determined the 4 types of plane involutions.

G. Castelnuovo and F. Enriques(24) showed that these are the types of curves for which the first adjoints of order $2n-3$ exist but the second and higher adjoints of order $2n-6$, $2n-9$—do not exist; and from this showed that the bigenus P_2, P_3, etc., are all 0 for a surface mappable on such a double plane. When a rational surface mappable on a double plane is mapped on a simple plane, then between the 2 planes there is a (1, 2) correspondence, and the image of line on the double plane is a curve on the simple plane belonging to the involution which interchanges points associated in the correspondence.

A. Bottari(9) has considered the cases of multiple planes of order $n>2$. The corresponding involutions are now cyclic of order n and generated by Cremona transformations. The types of periodic Cremona transformations have been determined by S. Kantor(97) and A. Wiman(144) and the former has also found the simple curves which belong to the involution and the curves of invariant points. By mapping these curves on the multiple plane Bottari found the following types of multiple planes, namely with curve of branch points:

(1) For arbitrary n, $\mu n+2$ lines, μn of which pass through a point O.
(2) $n=3$, an arbitrary cubic
(3) $n=3$, $C_6:P^4$ with 2 consecutive double points
(4) $n=5$, and arbitrary C_3 and an inflexional tangent.

A. Bottari(10) extends the results of the previous paper to cases in which the cyclic involution is the product of 2 or more of the simple involutions already considered.

23. *General Conditions for the Rationality of a Surface.*—M. Noether(115) and (118) showed that a surface containing a rational pencil of rational curves is rational. The method used was to first map the rational curves on the planes of a pencil and then transform them either into lines or conics so as to get a surface with a $n-1$ or $n-2$ fold line. The former is mappable upon a plane by projection. The latter contains rational curves meeting each conic lying on the planes through the $n-2$ fold line in one point P. The surface can be mapped on a plane by projecting each conic by lines through the corresponding point P.

G. Castelnuovo(23). From Noether's theorem a surface whose plane sections are rational is rational. E. Picard(125) and G. B. Guccia(84) showed that such a surface is either ruled or a Steiner surface. G. Castelnuovo(26) found that any surface whose plane sections are elliptic curves is either ruled or is rational and in the latter case is of order $n\leq 9$ and the plane sections can be mapped on a plane as cubics through $9-n$ points.

G. Castelnuovo(27). A surface containing a net of elliptic curves is either rational or transformable to a ruled surface. A surface containing a net of hyperelliptic curves of genus p whose characteristic series is not special or composed of pairs of points of the $g_2{}^1$ of the generic curve is either rational or transformable into a ruled hyperelliptic surface of genus p. If the characteristic series is composed of pairs of points of the $g_2{}^1$ the surface can be transformed into a rational surface, a ruled hyperelliptic surface or a ruled elliptic surface. F. Enriques(75) showed that a surface whose plane sections are hyperelliptic curves of genus $p>1$ is either ruled and of genus p or a rational surface containing a pencil of conics.

F. Enriques(74) and (76) investigates the irrationalities which may occur in representing a rational surface on a plane and shows that a rational surface F_n may be reduced by the aid of the adjoints of order $n-3$ to one or another of the following types:

(1) a surface with rational plane sections,
(2) a surface with elliptic plane sections,
(3) a surface with hyperelliptic plane sections.

L. Godeaux(80) showed that if a surface possesses two pencils of rational curves, then since the curves of one pencil must meet any curve of the other in a rational involution, both pencils are linear and hence the surface is rational.

G. Castelnuovo(28) showed that in order that a surface shall be rational it is necessary and sufficient that $p_a=0$ and the bigenus $P_2=0$.

Castelnuovo gives the example due to Enriques of a sextic surface with the edges of a tetrahedron for double curve for which $p_a=0$ but $P_2=1$ and which is therefore not rational. He also gives another example of a special F_7 with $p_a=0$, $P_2=1$.

G. Castelnuovo and F. Enriques(81) showed that a surface for which the series of successive adjoints C', C'', of a system of curves C on the surface terminates is either rational or ruled and that this is the case if the grade n of a system of curves, genus p, on the surface is greater than $2p-2$.

G. Castelnuovo(22) and (25) in proving the rationality of plane involutions shows that a surface is rational on which any linear normal system of curves has its characteristic series complete and on which there exists a linear system of dimension $>$ the genus.

24. *Rational Surfaces with Plane Sections of Genus 3.*—G. Castelnuovo(39) shows that there are 4 types of surfaces in addition to the 3 types of quartics with no double curve.

(1) F_{16} of S_{14} and its projection, the hyperplane sections being depicted on a plane by quartics.

(2) F_8 in S_6 and its projections, plane depiction by sextics with 7 double points.

(3) F_{16} of S_{14} and its projections, plane depiction by C_6 with a P^4.

(4) F_{16} of S_{14} and its projections, plane depiction by $C_{8-\mu}$ with a $P_{6-\mu}$ and μ consecutive double points ($\mu=0, 1, 2, 3$).

25. *Rational Surfaces Containing an Infinite Number of Conics.*—The case of quartic surfaces was considered by E. E. Kummer(105). There are 3 cases, namely the quartics with a double line or conic and the Steiner surface. Besides the quintic with a triple line there are the quintics with a double quartic of genus one (A. Clebsch(45)) and with a double quintic of genus 1 with a triple point (E. Caporali(18)). For the sextic surfaces E. Togliatti(136) finds the surfaces with a 4-fold line; with a double C_7 of genus 5 the plane sections being depicted as quintic curves with a triple point and 10 simple points; with a double curve of order 8 of genus 3 with 2 triple points or a double quintic with a quadric secant triple line or a triple conic and a trisecant double conic. The plane sections are depicted as quartics with a double point and 6 simple points; with a double C_9 with 4 triple points, the plane sections being depicted as cubics through 3 points, see Bordiga(6).

C. Grimaldi(83) has classified the rational surfaces of order 7 with an infinite number of conics.

G. Castelnuovo(23) showed that rational surfaces with hyperelliptic plane sections of genus p are cut by the system of adjoint surfaces of order $n-3$ (of dimension $p-1$) in $p-1$ conics. The plane sections are depicted as hyperelliptic curves either C_{p+3} with a P^{p+1} and a P^2 or $C_{2p+2-\mu}$ with a $P^{2p-\mu}$ and $p-\mu$ double points consecutive to $P^{2p-\mu}$ on different branches where $\mu=0, 1, 2, \ldots, p$.

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CHAPTER XIII

THE MAPPING OF A RATIONAL CONGRUENCE ON A PLANE

1. *Introduction.*—A set of ∞^2 lines in S_3 such that through a general point pass m lines and in a general plane lie n lines of the set is a congruence (m, n) of order m and class n. A point through which pass ∞^1 lines lying on a cone of order i is an i-fold singular point, and a plane in which lie ∞^1 lines tangent to a curve is a singular plane. If the lines of a congruence can be put into (1, 1) correspondence with the points of a rational surface, the congruence is rational and can be mapped on a plane. A $(1, n)$ congruence is obviously mappable on a plane by the intersections of its lines with the plane. An (m, n) congruence which has a singular plane such that the lines in the plane are tangent to a curve of class $n-1$ can be mapped on the plane. P. Visalli(28). An (m, n) congruence which has an $m-1$-fold singular point can be mapped by the planes through the singular point. E. Bertini(1). G. Loria(15). The Cremona congruences of T. A. Hirst(8), (9), and (10) formed by the lines joining corresponding points on 2 planes whose points are related by a Cremona transformation and the congruences of E. Caporali(3) where one plane is replaced by a rational surface whose points are in (1, 1) correspondence with the points of the other plane, are both obviously rational. The case of a quadratic transformation is considered by W. Stahl(25), and the lines joining corresponding points of 2 monoids, by A. Del Re(7). I. Conti(4) discusses the lines joining corresponding points on 2 planes related by a (1, 2) correspondence. L. C. Kiefer(11) considers the lines joining corresponding points on a plane and a quadric. The $(2, n)$ congruences with only a finite number of singular points were classified by E. E. Kummer(12). The lines of the congruence are bitangent to a quartic surface, called the focal surface.

The 2 rays through a point of the focal surface coincide. The value of n is between 2 and 7. There are 2 types of (2, 6) congruence, one with a five-fold singular point and the other with four four-fold points.

The $(2, n)$ congruences with no focal surface but with singular curves were classified by R. Sturm(26) and R. Schumacher(22).

2. *Congruences with a Finite Number of Singular Points.*—The first $(2, n)$ congruence to be proved rational was the (2, 2) congruence which is the complete intersection of a linear and a quadratic complex, L. Cremona(5). The linear complex was mapped on S_3 by M. Noether(19) and

S. Lie(13). A second linear complex meets the first in a linear congruence whose image in S_3 is a quadric through a fixed conic. A quadratic complex meets the first linear complex in a (2, 2) congruence whose image is a quartic surface with the fixed conic double, and is therefore rational. For another proof, see E. Caporali(8).

Congruences of type $(8-n, 2)$, where $1 \leq n \leqq 6$ were studied by T. Reye(21) by aid of the (1, 8) correspondence between 2 spaces S, S' in which the planes of S correspond to ∞^3 quadrics in S' through n fixed points. The vertices of the cones of the system of quadrics lie on a quartic surface K_4, and the images of the cones in S are planes tangent to a corresponding surface F, of class 4 and order $16-2n$.

The lines joining associated points in S' correspond to bitangents of F through the corresponding point. The bitangents form a (28, 12) congruence having the focal surface F. If the quadrics have n basis points then to the n bundles of lines through the basis points correspond n $(8-n, 2)$ congruences which are therefore also mappable on a plane. The $(2, 6)_I$ congruence having 4 four-fold points belongs to a tetrahedral complex with the four points for vertices. When this complex is mapped on S_3 (G. Loria(14)), the congruence is represented by an F_6 with $4P_4$, the lines joining them being double lines. This surface is rational and its plane sections can be mapped on a plane as C_5 with $2P_2$ and $11P_1$. It should be noted that all $(2, n)$ congruences with a quartic focal surface can be mapped on a double plane, the curve of branch points being the intersection of the focal surface with the plane. The double plane is therefore rational and also the congruence. The mapping of these congruences is treated by R. Sturm(27).

3. *Congruences with a Curve of Singular Points.*—D. Montesano(17) considers the (4, 12) congruence consisting of the generators of the cones of a net of quadrics. The vertices of the cones lie on a C_6, $p=3$. If 6 of the 8 points common to the quadrics are chosen so as to coincide in pairs at A_1, A_2, A_3 and a 7th point A_4 is taken on the surface of coincidences of the web of quadrics through the 6 points then the 8th point common to the net of quadrics coincides with A_4 and the sextic is composed of 2 space cubics through $A_1A_2A_3A_4$.

The (4, 12) congruence is composed of two (2, 6) rational congruences each comprising the generators of the cones of the net of quadrics with their vertices on a cubic. The focal surface of each congruence is a quartic surface on which the cubic is double. Each line of the congruence and the cubic determine a quadric and the congruence can be mapped by the quadrics and the system of cones.

If the quadrics of the net are all tangent to a plane π at a point O and pass through P_1, P_2, P_3, P_4 the congruence is composed of the lines through O and a (2, 6) rational congruence. The vertices of the cones of the net lie on a cubic curve of π having O^2 and meeting the 6 lines P_iP_k. The focal surface consists of π and an F_3 having P_i^2 which touches π at O and contains the cubic curve. The cones touch F_3 in a pencil of cubic curves. Through a point on a cubic curve passes one line of the congruence. Hence the congruence can be mapped on a plane.

D. Montesano(16). The (2, 4) congruence of lines which meet 2 conics which meet in 2 points T, T' is rational because the planes through T or T' and the lines of the congruences are in (1, 1) correspondence.

D. Montesano(18) shows that the (2, 6) congruence formed by the quartic curve of intersection of two quadrics tangent to one another at a point O can be mapped on a plane because the planes through O and the bisecants are in (1, 1) correspondence.

The secants of a line k and a C_n meeting it in $n-2$ points form a $(2, n)$ congruence which can be mapped on a cone having the genus of C_n. Hence if C_n is rational the congruence is rational. The tangents to a F_n with an $n-2$-fold line k which meet k but are not tangent at a point of k form a $(2, 2(n-1)^2)$ congruence which is rational since the lines of the congruence and the points of F_n which is rational are in (1, 1) correspondence. If between the points of a line k and the planes through k there is a $(2, n)$ correspondence then a $(2, n)$ congruence can be set up by joining any point P to the 2 points Q, R which correspond to the plane kP. The pencils of rays through a point of k can be represented by the points of a C_{n+2} which has L^n and M^2, if the points of k and the planes through k are represented by lines through M and L respectively. The congruence is rational if C_{n+2} has $n-1$ other double points, that is, if the congruence has $n-1$ pencils of double rays.

Given on a quadric cone K vertex O a rational C_n with O^{n-1}. The lines through a point P of C_n lying in the tangent plane to K at P form a $(2, n)$ congruence contained in the quadratic complex of the tangent lines to k. The congruence can be mapped on any tangent plane of the quadric cone by the intersection of its lines with the plane.

4. *Congruences Defined by Quadrics and Projectivities.*—V. Snyder(23) considers the congruences belonging to the generators of a net of quadrics. A line l of one system of generators meets the generator of the other system through a basis point B of the net in a point P which is in (1, 1) correspondence with l. Hence if BP describes a rational cone, l will describe a rational congruence. V. Snyder(24) and H. B. Owens(20) consider congruences of generators of

$$a\lambda^2+b\lambda+c=0, \qquad a\lambda^3+b\lambda^2+c\lambda+d=0,$$

where $a=0$, $b=0$, $c=0$, $d=0$ are quadrics and the two systems of generators are rationally separable.

A. Del Re(6) discusses the lines obtained by the intersection of a plane p through a point S and the plane determined by a point P on a plane s and a line l' on a plane s' when P, p, and l' are projectively related. He also considers the lines joining the points in which 2 lines l' and l'' through 2 points $s's''$ meet a plane p through a point s, when s', s'' and p are projectively related.

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CHAPTER XIV

INVOLUTIONS ON IRRATIONAL SURFACES

1. *Definitions.*—To the groups of n points of an involution of order n on an algebraic surface F correspond the points of a second algebraic surface F^* which is called the *image* of the given involution I_n and is thus in $(1, n)$ correspondence with F.

Of particular interest are the points P^* of F^* for which two (or more) points P of the n points on F corresponding to P^* coincide. Such a point P^* is called a *branch-point* and P is a *coincidence.* For the purposes of this report, it will be convenient to divide involutions into two classes: (a) those which have infinitely many coincidences forming an algebraic curve, and (b) those which have a finite number (or zero) of coincidences. It is no essential restriction to suppose that no point of either the coincidence locus on F or of the branch locus on F^* is a fundamental point (corresponding to a curve on the other surface). Under these circumstances, the above classification holds also with respect to the branch locus.

We exclude from the following discussion surfaces which have a continuous group of transformations into themselves. These surfaces have been discussed by Castelnuovo-Enriques(5) p. 744, p. 759 (elliptic surfaces), Picard-Simart(63) p. 517, and by Severi(69) Vol. 2, p. 778. Hyperelliptic surfaces are also treated at length in Chap. 17, § 2 of this report.

I. Involutions with a Curve of Coincidences

2. *Relations Between the Invariants.*—F. Enriques has shown (18) that, if F and F^* are free from exceptional curves and if the geometric genus of F^* is positive, then, to the canonical curves on F^* correspond on F curves which, added to the curve of coincidences, form canonical curves on F. It is further shown by Enriques, loc. cit., and by F. Severi(70) that, if we denote by p_a and $p^{(1)}$ the arithmetic and linear genera of F, and by π_a and $\pi^{(1)}$ those of F^*, then we have

$$p^{(1)} = n(\pi^{(1)} - 1) + \tau + \tfrac{3}{2}\delta$$
$$24(p_a + 1) = 24n(\pi_a + 1) + 6\tau + 3\delta - 2\sigma - 6$$

wherein τ is the genus of the branch-curve K^* on F^*, δ is the number of intersections of K^* with the canonical curves of F^*, and σ is the number of points of F^*, three of whose corresponding points on F coincide.

In this formula, it is supposed that K^* has no multiple component and that there are no fundamental points on F^* which correspond to curves on F. A discussion of the case in which exceptional curves exist, and extensions of the formulas to the case of (m, n) correspondence between two surfaces will be found in Severi, loc. cit. That an involution on a regular surface is regular was shown by G. Castelnuovo(4).

3. *The Double Plane.*—Most of the involutions with a curve of coincidences that have been considered in the literature are rational and cyclic, i. e., are representable as multiple planes, for the most part, indeed, as double planes. The double plane

$$z^2=F(x, y)$$

is a rather simple, yet fairly general, type of surface. Its properties are easily amenable to treatment and it furnishes excellent examples in various parts of the theory of algebraic surfaces. On the geometric side, it has been quite fully considered by Castelnuovo-Enriques(5). We aim here only to supplement their remarks by some comments on its Picard integrals and the base invariants ρ and σ.

Picard, in his theory of integrals, has turned repeatedly for illustrations to the double, or multiple plane. See, e. g., (59), Chaps. 2 and 3; (63), (57), (61). Most of these cases are also discussed in (63) and to this we shall give more detailed references.

In (63), Vol. 1, Chap. 6, Picard deals with integrals of total differentials of the second and third kinds. As in most of his work, the discussion is based chiefly on algebra and analysis, with relatively little appeal to geometry. In particular, he shows that the integrals in question are reducible, respectively, to the types

$$\int\frac{Pdx+Qdy}{f(y)z} \quad \text{and} \quad \int\frac{Pdx+Qdy}{f(y)A_1A_2\ldots A_nz}$$

where f is a polynomial in y alone and $A_1, A_2, \ldots, A_n$ are polynomials in x, y. The polynomials A are irreducible, prime to F, and cut the surface in two distinct curves whose equations are

$$(1) \qquad A_i=0, \qquad z=\pm\frac{M_i}{N_i}, \qquad i=1, 2, \ldots, n$$

where M_i, N_i are polynomials in x, y. Algebraically, this means that A_i is a factor of $M_i^2-N_i^2F$. To form all integrals of the third kind it is thus necessary to discover the irreducible factors of polynomials M^2-N^2F. By reasoning which holds for the general algebraic surface, it follows that there is a maximum number $\rho-1$ of polynomials A such that there is no corresponding integral of total differentials with *one* of the curves that they cut for logarithmic curves. The number ρ is precisely the Picard-Severi invariant (Chap. 15, No. 20). In the light of Severi's later developments, Picard constructs a base composed of a plane section $y=\text{const.}$, and of $\rho-1$ curves (1). These $\rho-1$ curves correspond to as many distinct polynomials A, for the two curves defined by the two signs in (1) are dependent, their sum being a multiple of the plane sections.

4. *Some Special Double Planes.*—Picard examines some interesting special cases, notably, (58), (62), also (63), Vol. 2, p. 254, the surface

$$z^2=f(x)F(y), \tag{2}$$

where f, F are polynomials of degree 3, $2p+1$ without multiple roots. In general, in this case, the polynomials M^2-n^2fF are irreducible and the polynomials A in the integrals of the third kind can be removed by the subtraction of the logarithm of a rational function. In certain special cases, there are obtained hyperelliptic surfaces of rank two (birational transforms of an elliptic Kummer surface). In this connection see also Chap. 17, No. 22.

Picard also shows, loc. cit., and (60), that if, in (2) both f and F are arbitrary polynomials, then $\rho=1$. If we consider the associated curves C, C':

$$u^2=f(v), \qquad u^2=F(v), \tag{3}$$

then $\rho>1$ for (2) demands that the period matrices of their integrals of the first kind (Chap. 15, No. 4) have a common submatrix.

In (63), Vol. 2, Chap. 10, Picard determines his invariant ρ_0 (number of double integrals of the second kind) for some of the preceding surfaces.

An interesting connection, compare L. Godeaux (42), exists between the surfaces (2) and the curves (3). We may establish between them a correspondence by assigning to the pair of points (u, v) of C, and (u', v') of C', the unique corresponding point

$$x=v, \qquad y=v', \qquad z=uu'$$

of the surface and to the point (x, y, z) of the surface the two pairs of points

$$u=\epsilon\sqrt{f(x)}, \quad v=x; \qquad u'=\epsilon'\sqrt{F(y)}, \quad v'=y$$

of the two curves, where ϵ, ϵ' are ± 1, the signs being chosen so that $z=uu'$. The surface (2) is thus the image of an I_2 on the surface ϕ which represents the pairs of points of the curves C, C'. It is quite likely that, by means of the known results on the number ρ for surfaces which represent the pairs of points of two algebraic curves (Chap. 16, No. 6), it would present no great difficulty to determine ρ for all surfaces (2) and also for similar algebraic varieties.

In (56), p. 359, Lefschetz has applied his general results on the base to the double plane (1). He has determined the minimum base and the base invariant when one of the components of $F=0$ varies within a sufficiently general linear system of plane curves. This includes most of the double planes considered in the literature.

5. *Cyclic Multiple Plane.*—In addition to the double plane, Picard has repeatedly considered the m-fold multiple plane

$$z^m = x^m + P(y), \tag{4}$$

where P is a polynomial of degree m. In (63), Vol. 2, p. 285, he found that the surface has a Severi base composed of $(m-1)^2+1$ straight lines so chosen that only one set of m is coplanar. In Vol. 1, p. 109, he makes use of a special case of equation (4) to prove that non-singular algebraic surfaces possess two-dimensional cycles that are not ~ 0 (Chap. 15, No. 9).

In (31), L. Godeaux gives an example of a p-fold plane F (p prime) carrier of an I_p without coincidence points and such that its image F^* has its Severi number $\sigma = p$ (Chap. 15, No. 20). The number σ that he considers is computed on the assumption that the singularities of F have no consecutive two-cycles. Godeaux's example, suitably modified, is likely to lead to an F^* whose σ in the strict sense is p. Incidently, in his transformation T, $z'=z$ must be replaced by $z'=\epsilon^k z$, where $\epsilon^k \neq 1$, for, as it reads, it has coincidence points $(0, 0, z_0)$ where z_0^p is the value of ϕ for $x=y=0$.

De Franchis shows, in (10), that a double plane whose irregularity $q \geqq 2$ carries an irrational pencil of curves. Comessatti has shown that this is true for every triple plane (7).

M. Bottasso discusses, in (2), the cyclic multiple plane with an irreducible branch-curve and determines certain numerical invariants for a fairly general type of such plane. In (3), he treats the case $n=3$ in greater detail.

6. *Other Special Surfaces.*—Given an irreducible curve C, there may be associated with its point pairs two surfaces; the first, F, represents its ordered point pairs, the second, F^*, represents its point pairs with order disregarded. Then F^* is image of an I_2 on F and, from that point of view, has been investigated by M. de Franchis (8) and by F. Severi (67), (68) p. 35. The curves on F^* are images of the symmetrical correspondences on C (Chap. 16, No. 7) and the degree and genus of F^* curves are to be considered as invariants of the C correspondences. Severi investigated these curves, finds their relation to the correspondences, determines a minimum base on F^*, and shows that ρ is equal to the number of independent symmetrical correspondences on C. Referring to

Rosati's theorems (Chap. 15, No. 7) on these correspondences, this means that $\rho=k+1$, where k in the singularity index of the period matrix of the integrals of the first kind attached to C (Chap. 15, No. 7). Severi gives formulas for the degree and genus of a curve in terms of the numbers which relate it to the minimum base.

Remy, in (64), (65), investigates a surface Φ, image of an I_2 on the surface F^* which represents the non-ordered point pairs of a non-singular plane quartic curve C_4. The groups of the I_2 are such that the four corresponding points on C_4 are collinear. The surface is representable by means of θ-functions in three variables, subject to the vanishing of a certain θ-function. The method is transcendental throughout. Remy examines the curves on Φ ,its various integrals, and determines the related invariant numbers. See also Godeaux(50).

In (1), R. Baldus maps a double ruled surface R^* on a simple one R. If R is not rational, to a generator g^* of R^* corresponds either a generator g of R, taken twice, or two generators g_1 and g_2 of R. In the second case, the locus of invariant points is composed of generators; in the first case, there is a simple or composite curve of coincident points which meets each generator in two points.

O. Chisini(6) discusses an elliptic surface $p_g=0$, $p_a=-1$ which has a unimodular linear pencil $|K|$ of elliptic curves and an elliptic pencil of irrational curves C, such that $[K, C]=n$, determining an I_n on the surface. Particular attention is given to the case where n is composite. The I_n may or may not be cyclical. The surface can be mapped on an n-fold cylinder $\phi(x, y)=0$, with curve of branch-points consisting of one or more plane sections $z=\text{const.}$

F. Severi has considered(66) the I_2 on a quartic surface F_4 which contains a sextic curve C_6 of genus 2. He shows in particular, that the only involutorial birational transformations of such a surface into itself are those generated by the infinite nets of genus two on the surface. He shows further that a quartic surface which is transformed into itself by an involutorial homography depends on, at most, 11 moduli as does also, under the same conditions, a double plane of genus one with a branch sextic.

II. Involutions Having a Finite Number of Coincidences

7. *General Theorems.*—The involutions for which the number of coincidences is finite (or zero) have been studied in particular by L. Godeaux. This author has outlined, in the important paper(51), the chief features of the theory of such involutions.

Let I_n be an involution of order n on an algebraic surface F and let F^* be the image of I_n. We may suppose that F and F^* each lie in an S_3 and that their equations are, respectively,

$$f(x, y, z)=0 \qquad \text{and} \qquad f^*(X, Y, Z)=0.$$

Then there exists a rational transformation

$$X=\psi_1(x, y, z), \qquad Y=\psi_2(x, y, z), \qquad Z=\psi_3(x, y, z),$$

which defines I_n. Moreover, F. Enriques has shown in (18) for surfaces of genera one and later Godeaux in (38), (39) proved without restriction the following fundamental theorem: If an I_n on any algebraic surface has only a finite number (or zero) of coincidences, it is generated by a group of birational transformations of the surface into itself. We shall say that the involution is cyclic, Abelian, etc., according as it is generated by a cyclic, Abelian, etc., group of transformations.

A (simple) point of coincidence P is said to be *perfect* if the group which generates I_n leaves invariant the tangent lines to F at P; otherwise, P in *non-perfect*. The branch-points of F^* are said to be perfect or non-perfect according as the corresponding coincidences on F are, or are not, perfect.

On a normal surface F^*, image of an I_n with n a prime number, we have, from L. Godeaux (38): A perfect branch-point is a conical n-fold point the tangent cone at which is rational; a non-perfect branch-point is a biplanar double point formed by $(n-2)/2$ consecutive double points such that the last is an ordinary biplanar point. On the surface F we have: the perfect coincidences of I_n are $(n-1)$-fold points on the canonical curves of F in composition with I_n.

If we denote the arithmetic and linear genera of F by p_a, $p^{(1)}$, and those of F^* by π_a, $\pi^{(1)}$; if, further, α is the number of perfect, and β the number of non-perfect coincidences of I_n (n a prime), then we have, [L. Godeaux (38)]

$$\begin{aligned} 12p_a &= 12n\pi_a+(n-1)(n-5)\alpha-(n^2-1)\beta+12(n-1), \\ p^{(1)} &= n(\pi^{(1)}-1)+(n-2)^2\alpha+1. \end{aligned}$$

If, in addition, F^* has an infinite canonical system, not in composition with a pencil, its plurigenera π_i $(i=2, 3, \ldots)$ are connected with the plurigenera P_i of F by the relations:

$$12P_i=12n\pi_i+[(n-1)(n-5)+6i(i-1)(n-2)^2]\alpha-(n^2-1)\beta$$

$n=$ a prime number, $i=2, 3, \ldots.$

8. *Involutions of Given Order.*—In (42) L. Godeaux has determined the necessary and sufficient conditions that a simple, normal surface F^*,

of arithmetic genus $\pi_a \geqq -1$, free from exceptional curves, of order n, with hyperplane sections of genus π, such that $n-\pi>\pi_a+1$, and lying in a space of $n-\pi+\pi_a+1$ dimensions, shall represent an I_2 having a finite number of coincidences. F^* must have a conical double point at each branch-point. The number of these branch points is a multiple of four. Among the hypersurfaces which intersect F^*, in addition to its multiple curve, in the complete linear system of curves expressed as double the hyperplane sections (i. e., such that some curves of·the system are constituted of two hyperplane sections) some pass through the branch-points and touch F^* at each point of their common curve. Let the equations of F^*, in non-homogeneous coordinates be:

$$\phi_1(x_1, x_2, \ldots, x_r)=0, \qquad \phi_2(x_1, x_2, \ldots, x_r)=0, \qquad \ldots, \qquad \phi_{r-2}(x_1, x_2, \ldots, x_r)=0$$

and let the hypersurface $f(x_1, x_2, \ldots, x_r)=0$ touch F^* as stated in the theorem. Then the equations of F are:

$$\phi_1=0, \qquad \phi_2=0, \qquad \ldots, \qquad \phi_{r-2}=0, \qquad x_{r+1}=\sqrt{f}.$$

As a particular consequence, we have the result that all the coincidences of an I_2 are perfect.

The I_2 on irregular surfaces F whose image surfaces F^* are regular were studied by L. Godeaux in ([26]) and ([27]). F is the rational transform of a surface S of the same irregularity which belongs to the Picard variety of F and which is transformed into itself by a transformation of second species of that variety. To the points of S invariant for this transformation correspond on F the coincidences of I_2. The number of these coincidences cannot be zero.

The same author has also shown, ([30]), see, also, ([32]), that the algebraic surfaces of genus $p_g>1$ on which the canonical system is not formed of curves of a pencil but such that the bicanonical system is in composition with an I_2 having a finite number of coincidences, have the genera $p^{(1)}=7$, $p_a=0$, $p_g=3$, $P_2=7$, $P_3=19$, etc. As a projective model of these surfaces, one may take a bicanonical double surface of order 12, in S_6, with hyperplane sections of genus 10, having 28 conical double points as branch-points.

The necessary and sufficient conditions that a simple, normal, surface F^* shall be the image of an I_3 having a finite number of coincidences and lying on an algebraic surface were found by L. Godeaux([31]). F^* must have a certain number α of conical triple points at which the tangent cone is rational and a certain number β of ordinary biplanar double points. Among the hypersurfaces determining on F^* the system of curves three-fold to the hyperplane sections, at least one must have three-point contact with F^* along a curve which passes doubly through α_1 of the triple points, simply through the remaining $\alpha-\alpha_1$ triple points, and through the β double points. The number α_1 vanishes simultaneously with β.

It has been pointed out by L. Godeaux(49) that surfaces exist on which lie involutions having a finite number of coincidences of prime order greater than any preassigned positive number. That involutions of arbitrarily given order, with no coincidences, lie on hyperelliptic surfaces was pointed out by Enriques-Severi(19).

9. *Involutions of Genera One on Surfaces of Genera One.*—That any involution of genera one ($p_a=P_2=1$) which lies on a surface of genera one, free from exceptional curves, can have only a finite number of coincidences was noticed by F. Enriques(12). L. Godeaux has classified these involutions (25), (30), (34), (35), and (53). In making this classification, he first divides these involutions into two species. Those of the first species are cyclic or are generated by cyclic transformations having common coincidences. All the others are of second species. Further the image surface F^* of the involution is simple, normal, of genera one, of order $2\pi-2$ (π arbitrary but sufficiently large). It lies in an S_π, and its hyperplane sections are of genus π. The conditions given are necessary and sufficient, except in (*g*), that such a surface F^* shall represent an involution of the given type.

Of the first species, he finds the following types:

(a) An I_2 having 8 coincidences. F^* has 8 conical double points. Among the hyperquadrics through these 8 nodes, some touch F^* at each point of their common curve.

(b) Cyclic I_3 having 6 three-point coincidences. F^* has 6 ordinary biplanar points. Among the hypercubics through the double points, some have three-point contact with F^* at each point of their common curve.

(c) Cyclic I_4 with 4 four-point and 4 two-point coincidences. F^* has 2 conical and 4 biplanar, each with 1 consecutive, double points. Among the hyperquartics through the double points, some have four-point contact with F^* at each point of their common curve. This type is discussed in detail in (25).

(d) Cyclic I_6 with 2 six-point, 4 three-point, and 6 two-point, coincidences. F^* has 6 double points; 2 conical, 2 ordinary biplanar, and 2 biplanar, each with 2 consecutive. Among the hypersurfaces of order 6 through the double points, some have six-point contact with F^* along their common curve.

(e), (f) I_8 generated by two cyclic birational transformations of order 4 such that $T_1^2=T_2^2=(T_1^2T_2^2)$. In (e) there are 4 eight-point and 4 two-point coincidences and F^* has 1 conical, and 4 ordinary uniplanar, double points. In (f), there are 2 eight-point, and 6 four-point coincidences, and F^* has 2 ordinary uniplanar, and 3 biplanar, each with 1 consecutive, double points. In both cases, 3 families of hyperquadrics touch F^* along curves of genus $\pi-2$ and pass through the uniplanar points in such a way that each of the 3 consecutive nodes at the point lies on the hyperquadrics of a single family. In (f), the hyperquadrics of each family touch, at two biplanar points, the line common to the 2 tangent planes. Further, in each case, $\infty^{2\pi-3}$ sets of three hyperquartics are circumscribed to F^*, have third order contact with it at $2\pi-3$ simple points, and pass through the 5 double points. Each node consecutive to a uniplanar point lies on a single one of them. In (*f*), each hyperquartic touches, at 2 biplanar points, the line common to the 2 tangent planes. These types are discussed in detail in (30).

(g) An I_{12} generated by two transformations T_1 and T_2, of periods 4 and 3 respectively, such that $T_1^{-1}T_2T_1=T_2^2$. It has 2 twelve-point, 6 four-point, and 4 three-point, coincidences. The double points of F^* are: 1 ordinary biplanar, 2 biplanar each with 1 consecutive, 2 uniplanar such that 2 of the 3 singular tangents are consecutive.

Of involutions of the second species, he finds the following types:

(h), (i), (j) Involutions of order $2^k (k=2, 3, 4)$ having $8(2^k-1)$ two-point coincidences forming $2^{4-k}(2^k-1)$ groups of the involution. F^* has $2^{4-k}(2^k-1)$ conical double points which may be arranged in k sets of 8 which have, two by two, 4 common points. Among the hyperquadrics which pass through the double points of each set, some touch F^* along their common curve.

(k) Abelian I_9 having 24 three-point coincidences forming 8 groups of I_9. F^* has 8 ordinary biplanar double points which may be arranged in 2 groups of 6 having 4 common points. Among the hypercubics through the points of each group, some have three-point contact with F^* along their common curve.

The same author shows in (43) Art. 12 (correcting a statement made by him in (53)) that there does not exist on these surfaces an I_8 of genera one and species two generated by a transformation of period 4 and one of period 2.

The conditions under which a double plane of genera one is image of an I_3 on a surface of genera one were determined by L. Godeaux(44) He finds that the branch-curve in the double plane can be reduced to one of the following types:

(*a*) C^6 with 6 cusps which lie on a C^2.

(β) C^8 with two fourfold points, A_1 and A_2, and 6 cusps. Infinitely many C^4 have nodes at A_1, A_2, pass through the cusps, and touch C^8 in 2 variable points.

(γ) C^{10} with a seven-fold point A to which are consecutive 2 triple points, A_1, A_2 and 6 cusps. Infinitely many C^6 have A as a four-fold point, A_1, A_2, as double points, pass through the cusps, and touch C^{10} at four variable points.

(δ) C^{12} having a nine-fold point A, to which are consecutive 3 triple points, A_1, A_2, A_3, and 6 cusps. Infinitely many C^8 have A as a six-fold point, A_1, A_2, A_3, as double points, pass through the cusps, and touch C^{12} in 6 variable points.

A similar list of conditions that a double plane of genera one shall be image of a cyclic I_4 on a surface of genera one were given by the same author in (25).

That any quartic surface, subject only to the conditions of containing 6 ordinary biplanar points, is image of an I_3 lying on a surface of genera one was shown by L. Godeaux(24). He further finds, in (25), conditions under which a quartic surface represents an I_4 lying on a surface of genera one.

L. Godeaux has noticed, in (23) and (52), that, if a surface of genera one is image of an I_2 on a surface of the same genera and if it contains a linear system of curves of odd genus greater than 3, the generic curve not passing through the branch points, then that surface is image of other I_2 on other surfaces of genera one. Further, if a surface of genera one is image of an I_n on a surface of the same genera, and if it contains a linear system of curves of genus $\pi \geq 2$, the generic curve not passing through the branch-points, then that surface contains $n-1$ other linear systems of curves of genus $\pi - 2$.

That a quartic surface, subject to the single condition of containing a gauche C^6 of genus 3, contains infinitely many I_2 of genera one was stated by L. Godeaux(45), but these I_2 were shown not to be effective by V. Snyder and F. R. Sharpe in (72), see p. 290. L. Godeaux has determined, in (47), see, also, F. Enriques(17), conditions under which a double plane of genera one shall contain one or more I_2 of genera one.

10. *Involutions of Genera* $p_a = P_3 = 0$, $P_2 = 1$ *on Surface of Genera One.*—F. Enriques has shown in (11) that every surface of genera $p_a = P_3 = 0$, $P_2 = 1$, is image of an I_2 on a surface F of genera one and that this I_2 has no coincidences. L. Godeaux has added, (40), and (37), see Chap. 3, that this surface F contains two other I_2. One of these is of genera one; the other is rational and makes it possible to represent F on a double quadric with a branch C^8. The transformations generating

these 3 I_2 are permutable in pairs. He also points out, (43), see p. 70, that any surface of genera one which contains an involution of genera zero, bigenus one, also contains one of the same order but of genera one.

In the last named paper, he classifies the involutions of genera $p_a=P_3=0$, $P_2=1$, lying on a surface of genera one, as follows:

(a) I_2 having no coincidences. Every surface of genera $p_a=P_3=0$, $P_2=1$, is image of such an I_2.

(b), (c) I_4 having 8 two-point coincidences. (b) is cyclic; (c) is Abelian and is generated by two transformations of period two.

(d) Abelian I_6 having 6 three-point coincidences.

(e)–(i) I_8 as follows: (e) cyclic, (f) composed of 2 I_4 of which one is of genera one, the other of type (b), (g) composed of an I_4 of genera one and an I_2 of type (a), (h) composed of an I_2 of genera one and an I_4 of type (b), (i) composed of $3I_2$ of which two are of genera one and the third is of type (a). In (e), (f), (g), there are 4 four-point and 4 two-point coincidences. In (h) and (i), there are 24 two-point coincidences.

(j) Abelian I_{12} composed of a cyclic I_6 of genera one and an I_2 of type (a). It has 2 six-point, 4 three-point and 6 two-point coincidences.

(k), (l) I_{16} having 56 two-point coincidences. (k) is Abelian and is composed of $4I_2$ three of which are of genera one and the fourth is of type (a). (l) is composed of $2I_2$ of genera one together with an I_4 of type (b).

The surfaces of genera one which contain a trirectangular group of birational transformations into themselves were studied by L. Godeaux in four notes(48). He finds five cases and determines the genera of the component involutions in each case.

11. *Involutions of Genera* $p_a=P_3=0$, $P_2=1$, *on Surfaces of the Same Genera.*—The involutions of genera $p_3=P_3=0$, $P_2=1$, which have a finite number of coincidences and lie on a surface of the same genera were also classified by L. Godeaux(37); see also, (36) and (43), Art. 11, p. 76. He determines the various types, considers the generating birational transformations, and finds the necessary and sufficient conditions that a simple, normal surface F^*, of genera $p_a=P_3=0$, $P_2=1$, shall represent an involution of the given type, as follows:

(a) Cyclic I_2 having 4 perfect coincidences. F^* is birationally equivalent to a sextic surface having the edges of a tetrahedron as double lines and 4 conical double points. There are ∞^2 quartic surfaces circumscribed to the tetrahedron which pass through the 4 double points and touch F^* along their common curve. It is further shown(42) that F^* may be represented by a double plane of one of two specified types. These double planes are studied in greater detail in three notes(28), (29). In the second of these notes it is pointed out that F^* is also the image of a second I_2 lying on a surface F' of the same genera as F but birationally distinct from it. The branch-points for the two I_2 coincide.

(b) Cyclic I_3 having 3 non-perfect three-point coincidences. F^* has 3 ordinary biplanar double points. Among the hypersurfaces which intersect F^* in the system three-fold to the hyperplane sections (i. e., in the complete linear system such that some curves of the system are constituted by three hyperplane sections) some pass through the 3 double points and have three-point contact with F^* along their common curve. At the 3 double points, this curve of contact is on one nappe of the surface.

(c) Cyclic I_4 having 2 perfect two-point and 2 non-perfect four-point coincidences. F^* has 1 conical double point and 2 biplanar, each followed by one consecutive, double points. Among the hypersurfaces which intersect F^* in the system of curves four-fold to the hyperplane sections, some pass through the 3 double points and have four-point contact with F^* along their common curve.

(d) Abelian I_4 generated by two transformations of period 2. It has 12 perfect two-point coincidences. F^* has 6 conical double points forming 3 couples. Among the hypersurfaces intersecting F^* in the system of curves double the hyperplane sections, some contain two couples of nodes and touch F^* along their common curve.

(e) Cyclic I_6 having 3 two-point, 2 three-point, and 1 six-point coincidences. The double points of F^* are: 1 ordinary biplanar with 2 consecutive, one ordinary biplanar, 1 conical. Among the hypersurfaces intersecting F^* in the system of curves six-fold to the hyperplane sections, some pass through the 3 double points and have six-point contact with F^* along their common curve.

(f) Abelian I_8 generated by 3 transformations of period 2. It has 28 perfect two-point coincidences. F^* has 7 conical double points. Among the hypersurfaces intersecting F^* in the system of curves double the hyperplane sections, some pass through 4 nodes and touch F^* along their common curve.

The same author shows (43), No. 11, that an I_9 of the above genera cannot exist on these surfaces.

12. *Involutions on Other Special Surfaces.*—The involutions on the surfaces which represent the non-ordered pairs of points of a C^4 of genus 3 were studied by L. Godeaux in 4 notes(50). Such a surface always contains an I_2 with 28 perfect coincidences. The pairs of points of this I_2 correspond to 2 pairs of points of C^4 which are collinear. If the given surface is also transformed into itself by a cyclic transformation of period

greater than two, the C^4 is also transformed into itself by a transformation of the same period. The transformations of a C^4 into itself having been determined by S. Kantor(55), and A. Wiman(73), Godeaux is enabled to determine the involutions with a finite number of coincidences on such a surface, as follows:

(α) An I_2 on the surface corresponding to an arbitrary quartic.
(β) An I_3 and an I_6 on the surface corresponding to $x^3y+f_4(y)=0$.
(γ) An I_3 and an I_6 on the surface corresponding to
$$ax^4+bx^2y+cxy^3+dx+ey^2=0.$$
(δ) An I_7 and an I_{14} on the surface corresponding to $xy^3+y+x^3=0$.
(ϵ) An I_9 and an I_{18} on the surface corresponding to $x^4+xy^3+y=0$.

The I_3 of (β) has 6 perfect and 9 non-perfect coincidences; that of (γ) has 3 three-point coincidences. The I_6 of (γ) has 1 six-point, 2 three-point, and 27 two-point coincidences. Image surfaces of some of these involutions are also determined. These involutions are studied further by the same author in (22).

An I_3 which has 2 perfect and 5 non-perfect coincidences and lies on a surface of order 5 and genera $p_a=p_g=4$, $p^{(1)}=6$, in S_3, was studied by L. Godeaux(33). The image surface F^* of this I_3 is of genera $p^{(1)}=2$, $p_a=p_g=2$, $P_i=i(i-1)/2+3$.

A rational involution with a finite number of coincidences which lies on a non-rational surface was found by L. Godeaux(22). He projects a surface of Veronese from a point not in its S_5 and intersects the resulting $V_3{}^4$ by an hypercubic of the S_6 determined by it. Among the surfaces of section, consider those surfaces F which are invariant under a projective transformation of period 7 having 7 invariant points. Three of these invariant points are shown to lie on F. They constitute the (non-perfect) coincidences of the I_7 on F. The surface F may be transformed into a sextic surface in S_3 having a uniplanar triple point. Its genera are, $p^{(1)}=4$, $p_a=p_g=3$, $P_2=7$. For the genera of I_7 he finds $p_a=P_2=0$; hence the surface is rational. The surface of Humbert (which represents the pairs of points of a C^4 of genus 3 in such a way that to each point of the surface corresponds 2 pairs of points constituting a canonical group) is a special case of the above surface. If C^4 is a Klein quartic, the Humbert surface contains an I_7 with 3 coincidences. Hence, the I_7 having 6 coincidences, on the surface which represents simply the pairs of points of Klein's quartic, is rational.

In (46), L. Godeaux has shown that, if a regular surface F^* represents an involution of prime order p, free from points of coincidence and lying on an algebraic surface, then the divisor of F^* is a multiple of p. Illustrations of this theorem are given in the article cited and in (31). For further consideration of the number σ for involutions without coincidences, whether on algebraic surfaces or varieties, see S. Lefschetz(56) and M. de Franchis(9).

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CHAPTER XV

TRANSCENDENTAL THEORY

This chapter contains a brief sketch of those parts of algebraic geometry wherein transcendental methods predominate. We pass as rapidly as possible over the classical topics in order to dwell more at length on recent investigations. It will be found that Riemann matrices and topology have been considerably emphasized. Their growing importance in this field and their numerous contacts with others assuredly justify this procedure.

The bibliographical data, admittedly not complete, will naturally be most scanty in the parts already well covered in the literature. At the end will be found a glossary of new notations and terms of most frequent occurrence in this and the next two chapters.

§ 1. ALGEBRAIC CURVES

1. For the classical theory a long list of standard treatises will be found in Wirtinger(52). Two expositions quite in keeping with our point of view are Picard(23) vol. 2, Chaps. 13-18, Severi(34), Chaps. 7, 8, 9.

2. *Riemann Surfaces.*—We owe to Riemann the clear concept of an algebraic curve C, $f(xy)=0$, as a real continuum ∞^2, with a representation as an m sheeted Riemann surface K_y for $y(x)$ (m degree of f in y). Of greatest importance is the fact that if C' is a birational transform of C with K'_y for its Riemann surface, then K_y and K'_y are in continuous $(1, 1)$ correspondence (homeomorphism). Consequently any entity with homeomorphic invariance as regards K_y also possesses birational invariance as regards C.

Basic Property of K_y.—It is homeomorphic to a *normal* surface K, the two-sided disk with p holes or sphere with p handles. If n is the number of branch points of K_y we have $n=2(p+m-1)$ from which to calculate p. With C in arbitrary position, m, n are its order and class. p is the *genus* and its birational invariance follows from the fact that it is a topological invariant.

By means of $2p$ suitably selected cuts on K, along closed circuits (retrosections) $\gamma, \gamma_2, \ldots, \gamma_{2p}$, it is turned into a two-cell (homeomorph of the interior of a circle) plus its boundary taken twice.

3. A sum of oriented circuits on K is called a *linear cycle* or merely *cycle* of the surface. Two cycles g, g' such that g can be deformed into

g' plus certain pairs of oppositely oriented arcs are *homologous,* this being expressed by a *homology* $g \sim g'$. If 0 stands for a point, then also $g - g' \sim 0$. Such homologies can be combined like ordinary linear equations, the multipliers being always integers. The cycles $g_1, \ldots, g_r$ are dependent or otherwise, accordingly as there exist λ's not all zero such that $\Sigma \lambda_i g_i \sim 0$ or not.

On a two-cell every arc with its end points on the boundary is deformable into an arc on the boundary whose end points are the same. Applying this to the two-cell derived from K by means of the retrosections we have:

I. Every cycle on K is $\sim$ to a sum of multiples of the γ's.

II. On the other hand the γ's are independent. Therefore the maximum number of independent cycles (maximum actually reached) is $2p$.

Given $2p$ independent cycles $g_1, \ldots, g_{2p}$ for every other g there is a relation

$$\lambda g \sim \Sigma \lambda_i g_i, \qquad \lambda \neq 0.$$

If $g'_1, \ldots, g'_{2p}$ is another set of cycles then

$$A_\mu g'_\mu \sim \Sigma A_{\mu\nu} g_\nu, \qquad A_\mu \neq 0$$

and the g''s are independent if and only if $|A| \neq 0$.* There are systems of independent cycles such that $\lambda = 1$ for every g, for example, the system of $2p$ retrosections. A set of such cycles, say $\delta_1, \ldots, \delta_{2p}$ is called a *fundamental set* for linear cycles or simply fundamental set, abridged into F. S.† If δ'_μ are the cycles of another F. S. then

$$\delta'_\mu \sim \Sigma a_{\mu\nu} \delta_\nu, \qquad \delta_\mu \sim \Sigma b_{\mu\nu} \delta'_\nu.$$

Hence

$$\delta_\mu \sim \Sigma b_{\mu\nu} a_{\nu\rho} \delta_\rho$$

and since the δ's are independent these are identical relations. Therefore $\mathbf{ba} = 1$, hence $|b| \cdot |a| = 1$ and as both determinants are integers

$$|b| = |a| = \pm 1.$$

4. *Abelian Integrals.*—By an *abelian integral* attached to C is meant an integral of type

$$v(x) = \int_{x_0 y_0}^{xy} R(xy)\, dx,$$

---

* We shall greatly simplify matters by making use of matrices. We apply these conventions now quite generally adopted: A matrix of elements, a_{ij} will be called $\mathbf{a}$, its transverse $\mathbf{a}'$, its determinant (when it is square) $|a|$, the matrix of the conjugate elements $\bar{a}_{ij}$, $\bar{\mathbf{a}}$, the unit matrix $\mathbf{I}$. A square matrix $\mathbf{a}$ whose terms are integers with $|a| = \pm 1$ is called *unimodular.*

† Fundamental set = primitive set = minimum base. These terms often found in the literature are all equivalent.

where R is rational and the path of integration is on K. When C is birationally transformed into C', v is changed into an abelian integral v' attached to C' and the singularities of v' are the transforms of those of v and are of the same nature. In particular if v is holomorphic at every point of C, so is v' at every point of C'. v is then called integral of *the first kind,* by far the most important type. We shall denote them by the generic letter u. Any u can be put in the form

$$\int\frac{Q(xy)\,dx}{f'_y},$$

where Q is an adjoint polynomial of order $m-3$ (polynomial with an $(r-1)$-fold point at every r-fold point of f granted that f has only multiple points with distinct tangents; see Chap. 2). There are well known connections with the canonical series on C. A set of integrals $u_1, \ldots, u_k$ is linearly dependent or otherwise accordingly as there do or do not exist constants c_i not all zero, such that $\Sigma c_i u_i$ is constant, or $\Sigma c_i du_i \equiv 0$. There is the basic theorem: The maximum number of linearly independent integrals of the first kind is p. It follows that if $u_1, \ldots, u_p$ are linearly independent every other u is a linear combination of them plus a constant.

Let g_n^r be a linear series on C, $G = A_1, \ldots, A_n$ a variable group of it, A a fixed point of C, u an integral of the first kind of the curve. Then

$$(1)\qquad \sum\int_A^{A_i} du = \text{const.}$$

This is Abel's classical theorem for integrals of the first kind. It is often written in the form

$$(2)\qquad \sum\int_{B_i}^{A_i} du \equiv 0, \quad \text{modd. the periods}$$

where $G' = B_1 \ldots B_n$ is any other group of g_n^r, and the periods referred to are those of u relatively to the cycles of C. The theorem (2) has been inverted by Clebsch who proved that when it holds for every integral of the first kind then G and G' are equivalent groups, i. e., belong to the same complete g_n^r.

A related question is Jacobi's *inversion* problem. Given the independent set $u_1, \ldots, u_p$, and the arbitrary constants $v_1, \ldots, v_p$ the problem asks for a group of p points $G = A_1, \ldots, A_p$, such that

$$\sum\int_A^{A_j} du_i = v_i, \quad (i = 1, 2, \ldots, p).$$

For the very extensive literature on this topic, see (52).

From the geometric point of view the chief result is that for arbitrary v's, G is unique. The exceptional values correspond to $G =$ a special group, and then the point $(v_1, \ldots, v_p)$ of S_p belongs to a certain V_{p-2} (non-algebraic). This together with Clebsch's result leads quite rapidly to the Riemann-Roch theorem.

5. There are two other kinds of abelian integrals.—The second is characterized by behavior like a rational function over K, the third by the existence of a finite number of so-called *logarithmic* points in the vicinity of which the integral $v(x, y)$ behaves like a logarithm plus a rational function. More precisely let $A(\alpha, \beta)$ be such a point, t the corresponding local uniformizing variable ($t = (x-\alpha)$ at finite distance, $\frac{1}{x}$ at infinity when axes are arbitrary, $(x-\alpha)^{\frac{1}{q}}$ at a branch point). Then there exists a k such that $t=0$ is at worst a pole for $v - \frac{k}{2\pi i} \log t$. The constant k is the related logarithmic *period* or residue. Let $A_1, \ldots, A_n$ be the logarithmic points, k_i the period for A_i. Then $\Sigma k_i = 0$. Conversely if this sum is zero there is a v whose logarithmic points and related periods are the A's and k's.

6. Of very great interest is the theory of periods of integrals of the first kind. For if u is an integral of the first kind and g a cycle. $\int_g du$ the period of u as to g, is a birational invariant of C. Thus the integrals of the first kind appear as the natural tool for deriving quantitative propositions. Let $g_1, \ldots, g_{2p}$ and $u_1, \ldots, u_p$ be independent sets of cycles and integrals and let $\omega_{j\mu} = \int_{g_\nu} du_j$. Then each term of $\boldsymbol{\omega}$ is invariant. Of particular importance is the effect on $\boldsymbol{\omega}$ of changing the sets. If we pass from the g's to the g''s as in No. 2, we have

$$A_\mu \int_{g'_\mu} du_j = \Sigma A_{\mu\nu} \int_{g_\nu} du_j = \Sigma A_{\mu\nu} \omega_{j\nu}.$$

Hence if $A_{\mu\nu} = A_\mu a_{\nu\mu}$ the new period matrix is $\boldsymbol{\omega}\mathbf{a}$. If the two sets are fundamental $|a| = \pm 1$ as before. Let us now pass from the u's to the u''s with

$$u'_j = \Sigma a_{jk} u_k; \qquad |a| \neq 0.$$

Then

$$\int_{g_\mu} du'_j = \Sigma a_{jk} \int_{g_\mu} du_k = \Sigma a_{jk} \omega_{k\mu}.$$

Hence $\boldsymbol{\omega}$ is now changed into $\boldsymbol{\alpha\omega}$. Therefore all possible period matrices are represented by $\boldsymbol{\alpha\omega}\mathbf{a}$; $|a|$, $|a| \neq 0$. Two matrices such as $\boldsymbol{\omega}$ and $\boldsymbol{\alpha\omega}\mathbf{a}$, where $\boldsymbol{\alpha}$ and $\mathbf{a}$ are square, of orders p and $2p$, with determinants $\neq 0$ and a with rational terms, are called *isomorphic*, the passage from one to the other is called an *isomorphic transformation* or an *isomorphism*. When a is unimodular, the matrices are said to be *equivalent*. These notions are due to Scorza(32.1). Matrices corresponding to two different sets of $2p$ independent cycles are isomorphs; when the two sets are fundamental the matrices are equivalent.

Our chief concern is not so much with the properties of $\boldsymbol{\omega}$ as with the whole class of matrices, equivalent, or more generally isomorphic to it. The properties invariant with respect to isomorphism are then of essential importance. We have first Riemann's fundamental theorems:

I. When the F. S. is composed of the retrosections, the u's can be so chosen as to give ω the form

$$\|\delta_{jk};\ \tau_{jk}\|, \qquad (j, k=1, 2, \ldots, p)$$

where τ is a symmetrical matrix, and $\delta_{jj}=1$, $\delta_{jk}=0$ $(k\neq j)$.

II. Let $\tau_{jk}=\tau'_{jk}+i\tau''_{jk}$. Then $\Sigma\tau''_{jk}x_jx_k$ is a definite quadratic form.

The equivalent conditions for the general isomorphic ω can be put in the following form: There exists an alternate rational matrix $\mathbf{c}$, such that

III. $\boldsymbol{\omega}\mathbf{c}\boldsymbol{\omega}'=0$.

IV. $i\boldsymbol{\omega}\mathbf{c}\boldsymbol{\omega}'$ is a definite positive Hermitian matrix.

In this precise form III, IV, were given explicitly by Baker(2.1) p. 224 (1907), for a period matrix of multiply periodic functions. A substantially equivalent form was given later by Scorza(32), (32.1). Any rational alternate $\mathbf{c}$ satisfying III is called an *alternate matrix* of $\boldsymbol{\omega}$. When it satisfies both III and IV it is called a *principal matrix* of $\boldsymbol{\omega}$.

7. *Riemann Matrices.*—Matrices satisfying III, IV are of considerable importance in various theories and well deserve to be investigated for their own sake as first done by Scorza (1914) who calls them *Riemann* matrices (abridged notation: R. M.). The matrix $\boldsymbol{\omega}$ of No. 6 is the R. M. of the curve C. Scorza's results will be found treated at length in (32.1), (33). See also Krazer-Wirtinger(12) p. 827; Lefschetz(16), and Ch. 17, § 4 (résumés of Scorza's work). Scorza brought out clearly the importance of certain isomorphic invariant integers: (a) The *singularity index* k of $\boldsymbol{\omega}$, or number of linearly independent alternate matrices of $\boldsymbol{\omega}$. (b) The *multiplication index* h, analogous to k when $\mathbf{c}$ is merely a *rational matrix* of $\boldsymbol{\omega}$, i. e., no more constrained to be alternate. (c) The *simultaneity index* λ of two R. M.'s $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, or number of linearly independent relations $\boldsymbol{\omega}_1\mathbf{c}\boldsymbol{\omega}'_2=0$, $\mathbf{c}$ rational. The index λ is unchanged when both $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are replaced by two isomorphs. As a matter of fact Scorza, wrote $1+h$, $1+k$, where we have h, k. This complicates matters a good deal; h, k as here defined are the actual numbers occurring all the time and are far more convenient in every way.

Scorza has shown that h is also the number of linearly independent *multiplications* of $\boldsymbol{\omega}$, or relations $\boldsymbol{\alpha}\boldsymbol{\omega}=\boldsymbol{\omega}\mathbf{c}$, where $\boldsymbol{\alpha}$ and $\mathbf{c}$ are square of orders p, $2p$, and $\mathbf{c}$ is rational. This is an actual restriction on $\boldsymbol{\omega}$

unless $\mathbf{c}=s\mathbf{I}$, s a rational multiplier and then $\boldsymbol{\alpha}=s\mathbf{I}$ also. In this special case we have *ordinary*, otherwise *complex* multiplication.

Of great importance is the question of reducing $\boldsymbol{\omega}$ to the form $\begin{Vmatrix} \boldsymbol{\omega}_1 & 0 \\ \boldsymbol{\omega}_2 & \boldsymbol{\omega}_3 \end{Vmatrix}$ where $\boldsymbol{\omega}_1$ is an R. M. of genus $<p$ called a *submatrix* of $\boldsymbol{\omega}$. It is then further reducible to the type $\begin{Vmatrix} \boldsymbol{\omega}_1 & 0 \\ 0 & \boldsymbol{\omega}_2 \end{Vmatrix}$, Poincaré, Picard, Scorza(32.1), p. 298, Rosati(31.1). The R. M. is then said to be *impure*. In order that two R. M.'s possess a common submatrix (which may coincide with one of them) it is necessary and sufficient that their $\lambda>0$, Scorza(32.1), p. 301. Any R. M. is isomorphic to an essentially unique type

$$\begin{Vmatrix} \boldsymbol{\omega}_1 & 0 & \dots \\ 0 & \boldsymbol{\omega}_2 & \dots \\ & \dots & \boldsymbol{\omega}_n \end{Vmatrix}$$

where the matrices $\boldsymbol{\omega}_j$ are all pure and any two are either identical or non-isomorphic, Scorza(32.1), p. 308.

Scorza has also considered the question of the upper limits of his invariants(32.1), pp. 303, 309. They depend upon the corresponding reduced form. He finds in any case $k\leqq p^2$, $h\leqq 2p^2$, $\lambda\leqq 2p_1p_2$ where p_1, p_2 are the genera of the two matrices whose simultaneity index is λ.

Additional references: Krazer(11), Krazer-Wirtinger(12) for earlier developments. Summaries of Scorza's work are found in (12) p. 822, Lefschetz(16). See also Ch. 17, §§ 4, 5, 6. A treatment more or less parallel to Scorza's is due to Rosati(31), also Ch. 16, No. 13, who applied it mostly to singular correspondences. Applications to hyperelliptic surfaces and abelian varieties will be considered in Ch. 17. In § 4 in particular Scorza's theory is examined more thoroughly and indications as to his method are also given.

§ 2. LINEAR SYSTEMS OF CURVES ON SURFACES AND RELATED INVARIANTS

8. Unless otherwise stated F_m is assumed to be a non-singular irreducible surface in some S_k, or else with only the singularities of the generic projections of such a surface. These singularities may then practically be disregarded. Regarding the reduction of an arbitrary F to this type see Ch. 10.

A linear system $|C|$ of curves of F is by definition a system cut out by a linear family of V_{k-1}'s (hypersurfaces) of its carrying S_k. The *genus* $[C]$ of $|C|$ is the genus of its generic curve, its degree $[C^2]$ the number of intersections of two arbitrary curves of it. Given $|C|$, $|D|$, $[CD]$ represents the number of intersections of arbitrary curves of the two systems. $|C|$ is *complete* if not contained in an ampler linear system. The meaning of $|C+D|$, complete system sum of $|C|$ and $|D|$ is obvious, likewise that of $|C-D|$ whenever there is a C containing some D as a part. When $C-D$ is not *effective*, i. e., does not exist, $C-D$ is said to be a *virtual* curve, Severi(40). To such curves, to reducible effective curves, or to curves with accidental singularities there may be attached a *virtual* genus and a *virtual* degree or intersection number, Severi(40) by means of the following two laws: (a) $[CD]$ is a symbol which obeys the distributive law as to C and D, i. e., $[(C+C')D]=[CD]+[C'D]$ and similarly with respect to D. (b) $[C]$ is a symbol such that in all cases

$$[(C+D)]=[C]+[D]+[CD]-1.$$

The two laws hold for effective curves. The proof is immediate for (a), and for (b) has been given by Noether(22), Picard-Simart(24), vol. 2, p. 106, Lefschetz(18), p. 252. See also in this connection (14).

An important observation, Lefschetz(13), p. 19, is that when $C, D, F,$ are properly oriented cycles (see below No. 10), $[CD]$ is actually equal to the algebraic number of suitably signed crossings $(C \cdot D)$, or to the Kronecker-index of the cycles as to F. [Veblen(50); Weyl(51); Lefschetz(20)]. When the curves are virtual there are definite cycles corresponding to them for which also $[CD]=(C \cdot D)$, as follows from the fact that the Kronecker-index also obeys (a).

9. The canonical system $|K|$ defined in No. 16, plays an essential part in the theory of F. If p_g-1 is its dimension, p_g is the *geometrical* genus of F. It is also the number of linearly independent adjoint polynomials of order $m-4$ (adjoint polynomial = a polynomial vanishing on the double curve when F_m has no other singularities). The postulation of adjoint polynomials of order $m-4+k$, k sufficiently high (number of linearly independent conditions satisfied by the polynomial) is a polynomial $f(k)$. The number $p_a=f(0)$, *arithmetical* genus of F, is like p_g an important birational invariant of F and their difference $q=p_g-p_a$ is the *irregularity* of F. The surfaces whose $q=0$ are called *regular.*

The canonical system can be defined directly thus: Let $|C|$ be sufficiently general and without base points. There exists a maximal system $|C'|$, such that C' cuts out canonical groups on C, called the adjoint system of $|C|$. Then $|K|=|C'-C|$. The system $|iK|=|iC'-iC|$ may exist whether $|K|$ exists or not and the number P_i of its linearly independent curves is the i-canonical genus. P_i is also an invariant of F and these plurigenera have played an important part especially in the classification of various types of surfaces.

The definition of the canonical and pluricanonical systems indicates plainly that they are invariant systems for F under *all* birational transformations, hence p_g, P_i, are *absolute* invariants, i. e., unchanged by any such transformation. This is also true for q and p_a. There are, however, *relative* invariants or numbers invariant only under certain types of birational transformations. In general when F acquires r' *exceptional* curves (transforms of ordinary points) and loses r'' such curves these invariants increase by a multiple of $(r'-r'')$. The most important and best known is the invariant I of Zeuthen-Segre (references in Castelnuovo-Enriques (^{4}), p. 700) defined as follows: Let $|C|$ be an irreducible pencil with $[C^2]$ ordinary distinct base points and let its δ curves of genus $<[C]$ have just one more double point than the generic C, and therefore precisely the genus $[C]-1$. Then $I=\delta-[C^2]-4[C]$. Another important relative invariant is the linear genus $p_1=[K]$. It is a relative invariant because K must always be considered as including the exceptional curves. There are other relative invariants of an algebraic nature but they are expressible in terms of I, p_1 and the absolute invariants.

For further references and details on the various invariants just described, see Castelnuovo-Enriques (4), p. 690 and Baker (2.2).

When $|K-C|$ exists and has the dimension $\nu-1$, $|C|$ is *special* and ν is its *specialization* index. This is true whether C is effective or virtual. In order that C be effective it is sufficient that

$$s=[C^2]-[C]+1+p_a-\nu\geqq 0$$

and then the dimension of the complete system $|C|$ is $\geqq s$. Furthermore $|C|$ belongs to a continuous set ∞^q of such linear systems. With more precision: There exists a V_q, the Picard-variety of F (see Ch. 17, No. 25), such that to each point of it corresponds one and only one $|C|$ varying continuously and birationally with the point.

The preceding theorem is closely related to the proposition on Picard integrals of the first kind recalled in Nos. 16, 17. For complete references on the theory summarized in the present number see Castelnuovo-Enriques (^{3}), (^{4}); Severi (^{35}).

§ 3. Topology of F

10. The surface F with the simplified singularities assumed in No. 8 and throughout the rest of this chapter, has the important topological property of being representable as a multiply sheeted Riemann four space (complex projective plane) which is a manifold in the sense of Analysis Situs, O. Veblen (^{49}), p. 88, a property not necessarily shared by arbitrary surfaces.

For the analytical applications the important notion is that of the *k-cycle* Γ_k of F. By that we mean here a closed k dimensional continuum imbedded in F and suitable as integration domain for k-fold integrals. A sphere or an anchor ring more or less deformed and contained in F are simple examples of a Γ_2. There will be local parametric representations for the coordinates of the points of Γ_k as analytical functions of variables $u_1, \ldots, u_k$ whose order determines the orientation of Γ_k. When we transpose the u's an odd number of times Γ_k is considered as changed into $-\Gamma_k$. Given a set of k cycles Γ_k^i $(i=1, 2, \ldots, r)$, we write the homology with integer coefficients:

$$\Sigma\mu_i\Gamma_k^i \sim 0, \quad \text{mod. } F,$$

Poincaré(30), whenever the continuum at the left can be deformed into one of lesser dimension plus certain k dimensional continua each counted a total algebraic number of times equal to zero. The "mod. F" is often omitted. Also $\Gamma_k \sim \Gamma'_k$ stands for $\Gamma_k - \Gamma'_k \sim 0$. The equivalence of the two homologies may equally be justified as in No. 3. When a homology as above exists the cycles Γ_k^i are *dependent*, otherwise *independent*. These notions are treated more at length by Poincaré(30), Veblen(49), Chs. 3, 4; Lefschetz(13), Ch. 1.

The maximum number of independent cycles is a topological invariant called the k-th *connectivity index* R_k. There are five connectivity indices, corresponding to $k=0, 1, \ldots, 4$. A point is conveniently considered as a zero-cycle and when F is irreducible, if A, B are two points $A - B \sim 0$, for it is the boundary of some arc AB. Hence $R_0 = 1$. Similarly $R_4 = 1$ for F is its own unique Γ_4 and it is not ~ 0. There remain R_1, R_2, R_3. But $R_1 = R_3$, Poincaré(30); Veblen(49), Ch. 4, and their common value is an even integer $2q$. This follows from investigations on Picard integrals (see below), and was also established directly by Lefschetz(15), (13), p. 30. q is in fact the irregularity of F. (See Nos. 17, 18.) There remain then the only distinct indices 1, R_1, R_2. By considering the Euler characteristic, Alexander proved in (2):

$$R_2 = I + 2R_1 + 2 = I + 4q + 2$$

where I is the Zeuthen-Segre invariant. See also Lefschetz(15), (13), p. 40. The number R_1 is an absolute birational invariant, but R_2 like I, is only a relative invariant (No. 9).

Another interesting set of invariant integers is obtained thus: Let Γ_k be such that $t\Gamma_k \sim 0$ for some $t > 1$ but for no lesser integer. Γ_k is called a *zero-divisor* for k-cycles. The number σ_k of non-homologous zero-divisors including one zero-cycle, is finite and a topological invariant. It is the k-th *torsion index*, Poincaré(27), Lefschetz(13), Ch. 1. All σ's except σ_1 and σ_2 are unity and these two are equal. We denote their common value by σ. As we shall see this is the same as Severi's base invariant σ(43).

11. Since any R_k+1 k-cycles are always dependent we may find a set Γ_k^i, $(i=1, 2, \ldots, R_k)$, such that, whatever Γ_k, there is a homology:

$$\lambda\Gamma_k \sim \lambda\Sigma\lambda_i\Gamma_k^i, \qquad \lambda\neq 0.$$

The proof is the same as given by Klein(9), p. 543. The Γ_k^i set may be called an *intermediary* F. S. by analogy with a term used by Severi in his theory of the base. If we agree to neglect zero-divisors, or accept division of the coefficients of a homology by a common factor, they constitute an F. S. It is best to indicate this modified homology by a special sign $\approx$. Then

$$\Gamma_k \approx \Sigma\lambda_i\Gamma_k^i,$$

a relation whose precise meaning is that the difference between the two sides is a zero-divisor. Let G_k^j, $(j=1, 2, \ldots, \sigma_k-1)$ be the actual divisors. Then

$$\Gamma \sim \Sigma\lambda_i\Gamma_k^i + \Sigma\mu_j G_k^j.$$

Hence the cycles Γ_k^i, G_k^j constitute an F. S. for k cycles.

12. There is a series of results due to Picard, Poincaré, Lefschetz (references below), describing the relations between the cycles and linear curve systems. We consider merely a linear system $|H|$ of hyperplane sections, though a more general one could take its place. Let H_{x_1} be an arbitrary section $x_1=\text{const.}$, $((x_1, x_2, \ldots, x_k)$ cartesian spaces for the S_k that carries $F)$, and p the genus of an arbitrary H. With arbitrary axes and infinite H, H will never acquire any other singularity than a double point with distinct tangents. Let this happen for $x_1=a_1, \ldots, a_N$. The genus of H_{a_i} is then $p-1$.

Draw rectilinear cuts aa_i. As x_1 describes aa_i there exists a linear cycle δ_i of H_{x_1} that tends to a point as $x_1 \to a_i$. When x_1 turns around a_i every cycle of H_{x_1} is increased by a multiple of δ_i, or is unchanged (such is in particular δ_i itself).

Concerning linear cycles we have these theorems: Every linear cycle of F is $\sim$ to one on H_{x_1}. The curve H_{x_1} carries $R_1=2q$ invariant cycles which are independent cycles of F, hence also of H_{x_1}. However there may not exist any F. S. composed of them. Finally every cycle of H_{x_1} that is ~ 0 mod. F is $\sim$ to a sum of δ's mod. H_{x_1}.

As regards the two-cycles, let Δ_i be the locus of δ_i when y describes aa_i. Denote generically by (H_a) a two dimensional (open) part of H_a. Then for every Γ_2 there is a homology;

$$\Gamma_2 \sim \Sigma\lambda_i\Delta_i + (H_a), \quad \text{mod. } F, \tag{3}$$

with the related condition

$$\Sigma\lambda_i\delta_i^0 \sim 0, \quad \text{mod. } H_a \tag{4}$$

where δ_i^0 is the initial position of δ_i on H_a. Let x_1 describe any path $a_i x_1$ (x_1 not an a_i) with the δ's occupying final positions called δ_i. Then from (4) and by continuity considerations

$$\Sigma\lambda_i\delta_i \sim 0, \quad \text{mod.}\, H_{x_1}.$$

It will be observed that the sum in (4) constitutes precisely the boundary of the continuum $-(H_a)$.

The basic references for the last few results are Picard(24), vol. 1, Chs. 1-4, vol. 2, Ch. 11; Poincaré(26); Lefschetz(15), (18), Chs. 2, 3.

13. In (18), (19), (21), Lefschetz established a basic connection between the topology of F and the distribution of its systems of curves, or the Severi base theory. The chief feature of his work is the study of the properties of the curves when considered as two-cycles of the surface.

If C', C'' are two curves it will be natural to consider the cycle $C = C' - C''$. Any combination of such cycles is called an *algebraic* cycle. It is always $\sim$ to a difference of two effective curves.

There exists some surface other than F passing through C''—let k be its order. Then if $H_1 \sim kH$ is its complete intersection with F and $H_1 - C'' = E$ the residual of C'' as to it, $C = (C' + E) - (C'' + E) \sim G - kH$ where $G = C' + E$ is effective. Hence of the two effective curves whose difference is C, the second may be chosen as a multiple of the plane sections. When kH varies in its system it remains $\sim$ to itself, hence the particular curve kH may be any one of the linear system $|kH|$.

14. Since any $1 + R_2$ two-cycles are related by a homology, there is a maximum $\rho \leqq R_2$ to the number of independent curves (effective or virtual). ρ is in fact Picard's number(24), vol. 2, Ch. 9, also called the Picard-Severi base number, as was proved by Lefschetz(21), (19), p. 348; also (18) and below, Nos. 16, 21. Since any $\rho + 1$ curves are dependent, the same reasoning as for general cycles will lead to an intermediary F. S. $C_1, C_2, \ldots, C_\rho$, which together with certain zero-divisor curves $D_1, \ldots, D_{\sigma-1}$, will form an F. S., so that for any C we have

$$C \sim \Sigma\lambda_i C_i + \Sigma\mu_j D_j.$$

The number σ is the same as in No. 11, Lefschetz(19), p. 348.

When $\rho > 1$ there can always be constructed an F. S. whose curves are all effective. For let $\bar{C}_i = e_i C_i + x_i H$, $e_i = \pm 1$. There is an integer ξ such that for every i, when $x_i > \xi$, whatever e_i, $\bar{C}_i$ is effective. Assume then the x's so chosen. We have

$$C \approx \lambda H + \Sigma\lambda_i \bar{C}_i.$$

15. Furthermore there is a relation

$$aH + \Sigma a_i C_i \approx 0, \quad a \neq 0.$$

Since no multiple of H is ~ 0, Lefschetz(13), p. 20, and since $\rho > 1$, we may assume the initial set C_i so chosen that at least *two* integers a_i are $\neq 0$. All that is needed in order to have an effective intermediary F. S. is to be able to express H as $\approx$ to a sum of $\overline{C}$'s. This in turn merely requires that we find e's and positive x's $> \xi$ such that $\Sigma a_i e_i x_i = a - 1$, which can always be done since there are at least two terms at the left.

Since H is then $\approx$ to a sum of $\overline{C}$'s, this is true for every C. Hence $\overline{C}_1, \ldots, \overline{C}_\rho$ constitute an effective intermediary F. S. We have $D_i = \overline{D}_i - s_i H$, where $\overline{D}_i$ is effective and s_i merely higher than a certain limit. Also there is a ν such that

$$\nu H \sim \nu \Sigma \mu_i \overline{C}_i.$$

Take every s a multiple of ν. Then D_i is $\sim$ to a sum of the effective curves $\overline{C}_i$ and $\overline{D}_i$. Therefore every C is $\sim$ to a sum of $\overline{C}$'s and $\overline{D}$'s. Hence $\overline{C}_1, \ldots, \overline{C}_\rho, \overline{D}_1, \ldots, \overline{D}_{\sigma-1}$ is an effective F. S. For $\rho = 1$ a similar reasoning holds with C'_1, H, taking the place of $C_1, \ldots, C_\rho$, and the intermediary F. S. will be composed of two effective curves, the F. S. of $\sigma + 1$ effective curves.

The treatment here is related to Severi's construction of intermediary and minimum bases, (38), (43), (35), for his relation expressing algebraic dependence.

§ 4. Integrals of F

16. Abelian integrals have been generalized in two different directions by Noether and Picard; see Castelnuovo-Enriques(4), p. 714. The simple integrals, or integrals of total differentials, or also Picard integrals, are due to Picard. F being assumed projected on an S_3 into $f(xyz) = 0$, these integrals are of the form

$$\int R dx + S dy, \quad \frac{\partial R}{\partial y} = \frac{\partial S}{\partial x}, \tag{5}$$

where R, S are rational in x, y, z. (5) is of the first kind if it is holomorphic on F. The number of linearly independent integrals of the first kind is q [Castelnuovo, Enriques, Severi]. References and complete discussion: Castelnuovo-Enriques(4), p. 723, to which may be added Lefschetz(13), Ch. 4; Severi(35), p. 769: also (45.1), where he indicates in outline that an integral of the first kind (5), called for the present u, may be completely characterized by the set of double points of the loci $u = \text{const.}$

Noether introduced double integrals of F of the type

$$\iint R(xyz)\, dx dy, \quad R \text{ rational.} \tag{6}$$

It is of the first kind if finite over any analytical continuum of integration on F. Noether showed that in that case (6) is of the form

$$\iint \frac{Q(xyz)\,dxdy}{f'_z},$$

Q being an adjoint polynomial of order $m-4$. The number of linearly independent double integrals of the first kind is p_g. The curves which the Q's cut out on F outside of its double curve constitute the canonical system $|K|$. References Castelnuovo-Enriques(4), pp. 696, 716.

Picard has also introduced simple integrals of the second and of the third kinds. (5) is of the second kind when it behaves everywhere like a rational function. The number of linearly independent integrals of the second kind is $R_1=2q$, Picard(24), vol. 1, p. 150, additional references Castelnuovo-Enriques(4), p. 721. Finally (5) is of the third kind if it possesses logarithmic curves. There may be found on F sets of curves that cannot be logarithmic curves of any integral (5). The maximum ρ of their number is the Picard-Severi base number, No. 14. Further references Castelnuovo-Enriques(4), p. 727, to which may be added Lefschetz(15), (13), Note I; Severi(35), p. 773.

Picard also considered and extensively investigated double integrals of the second kind(24), vol. 2. The number ρ_0 of linearly independent integrals of that kind (in a sense that need not be described in detail here) is an absolute invariant given by

$$\rho_0=I+4q-\rho+2,$$

where I is as before the Zeuthen-Segre invariant number. Lefschetz(15), (13), Note I, greatly simplified the Picard theory. In particular he puts his formula in the form $\rho_0=R_2-\rho$, and in (17) shows that it is only a special case of a very general similar formula for algebraic varieties. Further references Castelnuovo-Enriques(4), p. 731.

§ 5. DISTRIBUTION OF THE CURVES ON F AND THE BASES

17. The best attack on the whole transcendental theory is due to Poincaré(25), (28), (29). His work was considerably simplified and then extended by Lefschetz(13), (19), (18), (21), who brought into play Analysis Situs and linked it with Severi's work on the theory of the base(35), (38), (42), (43), (45), (46). This outline is substantially as given in Lefschetz(18).

To H_y there belongs a set of p linearly independent integrals of the first kind

$$u_j=\int\frac{Q_j(xyz)\,dx}{f'_z},\quad (j=1, 2, \ldots, p)$$

where the Q's are adjoint polynomials and of degree $m-3$ in x, z. Among them, however, there are actually $p-q$ say the first $p-q$, of degree $m-3$ in x, y, z, while the remaining q are of degree $m-2$. Let $A_1, \ldots, A_m$ be the points at infinity of H_y (section $y=$const.)—they are its fixed points. Let C be an algebraic curve on F and $B_1, \ldots, B_r$ be a group of points on H_y that includes all points on C and also some of the A's each counted with arbitrary multiplicity. We consider with Poincaré the sums

$$v_j(y) = \Sigma \int_{A_1}^{B_k} du_j.$$

Upon investigating their behavior we find that they have this form:

$$v_j(y) = \frac{1}{2\pi i} \Sigma \lambda_k \int_A^{A_k} \frac{\Omega_{jk}(Y)\,dY}{Y-y}, \qquad j \leqq p-q;$$

$$v_{p-q+j} = \alpha_j.$$

Here the a's are the critical values of y, the α's constants and the λ's certain integers.

As a matter of fact when the Q's are not further restricted than as stated, the v's do not have quite the simple aspect here indicated. The situation is at first this: The R. M. of H_y is impure for $q>0$ and isomorphic to

$$\left\| \begin{matrix} \omega_1 & 0 \\ 0 & \omega_2 \end{matrix} \right\|$$

where ω_2 is of genus q, the cycles corresponding to its columns being precisely the invariant cycles and its terms all constant. As to ω_1 its columns correspond to cycles which are dependent upon the δ's. Its terms are then functions of y, and it has been shown by Severi that for y arbitrary it is pure(39), p. 530. Now the special choice of u's leading to the form given for the v's corresponds precisely to the case when ω is in the above reduced form, Lefschetz(13), p. 74.

At this point we connect up with Picard's integrals of the first kind. For to every u_{p-q+k} corresponds an integral of the first kind (5) that reduces to it on H_y, i. e., such that

$$R = \frac{Q_{p-q+k}}{f'_z}$$

and conversely. Therefore F possesses precisely q linearly independent Picard integrals of the first kind. This is essentially Poincaré's proof of that theorem already mentioned in No. 16. See Poincaré(29), p. 91; Severi(45); Lefschetz(13), pp. 74, 76.

18. To each C there is attached a definite set of v's corresponding to the case where the B's include all the intersections of C with H_y and no

other points. The set is of course only defined modd. the periods. At the very outset there is this proposition: If C' has the same abelian sums as C then it belongs to the complete linear system $|C|$; Poincaré(29), p. 98. He imposes the condition that C' is of same order as C, which is superfluous, Lefschetz(13), p. 77. Hence in order that C and C' belong to the same complete linear system it is necessary and sufficient that they cut H_y in two groups of points that belong to the same linear series; Severi(36), p. 55; Lefschetz(13), p. 77. See also Severi's closely related extensions of Abel's theorem to Picard integrals(36), (41), (45) and his inversion theorem for them (47.1).

We shall consider in a moment the necessary conditions on the v's in order that they be the abelian sums of some curve. For the moment let us just state that they do not involve the α's. In the v's corresponding to a given C let us vary the α's. There results for each choice of them a linear system of curves $|D|$ whose limit as the α's tend to their values for C may be not C but $C+kH$, $k>0$. We conclude: Given any C there exists a k such that $D=C+kH$ determines a complete continuous system of curves which constitutes an irreducible algebraic set of ∞^q linear systems. See Lefschetz(13), p. 80. This brings us quite close to the very precise result embodied in the theorem at the end of No. 9. For the present we return to the functions v_i and shall show how they lead very rapidly to the Severi bases.

We shall now be particularly concerned with the part played by the λ's. Poincaré in (29), (25), gave a set of equations linear in the Ω's and λ's. Working in a different direction Lefschetz obtained in (19), p. 343; (13), p. 66, the results to be described presently. If the v's are to be functions as obtained in No. 14 a first condition is that the λ's must be the coefficients of the Δ's in some Γ_2 of the reduced form of No. 12. Furthermore, (19), p. 346; (13), p. 71, the period of every double integral of the first kind as to that Γ_2 must be equal to zero. Hence these theorems:

I. The necessary and sufficient condition in order that the λ's shall correspond to a C or to a sum of multiples of the points A_i is that they define a reduced Γ_2 and that all the corresponding periods of double integrals of the first kind are zero.

II. The algebraic cycles or virtual curves, are completely characterized by the property that the corresponding periods of double integrals of the first kind are all zero.

This last proposition gives rise to the only systematic method for obtaining ρ. We can so choose the u's that the integrands of the first

p_g be the same as for certain double integrals of the first kind. The Γ_2 just considered will be determined up to a multiple of H_a by the relations

$$\Sigma\lambda_k\Omega_{hk}\equiv 0, \qquad (h=1, 2, \ldots, p-q).$$

We can determine a set of independent relations such as these, calculate the period of the double integral attached to u_h, $h\leqq p_g$,

$$\Sigma\lambda_k\int_a^{a_k}\Omega_{hk}(y)\,dy$$

corresponding to the Γ_2, form the matrix of these periods. Its rank is precisely ρ. This is manifestly a method of little practical value—a defect shared by many explicit solutions of mathematical problems.

19. *Equivalence of Curves and the Severi Bases.*—Two curves C, D, on F are *algebraically equivalent* or simply *equivalent,* and we write $C=D$, whenever a curve E can be found such that $C+E$ and $D+E$ are effective and that the complete linear systems $|C+E|$, $|D+E|$ are members of one and the same irreducible set of linear systems of curves on F. When F is regular this amounts to demanding coincidence of the two linear systems. It will be observed that this does not demand that C and D shall be effective. This all important relationship was introduced by Severi(38), who made a complete study of it in this paper and also in (43), (42), (35). He was led to it by his investigations of the surface image of point pairs of two algebraic curves(46). See also Albanese(1), (2.3) for an investigation of various types of continuous systems of curves.

If $C=D$, then $C+A=D+A$, whatever A. For in the definition E may be replaced by any other curve obtained by adding to it an arbitrary effective curve. Hence we can so choose E that $C+A+E$ and $D+A+E$ will also behave in accordance with the definition. From this the desired equivalence follows.

We conclude that $C=D$ and $C-D=0$ express one and the same thing. There follows also a definite meaning for the relation

$$\mu_1C_1+\ldots+\mu_rC_r=0$$

where the μ's are integers, expressing the *dependence* of the curves C_i.

20. The bridge between Severi's investigations and the theory discussed so far is the following theorem, Lefschetz(19), p. 347, (18), p. 81: The two relations

$$\Sigma\mu_iC_i=0, \qquad \Sigma\mu_iC_i\sim 0,$$

are completely equivalent. This gives at once: (a) A maximal set of independent algebraic cycles is also a maximal set of independent curves in the sense of Severi and in particular is composed of exactly ρ curves.

This is Severi's *ordinary* base, (33), p. 209. It implies also that the Picard-Severi ρ is the same as our ρ. See also Albanese(1.2). (b) There exists an F. S. for the Severi relation composed of $\rho+\sigma-1$ curves. They may all be chosen effective for $\rho>1$. For $\rho=1$ it may be necessary to take $\sigma+1$ effective curves in the set.

By II, No. 18, if Γ_2 is a zero divisor it is algebraic. Hence σ is the number of distinct curves C such that $C\neq 0$, $\mu C=0$, for some $\mu>1$. This actually identifies σ with the σ of Severi(43), which he defines thus: If C_1 is effective and sufficiently general there exist $\sigma-1$ effective curves $C_2, \ldots, C_\sigma-1$ such that $C_i\neq C_j$, $\mu C_i=\mu C_j$ for some $\mu>1$. Examples of F with $\sigma>1$ have been given by Severi, Godeaux, Lefschetz, references in (18), p. 248.

Conclusion.—The curves on F are distributed into discrete continuous sets of linear systems. To each set corresponds a definite set of characteristic integers λ in the v's and to each component linear system of each set a definite set of q characteristic constants a_j. The sets are derivable from a certain minimum base $C_1, \ldots, C_\nu$ ($\nu=\rho+\sigma-1$ or $\sigma+1$) in the following manner essentially due to Albanese(1), p. 203: For any C there is a set of positive μ's such that $|C+\mu H+\Sigma\mu'_i C_i|$, $|\mu H+\Sigma\mu''_i C_i|$ are complete and belong to the same continuous set of ∞^q linear systems.

21. Severi has given this equivalence criterium for two curves A, B, (38): In order that $\lambda A=\lambda B$ it is necessary and sufficient that A and B be of the same order and that

$$[A^2]=[AB]=[B^2].$$

He demands also that their common value be positive but this is not essential, Albanese(1), p. 201. From the criterium follows that in order that $C_1, \ldots, C_r$ be dependent it is necessary and sufficient that

$$\left\|\begin{matrix} [HC_1] & [HC_2] & \ldots & [HC_r] \\ [C_iC_1] & [C_iC_2] & \ldots & [C_iC_r] \end{matrix}\right\|$$

be of rank $<r$, Severi(38).

From this condition Severi derives the following result: In order that there exist a Picard integral of the third kind whose logarithmic curves are $C_1, \ldots, C_r$, it is necessary and sufficient that the curves be algebraically dependent. Hence ρ as defined by Picard (maximum number of curves not logarithmic for any integral (5), see No. 16) is also the maximum number of algebraically distinct curves on F in the sense of Severi, or of non-homologous curves in the sense of Lefschetz. This leads rapidly to the various Severi bases (intermediary, minimum), (43), (42). For further references and details see Castelnuovo-Enriques(4), p. 728, Severi(35), p. 764.

The value of ρ has actually been determined for certain important classes of surfaces, Castelnuovo-Enriques(4), p. 730; Lefschetz(19), p. 358, (13), Ch. 5.

§ 6. Extension to V_r

22. The chief differences are already present in the passage from $r=2$ to $r=3$, hence it will be sufficient to consider the latter case; for the general r see Lefschetz(19).

Severi has made a searching investigation(41.1) of the linear systems of surfaces on a V_3 and the related genera. The results are much more inconclusive than for F_m, and also much more difficult to derive. Castelnuovo-Enriques have shown(4.1) that the linear cycles and Picard integrals of total differentials of the first kind of V_3 and of its hyperplane sections are the same. Further references along this line in (4), p. 761.

Let $f(x, y, z, t)=0$ be the equation of V_3, assumed with no other singularities than those of the projection of a non-singular variety from a higher space. Its topology is in many respects similar to that of F and has been treated at length by Lefschetz(13), Ch. 5.

The surfaces of V_3 can be related to one another by homologies or by Severi equivalences, Severi(43), Lefschetz(19), (13). The chief results pertain to the relation between the surfaces and their sections by a given hyperplane section (or a surface variable within a sufficiently general linear system), and are practically all due to Lefschetz, loc. cit. Let H_z be a generic section $z=\text{const.}$ and call AB the intersection of the surfaces A, B of V_3. The relations

$$A=B; \qquad AH_z=BH_z \text{ on } H_z;$$
$$A\sim B, \text{ mod. } V_3; \qquad AH_z=BH_z, \text{ mod. } H_z,$$

are all equivalent, Lefschetz(19), p. 352; (13), p. 104; also for the first two relations Severi(47), p. 635. This leads at once to the bases of various types, which are the same whether $\sim$ or $=$ are considered. In particular there are related numbers ρ, σ.

To the critical values of z (those for which $z=\text{const.}$ is tangent to V_3) correspond vanishing two-cycles of H_z. If one of these is algebraic so are all the rest. When none are algebraic every F. S. for the curves of H_z is the intersection with H_z of an F. S. for the surfaces of V_3; Lefschetz(19), p. 354, (13), p. 106. In many cases this makes possible the determination of an F. S. for surfaces in an S_3. The vanishing cycles will certainly not be algebraic when V_3 possesses a triple integral of the first kind whose periods are not all zero, Lefschetz(19), p. 357.

The determination of ρ, σ and the bases can be carried out in some fairly simple but important cases, in particular for complete intersections.

See Klein(10); Fano(5), (6); Severi(44); Lefschetz(19), p. 358. In (48) Sisam extends Klein's theorem of (10) on the rectilinear congruences of S_3 to any S_r. This is equivalent to finding the F. S. for the hypersurfaces of a certain V_k whose points correspond to the lines of S_r, hence to determining its ρ and σ. As in Klein's case the F. S. is composed of a unique V_{k-1} image of a linear congruence of S_r. This has been extended to the congruences of linear spaces in any S_r by Severi(37). In (19), p. 362, Lefschetz determines the numbers ρ, σ for the variety image of the point pairs of two varieties. In the same paper, also in (13), Ch. 6, he has determined ρ, σ and the F. S. for abelian varieties, for which see Ch. 17.

Severi(41.1) has defined the arithmetic genus of any V_r. Lefschetz(14) solves this question: Given $C_1, \ldots, C_n, D_1, \ldots, D_k$, hypersurfaces of V_r, with $\lambda_i D_i = \Sigma \lambda_{ij} C_j$, to determine the arithmetical genera of the intersections of the D's in terms of those of the C's and of the λ's. Albanese(1.1) treats special cases of that problem and also determines the Zeuthen-Segre invariant of a sum of two surfaces on V_3 in terms of the other invariant characters of the two surfaces. Severi(41.1) indicates a number of unsolved problems concerning V_r.

GLOSSARY OF NEW TERMS, AND NOTATIONS OF MOST FREQUENT OCCURRENCE

$$\mathbf{a} = \|a_{ij}\|; \qquad \mathbf{a}' = \|a_{ji}\|; \qquad \bar{\mathbf{a}} = \|\bar{a}_{ij}\|.$$

When $\mathbf{a}$ is square its determinant is $|a|$. When furthermore the terms are all integers and $|a| = \pm 1$, a is called unimodular. $\mathbf{I}$ = the matrix unity.

F. S. = fundamental set = primitive set = minimum base.
R. M. = Riemann matrix.

$$\left.\begin{array}{l} h = \text{multiplication} \\ k = \text{singularity} \end{array}\right\} \text{index of an R. M.}$$

λ = simultaneity index of two R. M.'s.

It is not necessary to repeat here the definitions of the preceding terms, nor of what is meant by principal matrix, rational matrix, rational alternate matrix of an R. M., nor by isomorphic or equivalent R. M.'s. They will be found in No. 7. The R. M. of C^p is the period matrix of p independent integrals of the first kind relatively to an F. S. of cycles. In reading the literature it is important to bear in mind that our h, k, are Scorza's $1+h$, $1+k$.

Special notations for F_m: $|C|$ = linear system of curves. $[C]$ genus of the generic C; $[CD]$ number of intersections of curves C and D;

$[C^2]=[CC]$. p_g=geometric genus; p_a=arithmetic genus; $q=p_g-p_a$= irregularity; P_i=i-canonical genus; I=Zeuthen-Segre invariant; p_1=linear genus; ρ=Picard-Severi number, σ=Severi number=number of distinct virtual curves such that $C\neq 0$, $\mu C=0$, $\mu>1$. ρ and σ are also called *base* invariants. Γ_i=i-cycle. $\sim$ is the homology symbol with zero divisors included; $\approx$ the same with zero-divisors omitted. R_i=i-th connectivity index with $R_1=2q$. These terms are all defined in Nos. 9-14. Most of them are also applied to a V_r.

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CHAPTER XVI

CORRESPONDENCES BETWEEN ALGEBRAIC CURVES

I. GENERALITIES

1. An algebraic correspondence $T(m, n)$ between the points of two distinct curves C, C', or between those of the same curve C, is the same thing as an algebraic transformation of C into C' or of C into itself, whereby to each point of C correspond n points of C', to each point of C', m points of C. T may or may not impose restricting conditions upon the R. M.'s of the curves. In the first case the correspondence is *singular*, in the second, *non-singular*. A non-singular correspondence between the points of C is also known as correspondence with *valence*. If A is any point of C, G the group of points TA, there exists an integer γ, the *valence* of T, such that $G+\gamma A$ is a group of a fixed linear series on C. This type of correspondence, the earliest known, has already been treated in Chapter 7.

The groups of points of a $(1, n)$ correspondence between C' and C define an *involution* of order n, I_n, on C. It is rational or irrational accordingly as C' is rational or otherwise. Rational involutions, present on every C, are treated in Chapter 7.

An involution of order *one* defines a birational transformation of C into C', or of C into itself when $C'=C$. The various types of correspondences that we have just rapidly described will now be considered separately in detail. The notations and terminology of this chapter are those of the glossary at the end of Chapter 15.

§ 2. SINGULAR CORRESPONDENCES ON C^p

2. Historically speaking they have been introduced by Abel in connection with complex multiplication of elliptic functions but without much regard to the geometric aspects of the question. A. Hurwitz as we shall see must be regarded as the true creator of the theory. Before going into it we shall find it instructive to consider an elliptic carrier C, for example the cubic whose representation by Weierstrass functions is

$$x=\wp(u;\,\omega_1,\,\omega_2)\,; \qquad y=\wp'(u;\,\omega_1,\,\omega_2),$$

and examine the relation of the *multiplication* or *transformation* of elliptic functions with algebraic correspondences on C.

If the correspondence $(m, n): T$ assigns to the parametric point u the points $u_1, \ldots, u_n$, then by a familiar argument Σu_i is an integral of a rational function of $\wp$, $\wp'$, holomorphic over the period parallelogram hence of the form au, and $\wp(au)$, $\wp'(au)$ are rational functions of $\wp(u)$, $\wp'(u)$. Conversely if a is a constant such that $\wp(au)$, $\wp'(au)$ are rational functions of $\wp(u)$, $\wp'(u)$, there exists a correspondence T attached to it, namely the correspondence assigning to u the points au. Now the discovery of constants a with the property in question is the precise object of the theory of *transformation* or *multiplication* of elliptic functions, upon which we are thus thrown back. Referring to that theory; on every elliptic carrier there exist correspondences for which a is an integer: they are the *ordinary* correspondences, or correspondences with *valence*, the valence being in fact $-a$. In the exceptional case where the period ratio $\frac{\omega_2}{\omega_1}$ is a number of a quadratic imaginary domain there are present correspondences of a different type, the *singular* correspondences. There appears to be no reason for considering them here, as they have received, under one name or another, ample treatment elsewhere. Severi (74.1), p. 273, (79), has discussed them geometrically. For further details and references see Fricke (25), Krazer (44), p. 183, Fueter (25.1).

3. Leaving aside the case $p=1$, singular correspondences on an algebraic curve C have been introduced and thoroughly investigated by A. Hurwitz, first in connection with a special C^3 that arises in the study of modular equations (30), then in the general case (1886), (32), (34). There is a very complete treatment in Klein-Fricke (40), vol. 2, p. 518, and a more brief one in Baker (4), p. 637. See also Zeuthen (96), p. 285, Berzolari (7), p. 347.

4. On the algebraic curve of genus p, C^p, let $\delta_1, \ldots, \delta_{2p}$ be an F. S. of cycles, $u_1, \ldots, u_p$, a set of independent integrals of the first kind, $\boldsymbol{\omega}$ the corresponding R. M. If x and y are corresponding points in an (m, n) correspondence T on C^p, then as (x) describes δ_μ the corresponding points $y_1, \ldots, y_n$ describe paths whose sum is a cycle $\sim \Sigma a_{\mu\nu}\delta_\nu$. The matrix $\mathbf{a}$ defines the transformation on the cycles corresponding to T. There is also between the integrals of the first kind a relation

$$\sum_j u_k(y_j) = \sum_s a_{ks} u_s(x) + g_k \tag{1}$$

which leads to the matrix equation

$$\boldsymbol{\alpha}\boldsymbol{\omega} = \boldsymbol{\omega}\mathbf{a}. \tag{2}$$

Thus T defines a multiplication of $\boldsymbol{\omega}$ (Ch. 15, No. 7). Conversely for every multiplication (2), we know from the theory of Jacobi's inversion problem that there exists an attached correspondence T whose index

$n=p$. The relation (2) is an actual restriction for $\boldsymbol{\omega}$ if and only if the multiplication is complex. It is then and only then that T is a singular correspondence. In the non-singular case the integer matrix a is a multiple of the identity, or $\mathbf{a}=-\gamma\mathbf{I}$. Then from (2) follows $\boldsymbol{\alpha}=-\gamma\mathbf{I}$ and (1) becomes

$$\gamma u_k(x)+\Sigma u_k(y_j)=\text{const.}$$

These relations show that for γ positive, $\gamma x+\Sigma y_i$ is a group of a fixed linear series $g_{n+\gamma}$, while for γ negative γx plus some fixed points is a coresidual of the group Σy_j. This identifies T as a correspondence with the valence γ. Conversely the preceding relation subsists for any T with valence γ. Hence non-singular correspondences and valence correspondences are the same.

Valence correspondences, and with valence γ an arbitrary integer, exist on every C as follows from the remark made above and as first shown by Hurwitz(84). Severi has shown(76) that it is the only type present on a C^p with general moduli. From what we have said follows that it is the only type present whenever there is no complex multiplication, Scorza(70), Rosati(60). Severi's theorem is then an indirect proof that the R. M. of a general C is without complex multiplication. See, however, a recent discussion by Zariski(95.1), also Lefschetz(50.1), and Severi(77.3).

5. Hurwitz(84) has defined correspondences as independent or not, accordingly as their multiplications are or are not linearly independent and showed that the maximum number that are independent is $\leqq 2p^2$. It is, in fact, the invariant h of $\boldsymbol{\omega}$. Klein expressly introduces an F. S. for correspondences(40), vol. 2, p. 543. Hurwitz(84) also gives for the number of coincidences of T the expressions

$$U=m+n-\Sigma a_{\mu\mu} \tag{3}$$

$$U=m+n+\Sigma c_i\lambda_i \tag{4}$$

where the λ's are the coefficients in the expression of T in function of the correspondences of the F. S. and the c's are integers that depend upon those correspondences but not upon T itself. When there is valence $a_{\mu\mu}=-\gamma$ and (3) reduces to the Cayley-Brill formula (Ch. 7, No. 6). A formula that includes (4) has also been derived by Severi(79), (74.1), p. 240. From (3) Hurwitz also deduces an expression for the number of groups common to two correspondences. This method is based upon theta functions. Klein(40), vol. 2, Baker(4) replace these by the so-called prime functions.

Recently purely topological methods have been applied to coincidence formulas which have been shown to hold for correspondences that are merely continuous. Lefschetz has investigated the most general type of continuous correspondences on an n dimensional manifold and obtained formulas which reduce to those of Hurwitz in the case of algebraic curves, (46), (47), (50), (48). See also Chisini(16); Enriques-Chisini(23), vol. 3, p. 427; Alexander(1).

6. Severi has systematically developed (1903), (79), (74.1), Ch. 6, the geometric aspects of the Hurwitz theory. Let $T(x)$ denote the group of points assigned by T to x. He introduces the sum $T+T'$ and product TT' of two correspondences. The latter has its natural meaning, the former is defined as assigning to x, the group $T(x)+T'(x)$. If T, T' have valences γ, γ', TT' and $T+T'$ also have valences, respectively equal to $-\gamma\gamma'$ and $\gamma+\gamma'$. Operations of a somewhat different type have been considered by A. A. Bennett(5), (6). Severi also defines T^{-1} as assigning to x every y such that x belongs to $T(y)$. A relation $\Sigma\lambda_i T_i=0$ implies that the abelian sums of $\Sigma\lambda_i T_i(x)$ are all zero or congruent to periods, which is written $\Sigma\lambda_i T_i(x)\equiv 0$ (linear equivalence), with a well-known interpretation in terms of linear series. Any correspondence with valence γ satisfies the relation $T+\gamma I=0$. The existence of a minimum base follows then from the Hurwitz results. Severi shows that if U_i is the group of coincidence points of T_i then from the above equation between the T's follows the linear equivalence $\Sigma\lambda_i(T_i(x)+T_i^{-1}(x)-U_i)\equiv 0$, and therefore $\Sigma\lambda_i(m_i+n_i-U_i)=0$ where m_i, n_i are the indices of T_i and U_i its number of coincidences. From this to the expression (4) is but a step.

In the second part of (79) Severi applies the very suggestive notion of the surface F image of the non-ordered point pairs of two algebraic curves C^p, C^q to the study of their correspondences. T is then represented by an algebraic curve Γ on the F corresponding to the case when C^q is merely a copy of C^p. Known birational invariants of algebraic curves on a surface give rise to two new invariants of T: the *genus* and *degree* (or similar elements of Γ for F). See also De Franchis(22), Severi(77.4), Lefschetz(48). Further references to papers on the surface F are given by Severi(79), p. 5. To dependent correspondences and a minimum base in the sense of Hurwitz correspond dependent curves and a minimum base for the curves of F in the sense of Severi. In fact it is here that Severi found the starting point for his investigations on the bases for surfaces (Ch. 15, No. 19).

In the same paper Severi considers special correspondences, in particular the so-called symmetric type (T and T^{-1} coincide) and also the associated surface, image of an involution of order two on F. In (74.1), p. 281, (78), he shows that if μ is the degree of T, then

$$2p(2mn-\mu)^2 \geqq (U-m-n)^2$$

where the equality holds only when T has the valence $(U-m-n)/2p$.

This follows also from an inequality due to Hurwitz(34), p. 580. See also Castelnuovo(12); Rosati(66); Roth(67). Severi has studied in (76), (77) correspondences on an algebraic curve which varies within a sufficiently general linear system on an algebraic surface F. When F is regular every such T has valence. This is to be understood in the sense that exceptional curves of the system may well have singular correspondences, but not so its generic curve. Recently discussing Severi's theorem, Zariski(95.1) has shown that a sufficient condition in order that it hold is that the linear system on the regular F be *simple* (i. e., its curves through an arbitrary point of F must not pass ipso facto through other variable points). Severi shows(76), p. 521, that on C^p all correspondences of given indices m, n, are distributed into a finite number of continuous families.

In his latest book (74.1), Ch. 6 (1927), which has very complete historical and bibliographical data, Severi makes a searching investigation of algebraic correspondences and goes as far as it is possible to go without transcendental methods. His treatment is very complete on the algebraical side. He makes use of the same surface F as above and of the degree and genus of T. He associates with T its *complementary* correspondence T' defined as follows: Let g^n_{n+p} be the complete non-special series determined by a definite group $T(x_0)$ plus p arbitrary points of C^p. It will assign a unique residual group of p points to every group $T(x)$, and this group is by definition $T'(x)$. Manifestly $T+T'=0$. Severi uses T' in the proof of the Cayley-Brill formula, and investigates the properties of T' in relation to those of T. See also Severi(78), (77.2).

7. A systematic study of singular correspondences and related numerical invariants has been made by Rosati(1913), (66), (66.1), (60), (63). Rosati in fact developed the theory of Riemann matrices about the same time as Scorza (1915) and by a projective method dual to his, but with the treatment of singular correspondences as his chief object. Let two correspondences T, T' be called *equivalent* if the abelian sums for $T(x)$ and $T'(x)$ differ only by constants, i. e., if $T=T'$ in Severi's notation. Geometrically speaking T and T' are equivalent if $T-T'$ is a correspondence with zero valence. Then to a class of equivalent correspondences corresponds a unique multiplication (2) and conversely. To a sum or product of correspondences correspond the sum or product of the multiplications of their two classes. Hence classes of correspondences and multiplications may be designated by a common symbol to be combined according to the ordinary laws of addition and multiplication. If $T_1, \ldots, T_n$ are any number of such symbols, any polynomial $F(T_1, \ldots, T_n)$ whose coefficients are integers is the common symbol for a certain correspondence and its associated multiplication. It follows that the multipli-

cation algebra (Ch. 17, § 6) goes over into an algebra for correspondences, and it is this algebra which Rosati systematically investigates in his work.

Rosati(66), like Severi notably in (74.1), considers *virtual* correspondences, corresponding to virtual curves on the surface F defined in No. 6, and their characters are considered by both authors. Rosati defines and investigates at length(66) the *simultaneous* invariant of T, (m, n) and T', (m', n') as $mn'+nm'-\nu$ where ν is the number of their common couples. The simultaneous character K of T and T^{-1} is $m^2+n^2-(U+2d)$ where d is the number of common couples of T and T^{-1}. It satisfies the inequality $-\Omega \leqq K \leqq \Omega$, $\Omega = 2mn-\mu$. The equalities correspond respectively to $T=-T^{-1}$ or $T=T^{-1}$, (66) p. 389. Here T^{-1} is the inverse in the more general sense considered below.

Rosati(66.2), (66.3), introduces the notion of characteristic and minimum equations for a correspondence (see Ch. 17, § 6), and investigates their properties. He greatly generalizes the notion of *multiple valences* introduced by Severi(76), p. 533. The valences of T as defined by Rosati are the roots of the equation $|a+xI|=0$, or the negative multipliers of the multiplication T. A Rosati valence may also be defined(66.3) as any number γ such that there exists an integral of the first kind u having the property that $\gamma u(x)+\Sigma u(y_j)$ is constant. When the valences are integers, but not all equal, they become the Severi *plurivalences*.

The inverse T^{-1} of T already mentioned above plays an important part in Rosati's work. As he understands it, it is the class individualized by the Severi inverse (see No. 6), *not* the class belonging to the inverse of the multiplication of T. Let the F. S. on C^p be the retrosections, so that the fundamental principal form of $\boldsymbol{\omega}$ is $\mathbf{c}=\left\Vert \begin{matrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{matrix} \right\Vert$. Then if (2) is the multiplication T, the multiplication T^{-1} (associated with the correspondence T^{-1}) will have in place of $\mathbf{a}$ the matrix $-\mathbf{ca'c}$. This was stated by Hurwitz(34) and proved analytically by Rosati(61), p. 38, and topologically by Lefschetz(48). If T, T' are such that TT' has valence, i. e., if $TT'+\gamma I=0$, Rosati(60), p. 15, calls them *complementary*. The matrices $\boldsymbol{\alpha}$, $\mathbf{a}$ of T' are then $-\gamma\boldsymbol{\alpha}^{-1}$, $-\gamma\mathbf{a}^{-1}$. T is said to be *symmetric* if $T=T^{-1}$, *skew-symmetric* if $T=-T^{-1}$. To each rational alternate matrix of $\boldsymbol{\omega}$ corresponds a class of symmetric correspondences, to each symmetric rational matrix a class of skew symmetric correspondences. Hence the numbers of linearly independent symmetric and skew symmetric correspondences are $\mu_1=k$, $\mu_2=h-k$. These numbers are both $\leqq p^2$, Rosati(60), p. 14.

The valences of T and T^{-1} are conjugate of one another and roots of equal multiplicity of their respective equations. The valences of a symmetric T are all real, those of a skew symmetric T are all pure complex or zero. Their minimum equations have no multiple roots.

On various occasions, notably in (62), Rosati considered correspondences S, T, such that $lS=f(T)$, where f is a polynomial, its coefficients and l being integers. For a given T let S vary within the total set of similar correspondences. Then its j-th valence varies within a definite Dedekind order of the algebraic domain $K(\gamma_j)$, where γ_j is the j-th valence of T. The units of the order correspond to birational transformations of the Jacobi V_p of C^p, which under certain circumstances may have an infinite discontinuous abelian group G. When T^{-1} is not a rational function of T its order is in general different from that of T and $G'\neq G$. When the two orders, hence also the two groups coincide, certain cyclic transformations of V_p determined by the orders become Hermitian and therefore by a theorem due to Comessatti (21), they give rise to a birational transformation of C^p into itself. Rosati examines at length the application of this theory to C^2. See in this connection Ch. 17, No. 63.

When $\boldsymbol{\omega}$ is impure the matrix $\boldsymbol{\alpha}$ may be of rank $q<p$. Then there are two special sets of integrals associated with T. The first consists of q integrals having the property that all abelian sums $\Sigma u(y_j)$ are linear combinations of these q integrals. The second consists of $p-q$ integrals for which the sums are constant. A more general circumstance would be that these sums are equal to the corresponding integral multiplied by an integer γ, which is a Severi plurivalence. The number of independent correspondences corresponding to assigned pairs of sets of integrals is $\leqq 2q^2$. The sets are intimately bound up with the reductions of $\boldsymbol{\omega}$ to its normal form in the sense of Ch. 15, No. 7; Ch. 17, No. 39. The presence of the first set indicates that there is an isomorph $\begin{Vmatrix} \boldsymbol{\omega}_1 & 0 \\ 0 & \boldsymbol{\omega}_2 \end{Vmatrix}$ such that when $\boldsymbol{\omega}$ is reduced to it, (2) becomes a multiplication $\boldsymbol{\alpha}_1\boldsymbol{\omega}_1=\boldsymbol{\omega}_1\mathbf{a}_1$ on $\boldsymbol{\omega}_1$ alone. In a slightly more general case we would have $\boldsymbol{\alpha}_1\boldsymbol{\omega}_1=\boldsymbol{\omega}_1\mathbf{a}_1$, $\gamma\boldsymbol{\omega}_2=\boldsymbol{\omega}_2\gamma$, where γ is a plurivalence. As this last relation is identically verified in $\boldsymbol{\omega}_2$, γ is an arbitrary integer. See in this connection Severi (76), Rosati (60), (66.2).

8. Castelnuovo has shown (10) that if C^p possesses a continuous system ∞^1 of correspondences with one index m then $p\leqq\binom{m+1}{2}$; when equality holds, C^p is the birational transform of a non-singular plane curve of order $m+2$. This holds with the restriction that the ∞^2 groups of points which by means of the system correspond to all points of C^p must not be compounded with the groups of an involution.

Correspondences on hyperelliptic curves have been studied by Wiman (94), first paper; Kantor (38); Holmquist (28); Torelli (83).

Correspondences on a C^2 have been studied by Rosati(60), p. 18, (64).

The most interesting special investigations on singular correspondences have originated in modular equations, Klein(43); Hurwitz(30), (33). Extensive elaboration with further developments by Klein(40), vol. 2, pp. 597-704. If in place of the general modular group, congruence subgroups modulo m are considered, the fundamental polygon of the complex variable, say Z, is depictable on a Riemann surface whose genus p depends upon m. Let C be an algebraic curve parametrically representable, in uniform manner by automorphic functions of Z attached to the polygon. In the Z plane the relation $Z' = nZ$ (n integer prime to m), defines a transformation giving rise on C^p to an algebraic correspondence whose indices are both $n\Pi\left(1+\frac{1}{q_i}\right)$, where the q's are the distinct prime factors of n(40), vol. 2, p. 599. In many cases the correspondence turns out to be singular, for example when $m = 11$ and n is not a quadratic residue of 11(40), vol. 2, p. 618. This is closely related to the theory of modular equations. There is worth while work to be done by way of investigating these singular correspondences, whose properties would help to enlighten the general theory. Further types of singular correspondences arise also in connection with curves invariant under a cyclic group, § 6.

§ 3. CORRESPONDENCES BETWEEN C^p AND C^q

9. Their theory is in close parallel with the case already considered. If $\boldsymbol{\omega}$, $\boldsymbol{\Omega}$ are the R. M.'s of the curves, to the correspondence T is attached a relation

$$\boldsymbol{\alpha\omega} = \boldsymbol{\Omega}\mathbf{a} \tag{5}$$

analogous to (2). This is not a restriction on the matrices when and only when $\boldsymbol{\alpha} = 0$, in which case also $\mathbf{a} = 0$. Then if x is a generic point of C^p the group $T(x)$ on C^q belongs to a fixed linear series and similarly on C^p. Such a correspondence is said to be *ordinary* and may be set up between any pair of curves. When (5) is an actual condition it implies that $\boldsymbol{\omega}$, $\boldsymbol{\Omega}$ have a common submatrix (Ch. 15, No. 7). The correspondence is then *singular*. The number of independent singular correspondences is equal to the simultaneity index λ of $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$, Hurwitz(34); Severi(78).

Rosati(61), (63), has more especially investigated what happens when the R. M.'s are impure. In that case (5) may actually be reduced to a multiplication of a common submatrix $\boldsymbol{\omega}_1$ of genus p_1 of $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$. There are $p - p_1$ integrals of the first kind invariant on the groups of T on C^p and similarly $q - p_1$ on C^q. The curve Γ image of $T(m, n)$

(image of the pairs of associated points of the two curves) carries two involutions I_n, I'_m whose images are C^p and C^q. Conversely a curve Γ with two distinct involutions can be considered as the image of a correspondence between two other curves in the manner just described. Rosati studies these involutions and their products especially in relation to the R. M. of Γ. In particular he finds that if they have the same genus they cannot leave the same set of integrals of the first kind invariant. Each is then individualized by a definite mode of decomposition of the R. M. To this we return in No. 10. Rosati also considers(61), p. 48, (62), p. 8, the correspondences $T^{-1}T$ on C^p and TT^{-1} on C^q. Their valences are real and $\leqq 0$, those $\neq 0$ being equal and of same multiplicity. On Γ the products of I, I' in one or the other order give rise to correspondences whose valences are also real, those $\neq 0$ being again the same as for TT^{-1} and with same multiplicity.

The algebraic aspect of correspondences between two curves has been searchingly investigated by Severi(74.1), Ch. 6, especially in regard to their representation by one or two equations. See also Hurwitz(34), Snyder-Sharpe(82) who give examples. Severi, loc. cit., gives a very complete discussion of various properties of correspondences, also very complete historical and bibliographical indications.

Further references: Severi(78), (77.1), where he shows that the degree of T, of indices m, n, is $\leqq 2mn$, the maximum being reached only when T is ordinary.

10. Regarding special types, Chisini in (14) has completely classified all (2, 2) correspondences. See also § 4 for special (1, n) types, involutions and Klein's regular Riemann surfaces.

R. Torelli(87), (88) has made a special study of correspondences between curves of same genus p with the chief object of determining when they are birationally equivalent. His work has recently been summarized by L. Godeaux(26). His chief result is that if two curves of the same genus have the same R. M. relatively to their retrosections they are birationally equivalent. That the period matrices must correspond to retrosections is essential when the moduli of the curves are not general [Rosati(64)]. Severi showed in a supplement to (21), p. 452, that when the singularity index k of the period matrix ω of a C^p is unity (non-singular matrix) the Jacobi variety V^p entirely individualizes C^p. The central point in his proof is the fact that the hypersurface cut out on V^p by a θ of order one is an F. S. for all hypersurfaces of V_p. This is a special case of a general theorem due to Lefschetz(49), p. 389. See also Ch. 17, § 3. Comessatti has proved this general theorem(21): When there exists between the Jacobi varieties V_p, V'_p of C^p, C'^p a hermitian birational transformation

T, or a transformation into one another of the θ's of order one corresponding to their normal period matrices, there is associated with T a definite birational transformation of the curves. This theorem was proved for $p=2$ by Enriques-Severi(24), p. 375. It does not imply any restrictions on the R. M.'s. A θ of order one cuts out on V_p a hypersurface V_{p-1} image of all sets of p points of C^p that belong to an assigned g^{p-1}_{2p-1}. Geometrically then the theorem is that if TV_{p-1} is a similar hypersurface for V'_p, T determines a birational transformation of C^p into C'^p. Comessatti gives also a new proof of Torelli's theorem and shows that any singular birational hermitian T of V_p into itself induces a birational transformation of C^p into itself. See also in this connection Lefschetz(49), p. 459.

§ 4. IRRATIONAL INVOLUTIONS AND RELATED TOPICS

11. Let C^q be the image of the involution I_n, $q>0$. It defines an $(n, 1)$ correspondence between C^p and C^q or a rational transformation from the first curve into the second. We are particularly concerned, however, with all such correspondences for an assigned C^p, and variable C^q. There are also the related questions of regular Riemann surfaces and reducible abelian integrals to be examined in their turn.

Weber has shown(92) that when two curves of same genus $p>1$ are in simply rational correspondence the correspondence is actually birational. See also Picard(57), p. 542 (also previous two editions of same work); Enriques-Chisini(23), p. 456. This means that on C^p an I_n of genus p is of order *one*. This has been proved topologically by J. W. Alexander(1), p. 175. His stated restriction that I_n has for image C^p itself is not used in his proof. Rational transformations of an elliptic curve into itself have been completely classified by Torelli(85) and Scorza(69). In (54) Painlevé has made a thorough investigation of a rational transformation T of C^p into C^q, $q>0$. Following a method applied by Picard to birational transformations (§ 6), he considers the effect of T on differentials of the first kind, then takes their ratios, obtains Weber's theorem and proves that the number of transformations with $q>1$ is at best finite. For $q=1$ there may be a discreet set and only a continuous family when $q=0$. See also Humbert(29); Painlevé(54.1). (55); DeFranchis(22); Berzolari(7), p. 350; Severi(74.1), p. 288; Enriques-Chisini(23), p. 477. DeFranchis and Enriques-Chisini gave this very simple proof of the Painlevé theorem: (a) From Zeuthen's expression $U=2p-2-n(2q-2)$ for the coincidences (Ch. 7) they derive $n \leq \frac{p-1}{q-1}$ when $q>1$. (b) For a given n the algebraic nature of the problem insures that the number of distinct continuous families is finite.

(c) When $q>0$ each family consists of a single I_n for the number of common pairs of transform points for two involutions in the same family is found on computation to be negative. From this to the theorem is but a step. Further references on involutions are Rosati(61), (62); Severi (74.1), Ch. 6 and this chapter No. 9.

12. *Regular Riemann Surfaces.*—The inverse and closely related problem consists in the determination of all curves C^p which carry an I_n whose image is C^q. A Riemann surface K^p for C^p is derived from a surface K^q attached to C^q about like an ordinary Riemann surface from the complex plane. We draw on K^q non-severing cuts issued from a fixed point, take n copies of the configuration and unite them along the borders of the cuts. The cuts and mode of uniting are governed by the sole condition that the final configuration be itself a connected Riemann surface. K^p so obtained has been called *regular Riemann surface* by Klein, who appears to have been first to construct that type. The groups of its I_n are the sets of n points of K^p that correspond to one and the same point of K^q. For the main references along this line see Krazer-Wirtinger(45), p. 843, to which may be added Enriques-Chisini(23), p. 427. In (56) Poincaré has considered the question from the point of view of automorphic functions and the related fundamental polygon. For polygons corresponding to modular functions see Klein(40), vol. 1, p. 333. Of interest is the observation that I_n may well be without coincidences, Poincaré(56), p. 318; Enriques-Chisini(23), p. 430. Then the cuts on K^q are all closed circuits. Chisini(17), (18), and Enriques-Chisini(23), pp. 430-456, have investigated at length the case where the permutations of the n copies of K^q along the cut circuits are powers of one and the same cyclic permutations. Then if $f(x, y)=0$ represents C^q, C^p can be represented as $f(x, y)=0$, $z^n=F(x, y)$, where F is a polynomial.

13. *Reduction of Abelian Integrals.*—As applied to integrals of the first kind it consists in the discovery of a set of $q<p$ integrals of the first kind of C^p whose period matrix is of the form $||\boldsymbol{\Omega}; 0||$ where $\boldsymbol{\Omega}$ is a submatrix of genus $q<p$ of the R. M. $\boldsymbol{\omega}$ of C^p. In other words the question is to find if $\boldsymbol{\omega}$ is pure or impure and in the latter case to find its submatrices. As a more special problem we may wish to find if the q integrals are rationally transformable into a set of integrals of a certain C^q. More precisely given $C^p: f(x, y)=0$ we wish to know if it possesses a set of $q<p$ integrals of the first kind $v_i(x, y)$ $(i=1, 2, \ldots, q)$ such that a rational transformation

$$(6) \qquad x=R(x', y'), \qquad y=S(x', y')$$

reduces them to q distinct integrals attached to the transformed curve C^q; $F(x', y')=0$. C^q represents an involution of genus q on C^p, so that there is an intimate connection between this problem and the determination of irrational involutions. The periods of the v's are linear combinations of $2q<2p$ of their number and those relatively to a suitably chosen set of $2q$ cycles give rise to a submatrix $\boldsymbol{\Omega}$ of $\boldsymbol{\omega}$ isomorph of the R. M. of C^q.

It must be kept in mind that the reduction in the general sense does not imply reduction by means of a transformation (6). In other words when $\boldsymbol{\omega}$ is impure it does not follow that its submatrix $\boldsymbol{\Omega}$ is the actual isomorph of the R. M. of a C^q, $q<p$ or, if we please, it does not follow that C^p carries an irrational involution of genus $<p$. This was first pointed out by Wirtinger, also mentioned by Poincaré and Severi. See (45), p. 843.

It is in this connection that Poincaré and Picard proved for the first time the fundamental theorem on impure matrices, according to which, if $\boldsymbol{\omega}$ is reducible to

$$\left\| \begin{matrix} \boldsymbol{\omega}_1 & 0 \\ \boldsymbol{\omega}_3 & \boldsymbol{\omega}_2 \end{matrix} \right\|$$

where $\boldsymbol{\omega}_1$ is an R. M., then it is also reducible to the form

$$\left\| \begin{matrix} \boldsymbol{\omega}_1 & 0 \\ 0 & \boldsymbol{\omega}_2 \end{matrix} \right\|,$$

where $\boldsymbol{\omega}_2$ is another R. M., the first complete proof of which was given by Scorza(70), p. 298, where further references are found.

For additional references on reducible abelian integrals see Krazer-Wirtinger(45), p. 838; on combinations of systems of reducible integrals also Severi(74), Chap. 10.

§ 5. IRRATIONAL SERIES

14. Their study is closely related to that of correspondences between two curves. The series, assumed on C^p, is usually denoted by γ_n^1, where n is the number of points in each group (all variable unless otherwise stated). The *index* ν is the number of groups of γ_n^1 which contain a given point, the *genus* q that of the algebraic curve C^q image of γ_n^1. The curves C^p, C^q are in (n, ν) correspondence. Here, however, C^q is a variable curve and the object is to investigate properties common to all such correspondences for C^p fixed. The necessary and sufficient transcendental condition for the existence of a γ_n^1 with C^q for image is that the R. M.'s of C^p and C^q have a common submatrix.

R. Torelli has investigated these series(86), (84). He defines a class of series: it consists of a maximal set whose groups can be paired off so as to give constant abelian sums or differences. Any two such groups of associated series of the same class either have a sum which is a group of a fixed linear series, or else are coresidual relatively to such a series. Torelli considers the effect produced on the Jacobi variety V_p when the curve possesses series of genus $<p$, in particular as to the congruence of subvarieties that V_p may thereby acquire. The existence of such a series when not contained in a $g_n{}^r$ implies the existence on V_p of abelian varieties of genus $<p$, hence the R. M. is impure. Hence if it is pure, in particular if it has general moduli, there can be no $\gamma_n{}^1$ of genus $<p$ on C^p. This is also contained in Severi's theorem(76) according to which on a C^p with general moduli every correspondence has valence. Torelli also investigates the behavior of $\gamma_n{}^1$ regarding linear equivalence of its groups, examines classes containing an involution or a series composed with an involution and conditions under which an involution is cyclic (generated by a cyclic birational transformation). See also Godeaux(26). There are a number of investigations referring to the comparison of $\gamma_n{}^1$ with a $g_m{}^r$. Schubert has proved the relation

$$Z=m\nu\binom{n-1}{k}-\binom{n-2}{k-1}d-\mu\binom{n}{k+1},$$

where $2d$ is the number of double points of $\gamma_n{}^1$, $k+1\,(\leqq r)$ the number of conditions imposed upon a group of $g_m{}^r$ in order that it contain a generic group of $\gamma_n{}^1$, and finally Z is the number of groups of the latter which impose k or less conditions on the linear series. Schubert's proof is reproduced by Segre in (72), (71), § 13. For further related developments see also Castelnuovo(11). In the same reference will also be found the proof of the relation

$$z=\nu(n+p-1)-d,$$

where z is the *defect of equivalence* of $\gamma_n{}^1$ or number of its groups partially contained in a non-special g^{n-1}_{n-1+p}. Severi(75) has shown that if the ν groups of $\gamma_n{}^1$ which contains a given point belong to a fixed g_n, then so does $\gamma_n{}^1$ itself. Castelnuovo showed(13) that in order that $\gamma_n{}^1$ belong to a g_n it is necessary and sufficient that $z=0$. For these properties see also Berzolari(7), pp. 350-352, Severi(74), pp. 191-196. Extensions of the Castelnuovo criterium to multiple series have been given by Torelli(89), (90). The transcendental interpretation for that criterium is that the p abelian sums for $\gamma_n{}^1$ must be constant. This Comessatti generalizes in (20), by introducing the character Z_r, $r\leqq p-1$, $Z_0=z$, number of sets of r groups of $\gamma_n{}^1$ contained in a non-special $g^{\mu}{}_{\mu+p}$, $\mu=(r+1)(n-1)$. He shows that in order that precisely $p-r$ integrals of the first kind give rise to constant abelian sums it is necessary and sufficient that $Z_r=0$, $Z_s\neq 0$ for $s<r$. He derives various relations between these integers, but

obtains their explicit expression only for an involution of genus q when $Z_r = n^{r+1}\binom{q}{r+1}$.

Let T be the symmetrical correspondence derived from γ_n^1 by assigning to every point of C^p all those which belong to a group that contains it. By investigating sets of common zeros of intermediary functions attached to the R. M., Castelnuovo[13] showed that the Z's are the coefficients of the characteristic equation of $-T$. In [65] Rosati derives in a new manner Comessatti's result, relations between the Z's and invariants of certain correspondences associated with γ_n^1, also these results: If the representative curve Γ of the series and C itself have general moduli they are birationally equivalent. The Comessatti invariants of the two series determined by a correspondence between two curves on the curves are the same.

Characters analogous to the above but for multiple irrational series have been considered by Allen[2], [3]. He also applies them to the extension of a formula given by Jonquières for linear series.

§ 6. BIRATIONAL TRANSFORMATIONS

15. We shall consider exclusively those of a C into itself. When $p>0$ they give rise to excellent examples of curves with singular correspondences.

Schwarz has shown, [68] (1879), that when the set of transformations is continuous C^p is parametrically representable by meromorphic functions of an integral of the first kind if any exist, hence $p=0$ or 1. Klein showed by means of automorphic functions that the set cannot even be discrete when $p>1$, [39], p. 64; [41]. Of these results Picard has given very simple proofs, readily extended to surfaces, by means of ratios of differentials of the first kind, [59], [58], [57], p. 524 (also previous editions). Proofs of a more geometric character: Hettner[27], completed by M. Noether[52], [53]; Segre[71]; DeFranchis[22]; Chisini[15]; Weierstrass[93], p. 285; Severi[74], p. 143; [74.1], p. 173; Enriques-Chisini[23], p. 309. Important generalizations by Painlevé[54], including proofs of the same theorems also. In [35], [31] (1887), Hurwitz has treated the question of determining all curves with birational transformations. With a given transformation T there is associated a linear transformation T' of the integrals of the first kind. The groups of T and T' are isomorphic for $p>1$. T' determines in fact a multiplication of the R. M. ω of C^p, which is a complex multiplication unless the curve is hyperelliptic and T the birational transformation that permutes the points of each pair of its g_2^1. It follows that C^p, $p>2$, does not in general possess a birational transformation into itself. The multiplication is shown to be a so-called *principal transformation,* leaving invariant the fundamental alternate form of $\boldsymbol{\omega}$, or, if we please, a set of associated theta functions (Ch. 17, No. 50). It follows that T' is cyclic and so is T. A very similar result holds for simply rational transformations of C^p into itself, (**35**), p. 301. That the total number of birational transformations is finite Hurwitz(**36**) proves by considering their effect on the Weierstrass points whose number he shows to be $\geqq 2p+2$ and comparing with the number of fixed points of T whose number is at most $2p+2$ with special considerations for the hyperelliptic case. If n is the order of T, C^p is the birational transform of a curve $f(x^n, y)=0$. In any case for $p>1$, $n\leqq 10(p-1)$ and the total order of the birational group is $\leqq 84(p-1)$, in the hyperelliptic case in fact $\leqq 4p+2$. See also Klein(**40**), vol. 2, p. 554-556; Wiman, who lowers n down to $4p+2$ for all cases, (**94**), first paper; Berzolari(**7**), p. 352; Severi(**74**), p. 145, (**74.1**), p. 175; Wirtinger(**95**). For the related investigations of Torelli, Severi, Comessatti, Rosati, see No. 11.

For birational transformations of C^1 see Fricke(**25**), Segre(**73**), Chisini(**16.1**), Severi(**74.1**), p. 185.

Birational transformations for $p=2$ have been investigated by Bolza(**8**), (**9**), summary of his results in Enriques-Severi(**24**), p. 378. Reducing C^2 to the form $y^2=f(x)$ where f is a polynomial of degree 6. the problem is to determine all such polynomials invariant under a projective group and their groups. He finds six types of essentially distinct curves and determines their groups. See also Kantor(**38**); Wiman(**94**), first paper.

Klein has studied the group of the quartic of genus three that bears his name(**42**); (**40**), vol. 1, pp. 692-731. The curve can be reduced to the form $x^3y+y^3z+z^3x=0$. It is a normal curve for $p=3$, not hyperelliptic. Its total group is the well-known projective G_{168} and a thorough discussion of it is given especially in relation to the subgroups of G_{168} and their effect upon the configuration of the inflections and double tangents both real and complex. Ciani in (**19**), has investigated the projective groups of all plane quartics. Enriques-Chisini(**23**), pp. 312-339, have done likewise for the quartic normal curve for the genus three. By considerations of fixed points they show that the order of any transformation of the group (which is then the complete birational group), can only be 2, 3, 7. See also Kantor(**37**); Wiman(**94**), first paper.

Birational transformations for $p=4$, 5, 6, have been considered by Wiman(**94**), second paper. For $p=4$ also by Miss VanBenschoten(**91**); for $p=5$ by McKelvey(**51**); for $p=6$ by Snyder(**80**). Their method is to investigate the projective transformations of the normal curve. Snyder has also investigated all plane quintics invariant under a projective group and determined their groups(81). In (49) part II, Ch. IV, Lefschetz examines the birational group of

$$y^q=\Pi(x-a_i)^{\alpha_i}; \quad q \text{ odd prime,}$$

when the a's are arbitrary, the type including many found in the literature. The genus $p=\frac{1}{2}r(q-1)$ where $r+2$ is the number of critical points of $y(x)$. He determines a set of integrals of the first kind and the corresponding R. M. By considering the effect of a birational transformation upon the critical points and also upon the R. M., he determines the complete group of the curve. The case $r=1$ demands a special treatment. The groups G are cyclic of order q, unless the curve is hyperelliptic when G is of order $2q$, or reducible to the curve

$$x^my^n+y^mz^n+z^mx^n=0, \qquad m^2-mn+n^2=q,$$

when G is of order $3q$, except when $q=7$ where the curve is Klein's quartic already considered. Several curves of these last types have been discussed previously, notably for $q=13$, $m=4$, $n=7$, $p=6$ by Snyder(81). He also considers another curve of similar equation, with $m=2$, $n=3$, $q=7$, but its genus $p=5$ and as $2p=10$ is not divisible by $q-1=6$ it does not belong to the type considered by Lefschetz, so that although its group is of order $3q$ as shown by Snyder, there is no contradiction. The curves just considered are noteworthy also in that they give simple examples, for as large p as we please, of curves with pure R. M. possessing singular correspondences. Lefschetz determines the invariants h, k for all cases, hence the base numbers for the various correspondences, symmetrical or otherwise.

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CHAPTER XVII

HYPERELLIPTIC SURFACES AND ABELIAN VARIETIES

By abelian varieties are understood varieties that admit of a parametric representation by means of abelian functions. They are highly interesting and important both intrinsically and because they are found in the background of many questions of algebraic geometry. It is our purpose here to give an account of their theory. The chief topics to be treated are the geometry on abelian varieties and the questions that center around complex multiplication.

Abelian functions proper have been fully covered in the Encyklopädie article of Krazer-Wirtinger(48), and in practically all non-geometric questions we shall merely refer to them. This will notably be the case in § 1 where we give a rapid résumé of the foundations. As the contributions of the earlier mathematicians, Hermite, Weierstrass, Riemann and others, have largely been along that line it will cause little surprise to find scarcely any mention of them in the chapter and none in the bibliography.

The notations and terminology are those of the Glossary at the end of Chapter 15.

§ 1. MULTIPLY PERIODIC AND RELATED FUNCTIONS

1. *Theta-Functions.*—Consider the matrix

$$\mathbf{N}\ \|\delta_{jk};\ \tau_{jk}\|$$

$$(\tau_{jk}=\tau_{kj};\ \delta_{jj}=1,\ \delta_{jk}=0 \text{ when } k\neq j;\ j,k=1,2,\ldots,p).$$

Let $\tau_{jk}=\tau'_{jk}+i\tau''_{jk}$ and assume that for arbitrary real values of the x's:

$$\Sigma\tau''_{jk}x_jx_k>0. \tag{1}$$

This characterizes $\mathbf{N}$ as an R. M. in the normal form with the principal matrix $\left\|\begin{matrix}0 & 1\\ -1 & 0\end{matrix}\right\|$, (Ch. 15, Nos. 6, 7).

We shall repeatedly consider functions of p variables $u_1, \ldots, u_p$ and shall designate them by the abridged notation $f(u)$, $\phi(u)$, etc.

Let us introduce the pair of forms

$$L(m;\ u)=\Sigma m_ju_j;\ Q(\tau;\ m)=\Sigma\tau_{jk}m_jm_k.$$

By means of them we define the θ function of order one as:

$$\theta(u)=\Sigma e^{\pi i(Q+2L)}$$

where the summation is over all integral values of the m's. From (1) it follows that the series is convergent for all u's and therefore $\theta(u)$ is an entire function. Let us write $\theta(u+\delta_k)$, $\theta(u+\tau_k)$ for the values of the function at $u_1+\delta_{1k}, \ldots, u_p+\delta_{pk}$ or $u_1+\tau_{1k}, \ldots, u_p+\tau_{pk}$. Then θ satisfies the periodicity relations

$$\theta(u+\delta_k)=\theta(u)\,,$$
$$\theta(u+\tau_k)=e^{-\pi i(\tau_{kk}+2u_k)}\theta(u).$$

For this reason the elements of $\mathbf{N}$ are called the periods of θ, and $\mathbf{N}$ its *period matrix,* or R. M.

2. In conjunction with a period matrix $\boldsymbol{\omega}$, whether of the form $\mathbf{N}$ or not it is convenient to consider the fundamental parallelepiped U_{2p} or set of points u given by

$$u_j=\Sigma t_\mu \omega_{j\mu}, \qquad 0\leqq t_\mu<1.$$

In all cases to be considered $\boldsymbol{\omega}$ will be an R. M., hence $\left|\frac{\omega}{\omega}\right|\neq 0$. Thus for $\mathbf{N}$, the determinant is $|\tau''|\neq 0$ owing to (1). There is then a unique set of real values of the t's corresponding to any set of u's. To increase the u's by the periods $\omega_{j\mu}$ is equivalent to replacing t_μ by $t_\mu+1$. Hence modd. the periods there is a unique point of U_{2p} corresponding to every point u.

Let now $\boldsymbol{\omega}$ be $\mathbf{N}$ itself and let $(c_1, \ldots, c_p)$ be an arbitrary fixed point of U_{2p} with $h_1, \ldots, h_p, g_1, \ldots, g_p$, as the corresponding t's. We introduce a more general type of function defined by

$$\theta\begin{bmatrix} g \\ h \end{bmatrix}(u)=\Sigma e^{\pi i[Q(\tau;\, m+g)+2L(m+g;\, u+h)]}$$

and called function θ of characteristic $\begin{bmatrix} g \\ h \end{bmatrix}$. We verify immediately that

$$\theta\begin{bmatrix} g \\ h \end{bmatrix}(u)=e^{\pi i[Q(\tau;\, g)+2L(g;\, u+h)]}\cdot\theta(u+c).$$

Hence the periodicity relations are

$$\theta\begin{bmatrix} g \\ h \end{bmatrix}(u+\delta_k)=e^{2\pi i g_k}\cdot\theta\begin{bmatrix} g \\ h \end{bmatrix}(u),$$
$$\theta\begin{bmatrix} g \\ h \end{bmatrix}(u+\tau_k)=e^{-\pi i(\tau_{kk}+2u_k+h_k)}\cdot\theta\begin{bmatrix} g \\ h \end{bmatrix}(u).$$

Of particular interest is the case where $\mathbf{N}$ is the R. M. of a set of normal integrals of the first kind of a C^p with respect to its retrosections. When the u's are replaced by the integrals with a common upper limit of integration (x, y), θ becomes a function of (x, y) on C^p, and its investigation opens up various interesting questions and in particular leads to the solution of the Jacobi inversion problem. See Krazer-Wirtinger(48), Krazer(47), Ch. 9.

3. By means of the θ function of No. 2 we can describe the most general function of order n whose periods are

$$(\mathbf{N}') \quad \left\| \frac{\delta_{jk}}{e_k}\, ; \; \tau_{jk} \right\|$$

where the e's are positive integers with $e_1 = 1$, e_k a factor of e_{k+1} and the δ's and τ's as before. We observe at first that the functions built with the τ's all multiplied by ν:

$$f_\rho(u) = \theta \begin{bmatrix} \dfrac{g+\rho}{q} \\ h \end{bmatrix} (\nu u)$$

($\nu = ne_p$ and n an integer; $q_j = \dfrac{\nu}{e_j}$; $0 \leqq \rho_j < q_j$ and ρ_j an integer), have a periodicity behavior as regards $\mathbf{N}'$, independent of the integers ρ; (the subscript ρ in f_ρ stands for the complex $\rho_1\rho_2 \ldots \rho_p$). Hence

$$\Theta_n \begin{bmatrix} g \\ h \end{bmatrix} (u) = \Sigma C_\rho f_\rho(u)$$

has a behavior independent of the constants c and described by the equations

$$\Theta\left(u + \frac{\delta_k}{e_k}\right) = e^{-2\pi i g_k} \Theta(u)$$

$$\Theta(u + \tau_k) = e^{\pi i (\nu \tau_{kk} + 2\nu u_k + h_k)} \Theta(u).$$

For this reason the function is called: Θ *of order* n and characteristic $\begin{bmatrix} g \\ h \end{bmatrix}$ attached to $\mathbf{N}'$. By direct comparison of coefficients it is found that for arbitrary c's the function has no other periods than sums of multiples of the elements of $\mathbf{N}'$, Wirtinger(123), p. 97. Owing to this $\mathbf{N}'$ is said to represent a *fundamental set* or F. S. of periods.

The necessary and sufficient condition for the existence of our functions is again that (1) hold and $\mathbf{N}'$ is also an R. M. with an attached principal matrix $\mathbf{c}$, such that $c_{k,\,p+k} = -c_{p+k,\,k} = e_k$, the terms of different type being all zero.

The functions f corresponding to the different systems ρ are linearly independent and their linear combinations are the most general entire functions behaving as they do. They constitute a linear system whose dimension is

$$q_1 \ldots q_p - 1 = \frac{(ne_p)^p}{e_1 \ldots e_p}.$$

When all the e's are unity $\mathbf{N}'$ reduces to $\mathbf{N}$ and its θ's of order n become θ's of order n attached to $\mathbf{N}$.

Remarks.—I. We have denoted the theta functions belonging to $\mathbf{N}$ or $\mathbf{N}'$ by the respective symbols θ, Θ. In the sequel they shall all be denoted by the same letter θ and the particular type specified when necessary by a statement concerning the R. M. When nothing is said in regard to the latter it is assumed to be of the more general type $\mathbf{N}'$.

II. While $\mathbf{N}$ has the same aspect as the R. M. of a set of normal integrals of the first kind of a C^p relatively to the retrosections, the latter R. M. is very special. Indeed when $p>1$ it depends upon $3p-3$ independent parameters, while the first contains $\frac{p(p+1)}{2}$ arbitrary constants, and $\frac{p(p+1)}{2}-(3p-3)>0$ for $p>3$. Hence when $p\geqq 4$ the C^p matrix is not the most general.

For $p=4$ there is then a single relation between the normal periods τ_{jk} corresponding to a C^4. This relation has been explicitly determined by Schottky and Poincaré, references and further details Krazer(47), p. 418, Krazer-Wirtinger(48), p. 786. We are as yet completely in the dark as to the similar relations for $p>4$.

The most complete treatment of theta functions is by Krazer(47).

4. *Intermediary Functions.*—An intermediary function $\phi(u)$ (Jacobi function according to some authors) with an F. S. of periods

$$\boldsymbol{\omega}=||\omega_{j\mu}||, \qquad (j=1, 2, \ldots, p; \mu=1, 2, \ldots, 2p)$$

is an entire function such that

$$\phi(u+\omega_\mu)=e^{-2\pi i(\Sigma\epsilon_{j\mu}u_j+\eta_\mu)}\cdot\phi(u),$$

where $\epsilon_{j\mu}$, η_μ, are constants. We impose the condition that the points

$$u_j=\Sigma t_\mu\omega_{j\mu}, \qquad 0\leqq t_\mu<1$$

fill up an actual domain U_{2p} of S_{2p}, which is finite. This amounts to exacting that $\begin{vmatrix}\omega\\ \bar{\omega}\end{vmatrix}\neq 0$. All θ's behave thus and are therefore intermediary functions.

Since adding ω_μ then ω_ν or vice versa must give the same result we have a set of relations

$$\Sigma(\epsilon_{j\mu}\omega_{j\nu}-\epsilon_{j\nu}\omega_{j\mu})=m_{\mu\nu},$$

the second member being an integer, with $\mathbf{m}$ an alternate matrix. The H. C. F. n of these integers is the *order* of ϕ. For a θ it reduces indeed to the order as defined in No. 3.

Frobenius has shown(28) that $|m|>0$. Hence $\mathbf{m}^{-1}=\mathbf{M}$ is uniquely defined. He shows then by direct analysis that $\boldsymbol{\omega}$ is an R. M. with $\mathbf{M}$ as an attached principal matrix.

If we apply to the u's a linear transformation

$$(2) \qquad u'_j=\Sigma\lambda_{jk}u_k, \qquad |\lambda|\neq 0,$$

and at the same time adopt a new F. S. of periods, $\boldsymbol{\omega}$ is replaced by an equivalent matrix $\boldsymbol{\lambda\omega}\mathbf{a}$, $|a|=\pm 1$, $\boldsymbol{\epsilon}$ by $\boldsymbol{\lambda}'^{-1}\boldsymbol{\epsilon}\mathbf{a}$, $\mathbf{m}$ by $\mathbf{a}'\mathbf{ma}$ and n is unchanged. Let us choose $\mathbf{a}$ such as to reduce $\mathbf{m}$ to its canonical form $\mathbf{N}'$. Then it is found that $||\epsilon_{j\mu}||$, $(j, \mu=1, 2, \ldots, p)$ is symmetrical. Hence upon multiplying ϕ by a suitable function $e^{Q(u)}$, where Q is a quadratic polynomial, the similar matrix for the new functions is zero, and there is a transformation of variables (2) reducing $\boldsymbol{\omega}$ to $\mathbf{N}'$ with $\phi e^Q=\psi$ behaving like a Θ of order n.

Conversely any R. M. $\boldsymbol{\omega}$ is equivalent to a certain matrix $\boldsymbol{\lambda\omega}\mathbf{a}$, $|a|\equiv\pm 1$, of type $\mathbf{N}'$. An arbitrary $\Theta(u')$ attached to the latter when expressed in terms of the u variables gives rise to an intermediary function $\phi(u)$ with $\boldsymbol{\omega}$ corresponding to an F. S. of periods.

Thus the necessary and sufficient condition for the existence of intermediary functions attached to $\boldsymbol{\omega}$ is that it be an R. M. If then $\mathbf{M}$ is any principal matrix of $\boldsymbol{\omega}$ there is a whole class of intermediary functions attached to $\boldsymbol{\omega}$ for which the matrix $\mathbf{m}=q\mathbf{M}^{-1}$, q a rational number. The whole class is reduced to a class of Θ's by a linear transformation of the u's that reduces any particular one of them.

By reference to the dimension of the associated Θ's we find that the intermediary function ϕ belongs to a linear system of functions with the same behavior as to $\boldsymbol{\omega}$ whose dimension is $\sqrt{|m|}-1$. There are important enumeration formulas regarding the number of common solutions of sets of intermediary functions, due to Poincaré, Wirtinger, Lefschetz. See Krazer-Wirtinger(48), p. 837, also No. 34 of this chapter.

5. *Multiply Periodic Functions.*—By a $2p$-ply periodic function $f(u)$ with $\boldsymbol{\omega}$ for matrix of an F. S. of periods is meant a meromorphic function such that $f(u+\omega_\mu)=f(u)$. The existence of a single function $f(u)$ not a function of a smaller number of linear combinations of the u's, implies the existence of $p-1$ more of type $f(u+c)$, functionally independent of f and any others are algebraic functions of these p, rational functions of $p+1$ suitably chosen, Krazer-Wirtinger(48), p. 824. In fact we may choose as the p additional functions the first partials f'_{u_j}; Wirtinger(128), p. 90. On this basis it can be proved that $\boldsymbol{\omega}$ is again an R. M., a theorem due to Riemann and Weierstrass.

6. Conversely to any R. M. $\boldsymbol{\omega}$ correspond $2p$-ply periodic functions for which $\boldsymbol{\omega}$ represents an F. S. of periods. For let $\phi(u)$ be an arbitrary

intermediary function attached to $\boldsymbol{\omega}$, and W its locus of zeros in the u space. In the vicinity of any point u^0 of W, $\phi \equiv \phi_1 \cdot \phi_2$ the two factors being holomorphic at u^0, with $\phi_2(u^0) \neq 0$, so that $\phi_1 = 0$ represents W about u^0. Then ϕ_1 cannot be a function of $q < p$ linear combinations of the u's for every point of W. For a linear transformation (2) will not alter the situation, hence we may assume that ϕ_1 is a function of the variables $u_1, \ldots, u_q$ alone. Then the functions

$$\frac{\partial^2 \log \phi}{\partial u_h \partial u_{q+k}},$$

being periodic and finite, are necessarily constant. As a consequence, by expanding log ϕ in power series about a point not on R we find

$$\log \phi = Q(u_1, \ldots, u_p) + f(u_1, \ldots, u_q),$$

where Q is a quadratic polynomial. Hence

$$\phi = e^Q \cdot \Phi(u_1, \ldots, u_q).$$

Hence ϕ has infinitesimal periods and $\boldsymbol{\omega}$ does not correspond to an F. S. of periods for the function. Therefore ϕ_1 cannot behave as assumed.

It follows that for arbitrary constants c_{jk}

$$f(u) = \Sigma c_{jk} \frac{\partial^2 \log \phi}{\partial u_j \partial u_k}$$

is a periodic function of $u_1, \ldots, u_p$ and of no smaller number of variables. In fact, in general $\boldsymbol{\omega}$ defines an F. S. of periods for $f(u)$, Wirtinger([123]), p. 96. This shows then the existence of periodic functions, of the desired type attached to $\boldsymbol{\omega}$. Furthermore, every other such periodic function is a rational function of $f(u)$, f'_{u_j} and among these $p+1$ there is a set of p linearly independent. These functions are all obtained by a linear transformation of variables from ratios of $p+2$ suitably selected θ's of the same order and characteristic. It follows that every $2p$-ply periodic meromorphic function of p variables is obtained by a linear transformation of the variables from a ratio of θ functions of the same order and characteristic.

7. In our presentation we have followed the same lines as most authors. Appell in ([1]) proceeded in a distinctly different manner and while the details are only given for $p=1$, 2, it is very likely that his method can be extended to any p. Starting directly with a multiply periodic function $f(u)$ attached to $\boldsymbol{\omega}$ he shows that it may be written as the quotient of two intermediary functions with the same periods. In connection with the work of Frobenius([28]), Appell's method contains all the elements of a very elegant exposition of the subject.

Additional references of a general character on the topics of this section are Krazer-Wirtinger(48); Poincaré(70); Wirtinger(120) (the last two are summaries of the work of the authors on abelian functions); Osgood(58); Baker(1.1), Chs. 7, 8; Scorza(93.1), (102); Castelnuovo(10); Lefschetz(51), Part II, Ch. 1, (55), (50), Ch. 6; Jung(45.1).

8. *Definition of Abelian Varieties.*—Let $\phi_0, \phi_1, \ldots, \phi_n$ be linearly independent intermediary functions with the R. M. $\boldsymbol{\omega}$ and behaving alike relatively to it, i. e., with the same constants $\epsilon_{j\mu}$, η_μ, of No. 4. Then the matrix $\mathbf{m}$ is the same for all and they belong to the same linear system of intermediary functions with like behavior, whose dimension is $\sqrt{|m|}-1$. The total system can be simultaneously reduced by a linear change of variables to a set of θ's of the same order and characteristic.

Let $n>p+1$. Since the ratios $\dfrac{\phi_i}{\phi_0}$ are periodic functions, from the theorems recalled in Nos. 5, 6, it follows that any $p+2$ of the ϕ's are related by a homogeneous algebraic equation. Hence the locus of the system of points of S_n given by

$$\rho x_i = \phi_i(u) \tag{3}$$

is an algebraic V_q of S_n, called *abelian variety* attached to $\boldsymbol{\omega}$. There are two possibilities: (a) $q<p$. Then the functional matrix of the ϕ's is of rank $q<p$, and V_q is actually an abelian variety attached to a submatrix of genus q of $\boldsymbol{\omega}$, which is then impure. We shall say that V_q is a *degenerate* abelian variety attached to $\boldsymbol{\omega}$. (b) $q=p$. The functional matrix is now of rank p. To each u point corresponds a single point of V_p and to an arbitrary point of V_p correspond a finite number r of points of U_{2p}, or points u distinct modd. the periods. r is called the *rank* of the variety.

It has been proved explicitly by Lefschetz(51), p. 367, that (3) may be so chosen that V_p is without singularities, and in (1, 1) *non-exceptional* correspondence with U_{2p}. In particular its rank $r=1$. In place of (3) he takes θ functions and bases the whole argument on the fact that sufficiently general θ's have $\mathbf{N}'$ as F. S. of periods. The V_p thus obtained is not unique, but any two are birational transforms of one another. The birational transformation between them is here in fact a homeomorphism. In particular there are no fundamental subvarieties on the two V_p's which are transformed into one another. Whenever we shall speak of the abelian variety attached to $\boldsymbol{\omega}$ we shall mean unless otherwise indicated, any variety of the class just defined. The varieties of various ranks attached to $\boldsymbol{\omega}$ are images of involutions on the V_p of rank one.

For $p=1$ the variety is merely an elliptic curve. To investigate it from the present point of view amounts to studying its properties in relation to its parametric representation by elliptic functions. There is nothing to be added along that line to Fricke(26), Fueter(29). The case $p=2$ leads to hyperelliptic surfaces, the name being due to their close connection with curves of genus two. These surfaces have been of paramount importance in algebraic geometry and we shall therefore consider them at length in the following section.

§ 2. HYPERELLIPTIC SURFACES

9. *Generalities.*—A hyperelliptic surface F on an S_3, is representable as

$$x_i=\theta^i(u, v), \qquad (i=0, 1, 2, 3) \tag{4}$$

where the θ's are of the same order and characteristic and belong to a matrix $\mathbf{N}'$ of genus $p=2$ and might also be replaced by intermediary functions.

The Kummer surface is the first hyperelliptic surface to appear in the literature and it was Klein who first pointed out that it could be parametrically represented by means of θ functions in two variables. From that point of view it has been studied by numerous writers to whose papers references will be found in Krazer-Wirtinger(48), p. 747. Hudson(31.1) has given a very thorough and complete exposition of the Kummer surface. Another closely related special hyperelliptic surface is the Weddle surface, Krazer-Wirtinger(48), p. 751. As these two special surfaces are adequately covered in (48) we shall consider them only incidentally.

The Kummer and Weddle surfaces are both of rank *two,* for they admit parametric representations (4) by thetas that are *even* functions of u, v for the first, odd functions for the second. Thus the surface points (u, v), $(-u, -v)$ coincide. They are distinctly special hyperelliptic surfaces.

A surface of rank r represents an involution I_r on a surface of rank one belonging to the same period matrix. More generally surfaces of ranks r, s, belonging to the same matrix are in (r, s) correspondence. Hence the study of surfaces of the various ranks and of the involutions on the surface of rank one are equivalent problems and we do not separate them in the sequel.

10. The study of hyperelliptic surfaces of rank one was initiated by Picard in connection with his investigations on integrals of total differentials (Picard integrals). In (62), p. 312, (69), Ch. 14, (64), (66), (68), he gives necessary and sufficient conditions in order that $F(xyz)=0$ shall represent a hyperelliptic surface of rank one. Every Picard integral of the first kind is reducible to the form $au+bv$, hence the irregularity

$q=2$. The surface must then possess two functionally independent Picard integrals of the first kind. Every double integral of the first kind is reducible to $a\int\int du\,dv$, hence the geometric genus $p_g=1$. F must then have a single adjoint polynomial of order $m-4$ (m order of F) and Picard proves that, outside the multiple curve, the surface defined by that polynomial cuts F in a set of rational curves. These are sufficient conditions in order that F be hyperelliptic. Enriques has shown (20) that these conditions are equivalent to the following conditions on the genera: $p_a=-1$, $p_g=P_4=1$. See also Enriques-Severi (22), p. 303. Observe in passing that the rational curves arise from the fact that the θ's of the representation (4) vanish simultaneously at some isolated points whose neighborhoods the rational curves represent. The curves are exceptional curves of the surface. The representation may be so chosen that none will be present, Enriques-Severi (22), p. 321.

In (63), p. 223, Picard proves that when a surface is invariant under a continuous birational group in $s>1$ parameters, necessarily $s=2$, the group is abelian and F hyperelliptic of rank one. See also Painlevé (59), p. 255; Castelnuovo-Enriques (11), p. 746; Severi (104), p. 778, (106.1).

11. The systematic study of hyperelliptic surfaces with a matrix **N** was undertaken by Humbert (32) to (44), to whom we owe much of what is known about the surfaces, and also a secure foundation for later progress in abelian functions.

In conformity with Humbert and most subsequent writers we take **N** as

$$\left\| \begin{matrix} 1 & 0 & g & h \\ 0 & 1 & h & g \end{matrix} \right\|, \qquad g_1g'_1-h_1^2>0,$$

where g_1, g'_1, h_1, are the complex parts of g, g', h. It is also a period matrix for a set of normal integrals of the first kind of a C^2, relatively to four retrosections. The related surface of rank one, F, is a birational transform of the surface image of the point pairs of C^2. To every pair of points of C^2 corresponds a unique point of F, and conversely with the exception that to a certain definite point of F correspond all the groups of the canonical g_2^1 on C^2. F is called the *Jacobi surface of* C^2. The quantities g, h, g' are the *moduli* of C^2 or F. There is a more general type of surface corresponding to a matrix **N′** which we shall take here as

$$\left\| \begin{matrix} 1 & 0 & g & h \\ 0 & \frac{1}{e} & h & g' \end{matrix} \right\|, \quad e>1; \qquad g_1g'_1-h_1^2>0.$$

It is called a *Picard surface.* Humbert in all his investigations considers exclusively Jacobi surfaces. Actually, however, the two types may merge:

His *singular* surfaces, or surfaces whose **N** has a singularity index $k>1$ may well also correspond to matrices **N**′ for a different F. S. of periods.

12. Humbert's first paper, (39) (1893), is mainly taken up with the properties of general Jacobi and Kummer surfaces. A résumé of his results is given by Hudson(31.1). At the outset Humbert proves, however, by means of a theorem due to Appell(1), this basic proposition: On any hyperelliptic F the parametric equation of every algebraic curve is of the form $\phi(u, v)=0$, where ϕ is an intermediary function. Conversely every such equation represents an algebraic curve on F. In his proof he restricts himself to general (i. e., non-singular) R. M.'s, but this restriction is superfluous and he was well aware of it, as his later work shows.

In (39), Part II, Humbert makes an extensive investigation of the Kummer surface based on the parametric representation, due to Weber(116), which is (4) when the functions at the right are four linearly independent even θ's (the maximum number) of order two and characteristic zero. Humbert classifies the curves on the surface. They are all of even order, and with 32 families for each value of the order. He examines the mode of obtaining them by means of intersecting surfaces, the properties of these surfaces, the singular points of the curves, their genera. He shows that any two curves of the same family constitute the complete intersection of the Kummer surface with a surface of S_3. These investigations are the most important on the distribution of families of curves on a surface before Severi's work on the base (Ch. 15, No. 20).

In Part III of (39) Humbert considers the Jacobi surface F, proves the Picard necessary and sufficient conditions (see No. 10) in order that a surface of S_3 be a Jacobi surface and extends them to the Kummer surface. He then studies the curves of F. The θ's of order m and given characteristic cut out a linear system whose dimension is m^2-1, degree $2m^2$, genus m^2+1, and which varies within a continuous system ∞^2 when the characteristic varies. This anticipates the theorem of Castelnuovo, Enriques, Severi, Picard, Ch. 15, No. 9. Humbert also considers curves with singular points and their intersections, sections of F by various surfaces, notably its adjoints, examines the special F of the minimum degree 8 and its curves. This surface, however, is not in (1, 1) unexceptional correspondence with the points (u, v), (modd. the periods) or points of U_4 in the sense of No. 2, for the surface carries exceptional curves. Enriques-Severi(22), p. 321, have shown that the Jacobi surface of least degree without exceptional curves is an F_{18} in S_9. See also Krazer-Wirtinger(48), p. 871.

In (39) Part IV Humbert proves that every surface of rank two belonging to **N** is a birational transform of the Kummer surface. He examines the possible orders, observes that quartic surfaces other than Kummer's surface are among them, for example Weddle's surface, whose relation to Kummer's surface he also investigates.

Humbert's results have been extended to Picard surfaces by Traynard(112).

13. In the short notes (32) to (38) and in an extended paper (41), (42), (43), Humbert investigates from all sides the singular matrix **N**, its abelian functions and Jacobi surfaces. **N** will possess, besides its principal matrix $\begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}$, another rational alternate matrix. The relation between the moduli takes the explicit form

$$(5) \qquad Ag+Bh+Cg'+D(h^2-gg')+E=0,$$

where the coefficients are integers. Humbert says that the functions or surfaces are l-ply *singular* when there are l linearly independent relations (5). If k is the singularity index, $l=k-1$, hence $l \leqq 3$. Humbert considers particularly the case where $k=2$ and (5) is the only relation between the moduli, so that the multiplication index $h=2$ also.

Among the transformations of **N** into an isomorphic matrix of the same form, some do not, some do, imply relations between the moduli. The first are called *ordinary*, the second *singular* transformations. The latter are to be found only when $k>1$, i. e., when (5) is not trivial. Under ordinary equivalence transformations (5) is changed into a similar relation between the new periods but the number $\Delta=B^2-4AC-4DE$, called the ***invariant*** of (5), is unchanged. Δ is always positive and $\equiv 0$ or 1 mod. 4. If we think of (5) as a relation $\mathbf{NcN}'=0$, $\dfrac{\Delta}{4}$ or $\dfrac{\Delta-1}{4}$ is the pfaffian of **c**, Bagnera-deFranchis(2), which explains its invariance. See Scorza(99), first paper; also No. 35 of this chapter. **N** is pure or impure accordingly as Δ is not or is a perfect square. When **N** is impure F carries elliptic curves and conversely. Hence the necessary and sufficient condition for the presence of elliptic curves on F is that Δ be a perfect square.

By applying an ordinary equivalence to **N**, it may be reduced to a similar matrix with the corresponding relation (5) in the simplified form,

$$(6) \qquad g+\beta h+\gamma g'=0, \qquad \Delta=\beta^2-4\gamma.$$

Furthermore we may assume g_1 and $g'_1>0$. Then given any two integers λ, κ such that

$$(7) \qquad \delta=\lambda^2+\beta\kappa\lambda+\gamma\kappa^2>0, \qquad \lambda>0,$$

there are associated intermediary functions $\phi(u, v)$ not reducible to θ's by an ordinary transformation and behaving in accordance with the relations, Humbert(41), p. 262:

$$\begin{aligned}\phi(u+1, v) &= \phi(u, v+1) = \phi(u, v),\\ \phi(u+g, v+h) &= e^{2\pi i(\lambda u-\kappa\gamma v+\nu)}\cdot\phi(u),\\ \phi(u+h, v+g') &= e^{2\pi i[\kappa u+(\lambda+\kappa\beta)v+\nu']}\cdot\phi(u).\end{aligned}$$

The determinant of the matrix $\mathbf{m}$ of No. 4 attached to ϕ is here equal to δ^2. Therefore if $C_{\kappa\lambda}$ denotes the algebraic curve $\phi=0$, the dimension of $|C_{\kappa\lambda}|$ is $\delta-1$, its degree 2δ and its genus $\delta+1$. More generally for the intersections of curves of two distinct systems, Humbert(41), p. 270,

$$[C_{\kappa\lambda}C_{\kappa'\lambda'}] = 2\lambda\lambda' + \beta(\kappa\lambda'+\lambda\kappa') + 2\gamma\kappa\kappa'.$$

In (41) Humbert also investigates *singular* Kummer surfaces, i. e., attached to a singular $\mathbf{N}$, in particular he determines the properties of their curves. See also Hudson(31.1), Ch. 18.

14. In (41), Part 4, Humbert deals with modular equations related to singular hyperelliptic surfaces. If we set

$$z^2 = f(x) = x(1-x)(1-\xi x)(1-\eta x)(1-\zeta x),$$

the coordinates of a point on the Kummer surface can be expressed rationally in terms of x, y, $\sqrt{f(x)}$, $\sqrt{f(y)}$. If the surface carries algebraic curves that do not exist when ξ, η, ζ, are arbitrary there exist algebraic relations between these parameters, which are the algebraic moduli of the Kummer surface. When $k=2$ and there is only one relation between g, h, g' there is only one relation $F(\xi, \eta, \zeta)=0$, that is the point (ξ, η, ζ) is on a certain algebraic surface. This surface is invariant for all ordinary transformations of g, h, g', or change in moduli corresponding to a change of retrosections on the curve $z^2=f(x)$, or if we take the moduli such as to correspond to a singular relation (6) for a narrower set of transformations, constituting a group G. Thus certain algebraic functions of ξ, η, ζ, will be automorphic functions of, say g, h, invariant under the transformations of a certain modular group. This aspect of the question is not considered by Humbert in the paper in question, nor is it as yet of importance for our present topic. What is of interest, however, is that he actually forms the equation $F=0$ for some special values of Δ, and gives a theoretical (very geometric) method for obtaining it in the general case, in terms of certain families of curves related to the hexagon $y+2ax+a^2=0$, $f(a)=0$.

For related investigations but with little geometrical content see Krazer-Wirtinger(48), p. 860, Hecke(30).

15. In (42) Humbert considers all questions that center around transformations of multiply periodic functions. Let F be the same Jacobi

surface as before, F' another with parameters U, V. In dealing with these questions and more in conformity with our habitual procedure, we do not restrict the R. M.'s to the type $\mathbf{N}$ and denote them by $\boldsymbol{\omega}$, $\boldsymbol{\Omega}$.

Geometrically the question is to determine whether F' is the image of an involution on F, or a rational transform of F. The method used by Hurwitz for correspondences on curves (Ch. 16, No. 3) gives for the equations of the transformation

$$(8) \qquad U = \alpha_{11}u + \alpha_{12}v + \beta_1; \qquad V = \alpha_{21}u + \alpha_{22}v + \beta_2$$

and the matrix equation $\boldsymbol{\alpha\omega} = \boldsymbol{\Omega}\mathbf{a}$ where $\mathbf{a}$ is a matrix of order four whose terms are integers. Humbert obtains this relation by the analytical condition that the periodic functions attached to $\boldsymbol{\Omega}$ be rational in those attached to $\boldsymbol{\omega}$. If the matrix relation imposes no restrictions on $\boldsymbol{\omega}$, nor on the moduli g, h, g', when it is reduced to the form $\mathbf{N}$, the transformation is *ordinary*, otherwise it is *singular*.

The problem might be slightly generalized by considering merely algebraic correspondences instead of involutions or algebraic transformations instead of rational transformations. Then the resulting matrix equation would be the same. See in this connection § 4 of this chapter.

Humbert shows that singular transformations exist when and only when $\boldsymbol{\omega}$ is singular. The order δ of the involution I_δ on F is then equal to $|a|$, (42), p. 291, a number always $\geqq 0$. To each solution λ, κ, of (7) corresponds a definite transformation with associated I_δ singular when $\kappa \neq 0$. The transformation is only defined up to the product by an ordinary transformation. Humbert gives the explicit coefficients α in (8), (42), p. 305. The effect on the θ's of order m is to change them into intermediary functions whose indices are $m\lambda$, $m\kappa$.

Of particular interest are the transformations whose $\delta = 1$, or birational transformations of the Jacobi surface, Humbert(42), p. 313. Their equations may be given the simple form

$$(9) \qquad U = \lambda u - \gamma\kappa v; \qquad V = \kappa u + (\lambda + \beta\kappa)v.$$

The ordinary transformations correspond to $\kappa = 0$. For (7) we can also substitute the Pellian equation

$$(10) \qquad \xi^2 - \Delta\kappa^2 = 4; \qquad \xi = 2\lambda + \beta\kappa;$$

where due to the nature of Δ, if ξ is an integer so is λ and conversely. Thus to every solution of (10) corresponds a definite transformation (9) and conversely. By a well-known property of (10) it follows that all transformations (9) are powers (positive or negative) of the transformation T_1 corresponding to the least positive solution ξ_1, κ_1 of (10). Moreover if $T = T_1{}^n$ its characteristics ξ, κ are given by

$$\frac{\xi + \kappa\sqrt{\Delta}}{2} = \left(\frac{\xi_1 + \kappa_1\sqrt{\Delta}}{2}\right)^n.$$

Of course Δ must not be a perfect square. This we may well assume for in that case there are no birational transformations. The most general birational transformation is obtained by replacing U, V, in (9), by $\epsilon U + \beta_1$, $\epsilon V + \beta_2$; $\epsilon = \pm 1$; β_1, β_2 arbitrary constants.

Humbert makes also the very interesting observation, (42), p. 317, that two Jacobi surfaces, F, F_0, may be birationally equivalent without its being the case for the associated curves of genus two. When that happens $\mathbf{N}$ is a period matrix with respect to retrosections for one of the curves but not for the other, Rosati (86), Comessatti (17). Humbert gives this precise theorem (42), p. 324: When $x^2 + xy + y^2$ can represent -1 there is a unique C^2 associated with F, and all surfaces $T_1{}^nF$ have the same moduli. When the form cannot represent -1 there are two birationally distinct curves with the respective Jacobi surfaces $T_1{}^{2n}F$ and $T_1{}^{2n+1}F$.

16. Humbert was the first to study complex multiplication for the genus two in all its generality, (42), p. 327. The situation is as in No. 15 and (8) are the equations of the problem, with the matrix condition $\mathbf{a}\boldsymbol{\omega} = \boldsymbol{\omega}\mathbf{a}$. Geometrically we have to find an I_δ on F having for image F itself. The more general problem is to find algebraic correspondences between the points of F. But we shall take the situation as considered by Humbert and assume that the correspondence is an involution.

In conformity with most writers we call a multiplication or transformation *ordinary* when $\mathbf{a}\boldsymbol{\omega} = \boldsymbol{\omega}\mathbf{a}$ is satisfied whatever $\boldsymbol{\omega}$. The equations (8) are in this case:

$$U = au + \beta_1; \qquad V = av + \beta_2$$

where a is an integer. All other multiplications will be described as *complex*, or the transformations as *singular*. Humbert, however, calls "ordinary transformation" the somewhat more general type, characterized by the property of leaving invariant the set of all θ's. To be precise it has then the property that if the change of variables (8) be applied to $\theta(U, V)$, the function becomes a $\theta(u, v)$. That type of transformation was called *principal* by Frobenius (27), *hermitian* by Enriques-Severi (22), p. 369. In terms of matrices the transformation is *principal* if there exists a principal matrix $\mathbf{c}$ of $\boldsymbol{\omega}$ such that $\mathbf{aca}' = l\mathbf{c}$, where l is a rational multiplier. Then $\mathbf{c}$ is also a principal matrix of $\boldsymbol{\omega}\mathbf{a}$.

Humbert shows that complex multiplication is present, when and only when, $\boldsymbol{\omega}$ is singular. He considers the three cases of 1, 2, 3, singular relations (5) and finds five principal categories according to the number of distinct singular relations and additional relations between the moduli, (42), p. 357. He thus determines all possible values of the singularity index k and related matrices and indirectly lays down the basis for the determination of h, but this last index actually introduced by Scorza, was not considered by Humbert.

The indices h, k were determined for Jacobi surfaces by Rosati(82), Part II, under another name and in connection with his investigations on correspondences. He also showed the existence of the various types. As every R. M. has an isomorph **N**, Rosati's determination holds for Picard surfaces as well. See also Scorza(94), p. 77, Lefschetz(51), p. 432. Scorza in particular enumerates 9 precise types. It must be remembered that his h, k are less by one unit than ours. Lefschetz gives also explicit forms for the three types of pure R. M.'s. The impure types can be constructed automatically. See also in this connection Cotty(19), Hecke(31), Krazer-Wirtinger(48), p. 857, and this chapter No. 55.

17. An important type of multiplication corresponds to birational transformations of the Jacobi surface. Humbert(42), p. 369, first seeks principal transformations and reduces the problem to finding curves $y^2 = f_6(x)$, such that the roots of $f_6(x) = 0$ are invariant under a linear group. This question has been solved by Bolza(7), who showed that f_6 belongs to any one of six types. Humbert then considers non-principal birational transformations and determines them for every value of the singularity index k. In particular when $k=2$, and in the notations of No. 13, any T is represented by equations (9) with arbitrary constants added on the right-hand side. Let $\boldsymbol{\alpha}$ be the matrix of the u, v coefficients, and $\boldsymbol{\alpha\omega} = \boldsymbol{\omega}\mathbf{a}$ the associated multiplication. Since the terms of $\boldsymbol{\alpha}$ are real we have $\boldsymbol{\alpha}\bar{\boldsymbol{\omega}} = \bar{\boldsymbol{\omega}}\mathbf{a}$ with $\mathbf{a}$ unimodular. Hence

$$\left\|\begin{matrix}\boldsymbol{\alpha} & 0\\ 0 & \boldsymbol{\alpha}\end{matrix}\right\| \cdot \left\|\begin{matrix}\boldsymbol{\omega}\\ \bar{\boldsymbol{\omega}}\end{matrix}\right\| = \left\|\begin{matrix}\boldsymbol{\omega}\\ \bar{\boldsymbol{\omega}}\end{matrix}\right\| \cdot \mathbf{a}; \qquad |a|^2 = |a| = 1;$$

$$|a| = \lambda^2 + \beta\kappa + \gamma\kappa^2 = \pm 1.$$

The related birational transformations of the Kummer surface are again represented by (9) with arbitrary half periods added at the right. They are the products of the powers of a single singular birational transformation multiplied by the 16 ordinary birational transformations of the surface (its six projectivities when in an S_3). Therefore the total group of the surface is discontinuous. This constitutes in fact the first example of a surface with an infinite discontinuous group; it is due to Humbert(42), p. 372; (32). See also Painlevé(60), Castelnuovo-Enriques(11), p. 752, Enriques(21).

For further information on the groups of Jacobi surfaces see Comessatti(17), Rosenblatt(91) and the papers of Enriques-Severi and Bagnera-de Franchis to be considered in the sequel. Scorza(94), Part II, determines the complete group for every one of the types of hyperelliptic surfaces in his list p. 340. For his type V ($k=2$, $h=4$), Lefschetz in (51), p. 440, has derived his result by a direct arithmetical method.

18. We now come to the important investigations of Enriques-Severi (1907), which are contained in two preliminary notes (24) and in a comprehensive paper in two parts(22), (23). The authors first observe that every Picard surface F is representable by θ's with an R. M. $\mathfrak{N}$, whose e is the *divisor* of F. The Jacobi surface corresponds to the divisor unity. Every hyperelliptic surface of divisor e and rank r is the image of an I_{er} on a Jacobi surface, or of an I_r on the Picard surface corresponding to the same matrix. Among the birational transformations of F of rank one there are two types present whatever the matrix; they are the ordinary transformations: (a) of the first kind or

$$u+u'=\text{const};\qquad v+v'=\text{const}.$$

(b) of the second kind or

$$u-u'=\text{const};\qquad v-v'=\text{const}.$$

The latter form a continuous two-parameter, abelian group. Those of the first kind form a cyclic binary group when the constants are zero. The transform of (u, v) is then $(-u, -v)$ and the two points together are a group of an I_2 whose image is a generalized Kummer surface with 16 nodes corresponding to u, v both equal to half periods. Transformations of the second kind give rise to cyclic groups of any assigned order, when the constants are $\frac{1}{n}$-th periods and the associated I_n are enumerated for the Jacobi surface(22), p. 293. The image of such an I_e on a Jacobi surface is a Picard surface whose divisor is e.

The curves of lowest genus carried by a surface of rank one are particularly important(22), p. 304. Rational curves occur only in finite number and can be removed by a birational transformation(22), p. 322. If there exists a single elliptic curve there are two irrational pencils of them. Each is characterized by the constancy of a linear combination $au+bv$ on its curves. Therefore in general there are no elliptic curves. An interesting Jacobi surface with elliptic curves is the image of the point pairs of two curves of genus one. The surfaces that carry elliptic curves are completely characterized by the presence of an irrational pencil. Both pencil and curves are then necessarily elliptic.

A Picard surface F always carries curves of genus two. They are obtained from an arbitrary linear system $|C|$, of genus p, by imposing $p-2$ double points on C. The number of such curves in $|C|$ is finite. By applying the ordinary transformations of the second kind to them there is obtained a continuous system ∞^2 whose image is F itself. When F is a Jacobi surface the curves have no double points and are birationally equivalent to the basic C^2 associated with the surface. The continuous system is then of degree two and index two (i. e., there are two curves of it through an arbitrary point of F). This property completely characterizes Jacobi surfaces. Every Picard F of divisor e carries an ∞^2 system of curves of genus two with $e-1$ double points, the degree and index being both $2e$. Both for the Picard and Jacobi surfaces these systems are cut out by the θ's of lowest order. On the Jacobi surface the system is complete, on the Picard surface it belongs to a continuous system ∞^2 of linear system $|C|$, whose dimension is $e-1$, genus $e+1$, degree $2e$. On the most general surfaces (i. e., with general period matrix) every curve belongs to a linear system $|mC|$. This is merely another way of stating the fact that it is cut out by a θ. For $e>3$ the consideration of these linear systems shows that F is birationally transformable into a surface of degree $2e$ on an S_{e-1}, carrying no exceptional curves. The authors henceforth assume that F has no exceptional curves. Strictly speaking they are considering a suitable birational transform in some space, (1, 1) image of the points of the fundamental domain U_4 (No. 8).

19. The authors now proceed to classify involutions on the Jacobi F, or, which is the same, surfaces of rank >1 belonging to a matrix $\mathbf{N}$. The only types are rational or elliptic surfaces, or F of divisor >1 and rank one attached to some R. M., or else regular surfaces such that either $p_g=P_3=0$, $P_2=1$, or $p_g=P_4=1$ (22), p. 324. Invariance under ordinary transformations leads to a classification according to the values of the irregularity q.

Of more importance is the number of coincidences of an I_n on any F of rank one. When it is infinite, I_n is rational or has for image an elliptic ruled surface. When there are no coincidences, I_n has for image a surface of rank one or an elliptic surface (surface with an elliptic group). Hence the most interesting case corresponds to an I_n with a finite, non-zero, number of coincidences with $P_2 \neq 0$, (22), p. 329.

Enriques-Severi give this basic theorem on involutions (23), p. 331: Every hyperelliptic I_r on F of rank one is generated by a birational group G_r of order r of the surface F. An equivalent statement is that every surface of rank r is the image of an I_r on the related surface F of rank one, generated by a birational G_r of F. The group of I_r to which any assigned point P of F belongs consists of the transforms of P by the operations of G_r. The operations of G_r are of the form

$$(11)\quad u'=a_iu+b_iv+c_i;\qquad v'=a'_iu+b'_iv+c'_i;\qquad (i=1, 2, \ldots, r).$$

With the transformations of G_r there are associated definite multiplications of the R. M. ω. If ω is arbitrary they can only be ordinary transformations. Hence the generic F of rank one carries only involutions of order two representable by a generalized Kummer surface of the same divisor as F, or else involutions representable by surfaces of different divisor but of rank one. The generalized Kummer surfaces are of degree $4e$ on an S_{2e} with 16 double points and singular hyperplanes giving rise to a configuration similar to Kummer's and these properties completely characterize them(22), p. 351.

The next important and simple type of I_r has the irregularity $q=1$ and its image is therefore an elliptic surface, and is completely characterized by $p_a=-1$, $p_g=0$, $P_i=0, 1$; (22), p. 361. The authors also determine the divisors for these surfaces and their period matrices (22), p. 361. See also Bagnera-de Franchis(4).

20. In the classification of involutions associated with a G_r Enriques-Severi limit themselves to Jacobi surfaces. Like Humbert(42) they base themselves on Bolza's work on $f_6(x)$, (7), and thus obtain eleven different families(22), p. 392, the first being composed of Kummer surfaces. In (23) they construct models of the remaining ten types which are of order $2r$ in S_{r+1} characterized by definite configurations of singular points and hyperplanes tangent along certain rational curves. The singular points correspond to coincidences of the related I_r on the Jacobi surface and are fully described in each case, the authors giving also schemes analogous to Humbert's(39), to represent the associated configurations of singular points and tangent hyperplanes.

The surfaces of the eleven types can all be projected into quartic surfaces of S_3, or double planes $z^2=f_6(x, y)$, with characteristic configurations of singular points and tangent planes in the first case, multiply tangent curves to $f=0$ in the second(23), p. 397. See also Piazzola-Beloch(61.1), (61.2).

21. Bagnera-de Franchis in (2) to (6), most of which appeared at the same time as the work of Enriques-Severi (1907-09), investigated all questions concerning hyperelliptic surfaces from the most general point of view. They made relatively little use of special geometric properties and considered directly the R. M. Owing to this their methods have lent themselves exceptionally well to extensions to abelian varieties. Their chief contribution is the complete classification of all hyperelliptic involutions and the determination of the Picard-Severi number ρ for all hyper-

elliptic surfaces. Their starting point(6) is the theorem of Enriques-Severi on the group G_r associated with any hyperelliptic I_r (No. 17). Let $\boldsymbol{\omega}$ be the R. M., F the corresponding surface of rank one. Let T be a cyclic transformation of order n, such as (11), but in the more convenient form (8). We have (No. 17), $\boldsymbol{\alpha\omega}=\boldsymbol{\omega}\mathbf{a}$, where $\mathbf{a}$ is unimodular. Since $\boldsymbol{\alpha}^s$, $\mathbf{a}^s$ correspond to T^s like $\boldsymbol{\alpha}$, $\mathbf{a}$, to T (Ch. 15, No. 7), $\boldsymbol{\alpha}^n=\mathbf{a}^n=\mathbf{I}$. Therefore the roots of the characteristic equations $|a-xI|=0$, $|a-xI|=0$, are n-th roots of unity. From this it follows that with a suitable linear change of variables the equations of T are $u'=\lambda u+v$, $v'=\mu v$, where λ, μ, called multipliers of T, are n-th roots of unity and $n\nu$ is a period, Bagnera-de Franchis(6), p. 265. When $\lambda=\mu=\pm 1$, T is an ordinary transformation—a type which need not be considered. When neither is $+1$ the equations can be given the simpler form $u'=\lambda u$, $v'=\mu v$. Since $|a-xI|=0$ is a biquadratic whose roots are n-th roots of unity we must have $\phi(n)\leqq 4$, where ϕ is the Euler function of number theory. It follows that n can only have the values 2, 3, 4, 5, 6, 8, 10, 12.

Another important result is obtained by considering the effect of T on the integrals of the first kind(6), p. 263. When λ and μ are both $\neq 1$, T leaves no Picard integral of the first kind invariant and the associated I_n is regular. Returning to the group G_r of operations (11), its I_r is regular if there exists no linear change of variables reducing simultaneously all its operations to the form

$$u'=u+\nu_i; \qquad v'=\lambda_i v.$$

Should such a transformation exist and one of the multipliers λ_i be $\neq 1$, I_r is irregular with $q=1$. When all the λ's are 1, $q=2$ and I_r has for image a hyperelliptic surface of rank one.

The effect of T on $\iint du\,dv$ is to change it into $\lambda\mu\iint du\,dv$. Hence when $\lambda\mu=1$, I_n has its $p_g=1$, when $\lambda\mu\neq 1$, $p_g=0$.

Regarding I_r when $\lambda\mu=1$ for every operation of G_r, $p_g=1$, otherwise $p_g=0$. Thus by very simple considerations Bagnera-de Franchis go far towards classifying the involutions. By examining the relation $\boldsymbol{\alpha\omega}=\boldsymbol{\omega}\mathbf{a}$ for the possible operations T they obtain all $\boldsymbol{\omega}$'s corresponding to a cyclic T and the required classification of surfaces of rank $r>1$. They obtain explicit equations of a birational model for each type, in particular a double plane when the surface is regular. At the end of (6) there is a complete list of matrices and groups determined in the paper. For the construction of the various types see also Verzi(114).

22. In (5) Bagnera-de Franchis determine the Picard-Severi number ρ and the number ρ_0 of linearly independent double integrals of the second kind for all hyperelliptic surfaces. They apply throughout Humbert's

theorem(39) on the representation of the curves by intermediary functions. When the rank $r=1$ they find by direct examination that $\rho=k$, the singularity index of $\boldsymbol{\omega}$, and they completely characterize the classes of intermediary functions. Hence ρ can have the values 1, 2, 3, 4. For surfaces whose $r>1$ they consider the number $[\rho]$ analogous to ρ when the isolated singular points are not counted as infinitesimal curves, and the number which they call ρ, when they are so considered. Both numbers are determined for all cases. For the generalized Kummer surfaces $[\rho]=k$ also, but $\rho=16+k$. See also Picard-Simart(69), pp. 256, 447, Severi(108).

The various invariants can be determined by the topological methods due to Lefschetz(50), (51). His theorems described in Chapter 15, lead at once to the relation $\rho=k$ for surfaces of rank 1, 2.

In (56) Lefschetz examines real hyperelliptic surfaces and their invariants h, k. He defines a new invariant ρ', the number of algebraically distinct real curves for a real surface of rank one, or number of real curves between whose associated two cycles there is no homology. He finds six distinct types corresponding to as many sets of values h, k. See also for these real surfaces Comessatti(16).

Comessatti(17) in line with a certain theorem due to Severi(107) has connected the theory of the base and the systems of curves on a surface of singularity $k=2$ with the group of automorphisms of the quadratic form whose coefficients are the intersection numbers of the two curves of the minimum base. The form is closely related to, but not identical with that considered by Humbert in (42). Comessatti's exposition is especially interesting from the geometric point of view. In particular he considers certain projective models of the surfaces and examines their properties.

23. *Additional References.*—Brief treatments are given by Castelnuovo-Enriques(11), p. 748; Severi(104), p. 779; more extensive discussions by Hudson(31.1), Chs. 16, 18; Krazer-Wirtinger(48), pp. 855, 870. Bonomi(7.1) considers surfaces with an elliptic pencil. Scorza(95), examines the intersections of the elliptic curves when any are present. Humbert(34), Hudson(31.1), Spampinato(109), consider the elliptic Kummer surface. In a number of the following papers, interesting special cases will be found treated, notably birational transforms of the Kummer surface (in the strict sense or generalized) in the form of quartics with 13, 14 or 15 nodes: Traynard(111); Rémy(75), (76), (78), (79), (81); Chillemi(13), (13.1); Piazzola-Beloch(61). In connection with his work on Cremona groups Coble(14.2), p. 354, has extensively investigated a projective model of the Jacobi surface, an F_{18} in S_8 invariant under a collineation group of order 162 independent of the moduli.

§ 3. ABELIAN VARIETIES. GENERAL PROPERTIES

24. Their definition has already been given in No. 8. That they exist follows from the Riemann-Weierstrass theorem on the algebraic relation between $p+1$ periodic functions with the same periods. The first step beyond that was taken by Klein who in 1886-87 discussed the analogue of the Kummer surface for any p, represented in a certain S_n by setting the homogeneous coordinates proportional to even θ's, or even intermediary functions. See No. 62.

25. In ([67]), ([65]), Picard extended to any p the theorem described in No. 10, and showed that a V_p with a continuous abelian group in p parameters is an abelian variety of genus p.

The essential step of connecting abelian varieties with algebraic surfaces was taken by Castelnuovo([8]), ([9]). See also Castelnuovo-Enriques ([11]), p. 709; Severi([104]), p. 762. On a surface F of irregularity q let $|C|$, $|C_1|$, $|C_2|$ be three elements of a sufficiently general maximal (complete) set of linear systems, W. Then $|C+C_1-C_2|$ is also an element of W, from which Castelnuovo concludes that W is reproduced by the addition of C_1-C_2 to its curves. That means that W is invariant under a continuous abelian group in q parameters, hence it is an abelian V_q. The variety is independent of C provided it is sufficiently general; it has been called the *Picard variety* of F. Referring to Ch. 15, No. 17, V_q corresponds point for point to the sets of constants a_j, modd. the periods of the Picard integrals u_{p-q+j}. Construct for the linear cycles an intermediary F. S. (Ch. 15, No. 11) of which the first $2q$ cycles, $\Gamma_1^1, \ldots, \Gamma_1^{2q}$ are independent and the rest zero-divisors, and let $\omega_{j\mu}$ be the period of u_{p-q+j} as to Γ_1^μ. Then the a's are determined up to sums of multiples of the terms of $\boldsymbol{\omega}$ and V_q is the most general abelian variety attached to $\boldsymbol{\omega}$. A similar V_q may be defined for any algebraic variety of linear connectivity index $R_1=2q$. The Picard variety of an algebraic curve of genus p, C^p, $(q=p)$ is called the Jacobi variety of the curve. The counting of moduli shows that a Jacobi V_p is a very special abelian variety. Not so, however, the Picard V_q of an algebraic surface or algebraic variety of irregularity q. Indeed from theorems on the reductions of linear cycles of a variety to a surface or subvariety that it carries, Lefschetz([50]), Ch. 5, it follows that any abelian variety is the Picard variety of some algebraic surface or variety that it carries, for example, sections by suitable linear spaces. Furthermore any V_n, $n\geqq 3$, has the same Picard variety as its hyperplane sections. Hence the Picard variety is a perfectly arbitrary abelian variety. In this connection there is also an interesting theorem due to Severi([106]) according to which a surface without an irrational pencil is the rational transform of a surface of the same irregularity, carried by its Picard variety, a theorem which he has also extended to varieties.

Another important series of investigations are those on irrational series on a C^p, Ch. 16, § 5, notably those of Torelli and Comessatti. Many of the questions taken up by them are of interest not only for the Jacobi V_p of C^p but for the general abelian V_p as well. A similar remark applies to much of the work of Rosati, discussed at various places in Ch. 16. It is indeed plain that the properties of the Jacobi variety of C^p are of basic importance for the *singular* correspondences of the curve.

26. The comprehensive study of the general abelian variety V_p, $p>2$, was actually initiated by Scorza in 1914, and is largely the work of Scorza, Rosati, Lefschetz. Scorza's investigations bear mostly on Riemann matrices and complex multiplication, and Rosati's are along parallel lines with singular correspondences on a C^p chiefly in view. The work of Lefschetz leans more to geometry and function theory; in many places the three authors overlap. The most profitable thing to do here will be to give a comprehensive picture of the theory as it now stands.

27. *Hypersurfaces of V_p. Topology.*—By a *hypersurface* of V_p is meant any V_{p-1} on the variety. We take the V_p attached to ω and propose to describe its continuous systems of hypersurfaces in relation to the matrix. In this we follow Lefschetz(^{51}), Part II, Ch. 1; (^{50}), Ch. 6, whose method is based on Analysis Situs considerations of a fairly simple nature. For the notions of Analysis Situs involved, see Ch. 15, § 3.

Topologically V_p is homeomorphic to the generalized ring derived from U_{2p} of No. 2 by bringing into coincidence the boundary points whose u's differ by a set of periods—a familiar procedure for elliptic functions. Let P_0 be the vertex of U_{2p} whose u's or real coordinates t_μ of No. 2 are all zero, P_μ the vertex whose $t_\mu=1$, the others being again zero. There is a definite k dimensional face of U_{2p} carrying the simplex (generalized tetrahedron) $P_0P_{i_1}\ldots P_{i_k}$. This face sensed by means of the simplex as indicatrix, (^{50}), p. 3, is denoted by $(i_1\ldots i_k)$. The face is the image of a definite k-cycle of V_p also designated by that symbol.

A set of $\binom{2p}{k}$ k-cycles so obtained, no two composed of the same set of i's, constitutes an F. S. for k-cycles and as the cycles of the F. S. are independent, the connectivity and torsion indices of V_p are

$$R_k=\binom{2p}{k}, \qquad \sigma_k=1.$$

In particular Severi's invariant $\sigma=\sigma_1=1$. Therefore any intermediary F. S. for the hypersurfaces of V_p is also an actual F. S.

Observe that $(12\ldots 2p)$ is a $2p$-cycle which coincides with V_p or $-V_p$ depending upon the orientation of the variety.

28. Given two cycles Γ_h, Γ_k, there exists a definite cycle of $h+k-2p$ dimensions, their intersection, written $\Gamma_h \cdot \Gamma_k$. When $h+k=2p$ we consider the Kronecker-index $(\Gamma_h \cdot \Gamma_{2p-h})$ or algebraic number of intersections. The latter is computed on the basis that each crossing is affected with $+1$ or -1 and the sum of these units is the index. All doubtful cases are settled by the following theorems: (a) Intersection cycle or index obey the distributive laws with respect to both intersecting cycles. (b) The intersection cycle is ~ 0 or the index zero when one of the intersecting cycles is a zero divisor. This can all be extended to several intersecting cycles. See Lefschetz(50), Ch. 1, (52); Veblen(113); Weyl(118).

When V_p is sensed like $(12\ldots 2p)$ the intersections and indices of the cycles of No. 27 are, (50), pp. 10, 114:

$$(i_1 i_2 \ldots i_k i_{k+1} \ldots i_h) \cdot (i_1 \ldots i_k i_{h+1} \ldots i_{2p}) \sim (-1)^n (i_1 i_2 \ldots i_k), \tag{12}$$

$$((i_1 i_2 \ldots i_k) \cdot (i_{k+1} \ldots i_{2p})) = (-1)^n, \tag{13}$$

where the i's are the sequence $1, 2, \ldots, 2p$ with n transpositions. If at the left in (13) the i's are not all distinct, the index is zero. When V_p is not sensed as above the right sides must be changed in sign.

29. Let $A_1, \ldots, A_p$ be hypersurfaces of V_p intersecting at the point Q. We assume that they have no singularity there and that their tangent S_{p-1}'s intersect at Q only. They are then cut out in the space that carries V_p by certain polynomials and in the u space by certain intermediary functions $\phi_h(u)$ which have a common zero Q', (u^0), corresponding to Q, with a non-vanishing functional determinant at Q'. Set $\phi_h = \phi'_h + i\phi''_h$; ϕ'_h, ϕ''_h real. The locus of the u space where all functions ϕ', ϕ'' are zero except ϕ'_h is an arc on which we take a point Q'_{2h-1} in the direction of increasing ϕ'_h and very near Q'. There is also a similar point Q'_{2h} for ϕ''_h. We now assign to the u space the indicatrix $Q'Q'_1 \ldots Q'_{2p}$ and to the image A'_h of A_h the indicatrix obtained by removing Q'_{2h-1} and Q'_{2h} from the preceding one. Naturally the variety and A_h will receive for indicatrices the images of the preceding. Under the circumstances by reference to the situation at Q' in the u space, we find that the contribution of Q to the Kronecker index $(A_1 \cdot A_2 \cdot \ldots A_p)$ is always $+1$.

30. Given two sets of analytical functions behaving respectively at R, S, like the ϕ's, together with the sets of associated manifolds of the u space, treat each exactly as above. There is a continuous deformation of the space bringing R into S and the neighborhood of the first set of manifolds about R into that of the second at S with all orientations

preserved. When the functions of the sets are all linear the verification is immediate, and as the situation merely depends upon the first degree terms in the Taylor expansions, it holds in all cases. Hence the indicatrices so defined for all points of our manifolds are consistent, and moreover independent, for any one of them, of all the others. The only condition to be fulfilled is that the functional determinant of the functions in any set be $\neq 0$. The same conclusion is valid as regards the orientation of the hypersurfaces of V_p: The orientations of all the indicatrices of No. 29 for a given A are in agreement and the cycle A is therefore a perfectly definite oriented spread.

31. When one of the sets of functions of No. 30 is the set $u_1, \dots u_p$ there is obtained an indicatrix deformable into $PP_1 \dots P_{2p}$ of No. 27 if and only if

$$\Delta = \frac{(-1)^{\frac{p(p+1)}{2}}}{(2i)p} \begin{vmatrix} \boldsymbol{\omega} \\ \bar{\boldsymbol{\omega}} \end{vmatrix} > 0. \tag{14}$$

When $\boldsymbol{\omega}$ is replaced by $\boldsymbol{\omega}\mathbf{a}$, Δ is replaced by $\Delta|a|$. If $\Delta < 0$ we can take a new F. S. of periods corresponding to a matrix $\mathbf{a}$, $|a| = -1$ and Δ will have become >0. Hence we can and shall assume that $\boldsymbol{\omega}$ is so chosen that (14) holds. As a consequence $V_p \sim (1\ 2 \dots 2p)$ and formulas (12), (13), are valid as they stand.

32. Consider a definite analytical function $f(u)$ whose derivatives do not all vanish at R. Through R draw in the u space a small oriented planar element E not tangent to $f=0$. On E draw a small positive circuit ζ surrounding R. Then

$$\frac{1}{2\pi i}\int_{\zeta} d \log f = (E \cdot f), \tag{15}$$

where the index is that of E and $f=0$. For both sides are ± 1 and it is only a question of showing that their signs are the same. The true sign to be put at the right is unchanged under a deformation such as described in No. 30 combined with a deformation of E during which it never becomes tangent to f. This enables us to take $u_1 = f$ and for E an element of $u_2 = \dots = u_p = 0$, so oriented that both sides in (15) are $+1$. Therefore (15) holds in all cases.

33. We shall now apply the preceding results to the hypersurfaces of V_p. A first remark of considerable importance: the index $(A_1 \dots A_p)$ is greater than 0 and equals the virtual number of intersections $[A_1 \dots A_p]$ in the sense of Ch. 15, No. 8. It is in fact the actual number of intersections whenever they are distinct and each counts for one in the total number. For by applying transformations $u = u' + \text{const.}$ to the A's we may replace them if necessary by hypersurfaces in the same continuous systems, hence homologous to them, and therefore with the same

$(A_1 \ldots A_p)$ and $[A_1 \ldots A_p]$, whose intersections will be isolated and as in No. 29. Since each intersection contributes then $+1$ to each of these numbers, the latter are equal.

34. Let again ϕ be an intermediary function attached to $\boldsymbol{\omega}$. The second derivatives of $d \log \phi$ are periodic, hence rational point functions on V_p. Therefore $\phi = 0$ is the parametric equation of a hypersurface A of V_p. Since it is also a $2p-2$ cycle we have

$$A \sim \Sigma(-1)^n m_{\mu\nu}(\sigma \ldots \kappa), \tag{16}$$

where $\mu, \nu, \sigma, \ldots, \kappa$, is the sequence $1, 2, \ldots, 2p$ with n transpositions and the sets $\sigma, \ldots, \kappa$ are all different. By (13), (15), and as in (50), p. 120, we find in the notations of No. 4

$$m_{\mu\nu} = (A \cdot (\mu\nu)) = \Sigma(\epsilon_{j\mu}\omega_{j\nu} - \epsilon_{j\nu}\omega_{j\mu}).$$

Hence the integers $m_{\mu\nu}$ are the same as in No. 4. In any case by (13), whether A is cut out by an intermediary function or not, the first equality sign is valid, and the interpretation of $m_{\mu\nu}$ as a Kronecker-index is to the point. But this is not very important for as we shall see A is always cut out by some ϕ.

Having definite homologies for the hypersurfaces we can apply (12), (13), to obtain homologies or indices for their intersections. Let

$$A_j \sim \Sigma(-1)^n m^j{}_{\mu\nu}(\sigma \ldots \kappa),$$

where in case A_j is cut out by the intermediary function $\phi_j = 0$, $m^j{}_{\mu\nu}$ is the $m_{\mu\nu}$ of the function. Then

$$A_1 A_2 \ldots A_s \sim \frac{1}{2^s \cdot (2p-2s)\,!} \Sigma(-1)^n m^1{}_{\mu\nu} \ldots m^s{}_{\lambda\rho}(\sigma \ldots \kappa), \tag{17}$$

while for $s = p$ the number of intersections is

$$[A_1 \cdot A_2 \ldots A_p] = \frac{1}{2^p} \Sigma(-1)^n m^1{}_{\mu\nu} \ldots m^p{}_{\sigma\kappa}. \tag{18}$$

The sequence $\mu, \ldots, \kappa$ is the same as $1, 2, \ldots, 2p$ with n transpositions.

Every A determines a continuous system such that there are always sets of two, three, $\ldots$, p, intersecting in cycles of $2p-4, \ldots, 2$ dimensions or in isolated points. For the related cycles and intersections, conveniently denoted by $A^2, \ldots, A^{p-1}, A^p$, (17), (18) lead to especially simple expressions.

Let π be the pfaffian of the skew-symmetric matrix m, Kowalewski(46), p. 140. In its expression let $M_{\mu \ldots \rho}$ be the coefficient of the term $m_{\mu\nu} \ldots m_{\lambda\rho}$. It may be considered as a certain minor of π. Then

$$A^s \sim s\,!\Sigma M_{\mu \ldots \rho}(\mu \ldots \rho). \tag{19}$$

$$(A^p) = [A^p] = p\,!\pi. \tag{20}$$

35. If as in No. 31 we replace $\boldsymbol{\omega}$ by $\boldsymbol{\omega}\mathbf{a}$, $|a|=\pm 1$, π is replaced by $\pi|a|$ and therefore $\pi\Delta$ is unchanged. We have so chosen $\boldsymbol{\omega}$ that $(A^p)>0$, therefore $\pi \geqq 0$. Thus as a consequence of the choice of $\boldsymbol{\omega}$ the pfaffian of the $\mathbf{m}$ of every intermediary function is positive. Whatever the choice of $\boldsymbol{\omega}$ all the pfaffians have the same sign.

Maintaining our convention we have $\pi=+\sqrt{|m|}$. Therefore the dimension of the complete linear system to which ϕ belongs is $\pi-1$.

In order to verify whether $\boldsymbol{\omega}$ has been correctly chosen it is then merely necessary to compute a single pfaffian. When $\boldsymbol{\omega}$ is reduced to the form $\mathbf{N}'$ of § 1 some intermediary function becomes a θ_n and its pfaffian is $\dfrac{(ne_p)^p}{e_1 e_2 \ldots e_p}>0$ hence $\boldsymbol{\omega}$ is then such that (14) holds. Consequently any proper $\boldsymbol{\omega}$ is of the form $\mathbf{N}'\mathbf{a}$, $|a|=+1$. This is absolutely in line with Humbert's conventions, (41), (42), also those of Bagnera-de Franchis(2); see also Scorza(99), first paper.

Special cases of (18), (20), are due to Poincaré(70) and Wirtinger(120), (124), their treatment, especially Poincaré's, being related to ours. We have gone into the matter at some length as our procedure reduces most questions of enumeration on a V_p to an arithmetical algorithm. Further references are Baker(1.1), Ch. 10, (1.2); Lefschetz(51), p. 381, where the whole set of formulas was first given in its completeness, (50), Ch. 6; Castelnuovo(10), where interesting applications to correspondences on a C^p are also given. For $p=2$ see also Humbert(41), Comessatti(17). In (51), p. 389, Lefschetz gives various enumerative formulas for the arithmetical genera of the hypersurfaces of V_p and of their complete intersections.

36. Starting with (16) we can derive the theorem of Frobenius given in No. 4, according to which if ϕ is a non-degenerate intermediary function attached to $\boldsymbol{\omega}$, $\mathbf{m}^{-1}=\mathbf{M}$ is a principal matrix, hence $\boldsymbol{\omega}$ is an R. M., Poincaré(70), Lefschetz(51), Ch. 1, (50), Ch. 6. This can be inverted: Given $\boldsymbol{\omega}$ with the principal matrix $\mathbf{M}$, $\mathbf{m}$ is defined (up to an integer factor) and there is a whole class of related intermediary functions of every order $n>0$. They cut out hypersurfaces all algebraically equivalent (homologous) to a multiple of a fixed A (cut out by functions of order one of the class). From this Lefschetz derives, loc. cit.:

I. The intermediary functions of like behavior, hence with same $\mathbf{m}$ and π, cut out a complete linear system $|A|$ whose dimension is $\pi-1$ and which belongs to a continuous system ∞^p.

II. The base number ρ of V_p is equal to the singularity index k of $\boldsymbol{\omega}$.

III. Every hypersurface of V_p is cut out by an intermediary function.

IV. There exists in all cases an F. S. composed of ρ effective hypersurfaces.

The precise construction of the F. S. is as follows: We can find k linearly independent principal matrices $M_1, \dots, M_k$ of $\boldsymbol{\omega}$ and construct the class of intermediary functions for each. In each class we take an intermediary function of order one. It corresponds to an $\mathbf{m}=l\mathbf{M}^{-1}$ with the integer factor l so chosen that the terms of $\mathbf{m}$ are relatively prime integers. The k functions $\phi_1, \dots, \phi_k$ so chosen cut out the hypersurfaces of the F. S. Incidentally in every complete linear set of intermediary functions there exists a definite one of the form $\Pi\phi_i^{n_i}(u+c)$ where the n's are positive or negative integers.

These theorems were known for $p=2$. See Humbert(39), Bagnera-de Franchis(2), this chapter, Nos. 12, 22; also Castelnuovo(10) for the general case.

37. Just as for hyperelliptic surfaces the integrals of the first kind of various multiplicities (integrals of total differentials and their generalizations) can be expressed in terms of the u's. The s-fold integrals are all linear combinations of the integrals $\int\dots\int du_{i_1}\dots du_{i_s}$ and their periods are minors of order s of $\boldsymbol{\omega}$. Of chief interest are the simple integrals which are merely the u's. For the enumeration questions connected with these integrals see Lefschetz(51), p. 373.

§ 4. IMPURE MATRICES AND THEIR VARIETIES

38. The most important work along that line is due to Scorza. To give a good account of it we will find it advantageous to consider in some detail his very suggestive projective method, since some of its best applications have been in the theory of impure matrices. The chief references for Scorza's work are his notes(99) and major paper(94), Part I; Krazer-Wirtinger(48), p. 827; Lefschetz(54) (brief summary). For impure matrices see (48), p. 838, to whose bibliographical list should be added Krazer(47), Ch. 11; Severi(105), Ch. 10. There is an important parallel development initiated by Rosati(82), (88), almost simultaneously with Scorza (1914). In his work, based on a dual of Scorza's method, he considers throughout, period matrices of algebraic curves, and his chief aim is to investigate singular correspondences. There are many points of contact between the two authors but naturally Rosati's is far more special. An account of his investigations on correspondences has already been given in various parts of Chapter 16.

Given an R. M. $\boldsymbol{\omega}$, consider with Scorza the S_{p-1}, σ, of S_{2p-1}, carrier of the p points whose projective coordinates are $x_\mu=\omega_{j\mu}$, together with its conjugate σ carrier of the points whose coordinates are $x_\mu=\bar{\omega}_{j\mu}$. An alternate matrix $\mathbf{c}$ of $\boldsymbol{\omega}$ gives rise to a rational null system

$$\Sigma c_{\mu\nu}x_\mu y_\nu=0$$

with respect to which all point pairs of σ (or of $\bar{\sigma}$) are conjugate. It may also be considered as defining a line complex of S_{2p-1} carrying all lines on σ (hence also those on σ). When $\mathbf{c}$ is principal there are no real lines of the complex intersecting both σ and $\bar{\sigma}$. As ordinary rational $\mathbf{c}$ of $\boldsymbol{\omega}$ defines a rational reciprocity ρ with respect to which all point pairs of σ (or of $\bar{\sigma}$) are conjugate. The indices h, k are the numbers of linearly independent rational reciprocities or null systems of the preceding type.

Let us consider a definite principal matrix $\mathbf{c}_0$ with its reciprocity ρ_0 which is not degenerate since $|c_0| \neq 0$. Then $\eta = \rho\rho_0^{-1}$ is a rational projectivity leaving σ and $\bar{\sigma}$ invariant. There is a corresponding matrix relation $\boldsymbol{\alpha}\boldsymbol{\omega} = \boldsymbol{\omega}\mathbf{a}$; $\mathbf{a} = \mathbf{c}\mathbf{c}_0^{-1}$, or multiplication of $\boldsymbol{\omega}$ (Ch. 15, No. 7). Conversely such a multiplication defines an η, hence a rational reciprocity $\rho = \eta\rho_0$ of the above type. This proves Scorza's theorem(94), p. 280: The numbers of linearly independent rational reciprocities and of rational projectivities leaving σ invariant are equal. Their common value is the index of multiplication h. It is also the number of linearly independent multiplications (see below No. 46).

39. Let now $\boldsymbol{\omega}$ be isomorphic to $\begin{Vmatrix} \boldsymbol{\omega}_1 & 0 \\ \boldsymbol{\omega}_3 & \boldsymbol{\Omega}_2 \end{Vmatrix}$, where $\boldsymbol{\omega}_1$ is an R. M. of genus $p_1 < p$, i. e., let $\boldsymbol{\omega}$ be impure. Since isomorphs have the same σ there exists a rational S_{2p_1-1} whose intersection with σ is an S_{p_1-1}, σ_1, called an *axis* of $\boldsymbol{\omega}$. Of course S_{2p_1-1} intersects $\bar{\sigma}$ in $\bar{\sigma}_1$. The theorem of Poincaré, Picard, and Scorza(94), p. 298, Krazer-Wirtinger(48), p. 838, on the isomorphism of $\boldsymbol{\omega}$ to $\boldsymbol{\omega}_0 = \begin{Vmatrix} \boldsymbol{\omega}_1 & 0 \\ 0 & \boldsymbol{\omega}_2 \end{Vmatrix}$ signifies that to each axis σ_1 of $\boldsymbol{\omega}$ corresponds a *complementary* axis σ_2 (belonging to $\boldsymbol{\omega}_2$), and the two do not intersect.

It is important to observe, Scorza(95), p. 222, that $\boldsymbol{\Omega}_2$ like $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, is also an R. M., but not necessarily complementary to $\boldsymbol{\omega}_1$. Furthermore up to a transformation into an isomorph, $\boldsymbol{\omega}_2$ is uniquely determined by $\boldsymbol{\omega}_1$, Scorza(94), p. 304. This does not mean that σ_1 necessarily determines its complementary axis σ_2, but that all submatrices $\boldsymbol{\omega}_2$ corresponding to these σ_2's are isomorphic. From this follows the reduction of $\boldsymbol{\omega}$ to an essentially unique normal form

$$\begin{Vmatrix} \boldsymbol{\Omega}_1 & & \\ & \ddots & \\ & & \boldsymbol{\Omega}_k \end{Vmatrix}, \qquad \boldsymbol{\Omega}_j = \begin{Vmatrix} \boldsymbol{\omega}_j & & \\ & \ddots & \\ & & \boldsymbol{\omega}_j \end{Vmatrix}$$

where $\boldsymbol{\omega}_j$ is pure and $\boldsymbol{\omega}_k$, $\boldsymbol{\omega}_j$ are non-isomorphic for $k \neq j$. The number of essentially distinct modes of reductions to the normal form (number of distinct sets of pure complementary axes) is finite when the $\boldsymbol{\Omega}$'s are all pure, infinite otherwise, Scorza(94), p. 307.

When $\boldsymbol{\omega}$ is impure and possesses an axis σ_1 its representative space is invariant under a degenerate rational reciprocity ρ which assigns to every point of σ an S_{2p-2} passing through σ_1 and conversely when such a ρ exists σ_1 is an axis of $\boldsymbol{\omega}$ and $\boldsymbol{\omega}$ is impure. Hence in order that $\boldsymbol{\omega}$ be impure it is necessary and sufficient that it possess a degenerate rational reciprocity, or a rational matrix $\mathbf{c}$ whose determinant $|c|=0$. This remains true when reciprocities are replaced by null systems, or projectivities. In particular $\boldsymbol{\omega}$ is impure if and only if there is a multiplication $\boldsymbol{\alpha\omega}=\boldsymbol{\omega a}$ with $|a|=0$. See Scorza(94), p. 278.

40. Scorza(94), p. 301, determines the invariants of $\boldsymbol{\omega}$ in terms of those of the submatrices of its normal form. Let first $\boldsymbol{\omega}$ be isomorphic to $\boldsymbol{\omega}_0$ above, with $h_j, k_j; h, k; \lambda$, as the invariants of $\boldsymbol{\omega}_j$ and $\boldsymbol{\omega}$, and simultaneity index of $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ (Ch. 15, No. 7). Then

$$h=h_1+h_2+2\lambda, \qquad k=k_1+k_2+\lambda.$$

Since the λ of two isomorphs is their common h the repeated application of these formulas gives for $\boldsymbol{\omega}$ when in the normal form the indices:

$$h=\Sigma n_j^2 h_j; \qquad k=\Sigma((n_{j_2})h_j+n_jk_j),$$

where n_j is the number of components $\boldsymbol{\omega}_j$ in $\boldsymbol{\Omega}_j$ and h_j, k_j the invariants of $\boldsymbol{\omega}_j$.

Owing to our definition of h, k, these formulas are decidedly simpler than Scorza's. See also Lefschetz(51), p. 394.

In (94), Part I, §§ 6-9, Scorza examines at length the various possibilities for h, k. In any case $h\leqq 2p^2$, $k\leqq p^2$ and for a pure $\boldsymbol{\omega}$: $h\leqq 2p$; $k\leqq 2p-1$. The indices possess lacunary values but for every p there are R. M.'s for which they simultaneously assume their extremes $2p^2$, p^2, and for all others $h<2p^2$, $k<p^2$. The R. M.'s with maximal indices are reducible to the form $\mathbf{N}$ of No. 1, with $\tau_{jk}=\delta_{jk}\cdot i\sqrt{d}$, where d is a positive integer, Scorza(94), p. 312, de Franchis(25), Lefschetz(51), p. 400. The special R. M. with $d=1$ has an isomorph whose terms differ by as little as we please from those of any assigned $\boldsymbol{\omega}$, Poincaré, further details and references in Krazer-Wirtinger(48), p. 831. The R. M.'s of type $\mathbf{N}$ with $\tau_{jk}=\delta_{jk}\tau$ (τ an imaginary number) possess a number of interesting properties. In particular they have an infinite number of distinct elliptic submatrices $\|1, \tau\|$. The abelian V_p is the image of the groups of p points, each on one of p elliptic curves with the common modulus τ.

41. *Varieties with Impure R. M.*—Let again $\boldsymbol{\omega}$ be isomorphic to $\boldsymbol{\omega}_0$ of No. 39, with $\boldsymbol{\omega}_i$ of genus p_i. The initial variables u_j can be linearly transformed into new variables v_j such that the periods of $v_1, \ldots, v_{p_2}$ as to the last $2p_2$ columns of $\boldsymbol{\omega}_0$ and of $v_{p_1+1}, \ldots, v_p$ as to the first $2p_1$ columns are all zero. Expressed in terms of $\boldsymbol{\omega}$ and the u's, there are

p_i linear combinations of these variables with only $2p_i$ distinct periods. In view of No. 37, this explains the common description of such a set as "regular system of reducible integrals."

Another property of an impure $\boldsymbol{\omega}$ is that there are associated intermediary functions factorable into a product of functions of a smaller number of variables, all linear combinations of the primitive variables, Wirtinger(119), Krazer(47), p. 500, Krazer-Wirtinger(48), p. 839.

Let V_{p_i} be the abelian variety of $\boldsymbol{\omega}_i$. V_p carries a congruence of varieties V'_{p_i} birationally equivalent to V_{p_i}, their equations being $v_{p_1+j} = \text{const.}$ for $i=1$, $v_j = \text{const.}$, $j \leq p_1$ for $i=2$. These congruences have been investigated by Castelnuovo(8), (9); Scorza(93), (95); and on a Jacobi variety by Torelli(110), Comessatti(16) in connection with their work on singular correspondences, Ch. 16, No. 13. Scorza in his papers determines the number n of intersections of V'_{p_1} with V'_{p_2}. It can also be obtained by writing down homologies analogous to (16) for V'_{p_1} and V'_{p_2} and applying (13). V_p itself is in $(1, n)$ correspondence with the variety V'_p that represents the point pairs of V_{p_1} and V_{p_2} and whose R. M. is $\boldsymbol{\omega}_0$. It is birationally equivalent to V'_p when and only when $\boldsymbol{\omega}$ is equivalent, not merely isomorphic, to $\boldsymbol{\omega}_0$. To every reduction of $\boldsymbol{\omega}$ to its normal form corresponds a definite involution on V_p.

§ 5. Transformations of V_p

42. There are two phases to the question: Geometric or transformation theory proper, and arithmetic or complex multiplication. We propose to consider the geometric phase first and to show its connection with complex multiplication. In § 6 we shall then treat the latter without concerning ourselves with its geometric implications.

Let T be an algebraic correspondence on V_p. By the same reasoning as that of Hurwitz for correspondences on a C^p (Ch. 16, No. 4), Scorza shows, (93), (94), Part II, that T leads to a multiplication

$$\boldsymbol{\alpha\omega} = \boldsymbol{\omega}\mathbf{a}, \tag{21}$$

the terms of $\mathbf{a}$ being here also integers. He considers various properties of these correspondences, in particular the valence when (21) reduces to an ordinary multiplication. He also shows that every multiplication determines a whole class of transformations, among which there are always rational transformations. Multiply if necessary both sides of the multiplication by an integer so chosen that $\mathbf{a}$ is as above a matrix with integer terms and let (21) be the form then assumed by the multiplication. The rational T will then have for equations:

$$u'_j = \Sigma a_{jk} u_k + \beta_j \tag{22}$$

similar to Humbert's system (8) of No. 14. The β's are of course arbitrary constants.

The ordinary transformations are here also those for which (21) is identically verified. They are classified as for hyperelliptic surface (No. 18) in transformations of the first kind

$$u_j + u'_j = \text{const.}$$

and transformations of the second kind

$$u_j - u'_j = \text{const.}$$

The latter alone form a continuous group of birational transformations, which is a p-parameter abelian group.

Let x_μ ($\mu = 1, 2, \ldots, 2p$) be the differences of the real coordinates (defined as in No. 2) of two points u corresponding by (22) to given u'. Then

$$\Sigma a_{jk} \omega_{k\mu} x_\mu = \Sigma \omega_{j\mu} a_{\mu\nu} x_\mu$$

must be a set of periods. The number n of distinct points of V_p corresponding to the point u' is then the number of distinct sets of rational x's between 0 and 1 (0 included) such that the expressions $\Sigma a_{\mu\nu} x_\mu$ are all integers. Hence $n = |a|$, Krazer(47), p. 58. Therefore T is $(1, n)$, Humbert(42), p. 289, Scorza(94), (93). T has $|a - I|$ coincidences (when their number is finite), Scorza(93).

The n points of V_p derived by T from a given point u of V_p constitute an I_n whose image is V_p itself. The u's of any two of them differ by constants. Therefore the n points are obtained from a given one of them by the operations of a birational group of order n of V_p whose transformations are all ordinary of the first kind. T may be considered as a birational transformation of V_p into another copy V'_p of itself taken as the image of I_n.

43. When we have a multiplication (21) we can deduce a whole class from it by multiplying the terms of $\boldsymbol{\alpha}$ and $\mathbf{a}$ by the same rational number m, the matrices being then $m\boldsymbol{\alpha}$, $m\mathbf{a}$. Hence we can always assume that the terms of $\mathbf{a}$ in (21) are relatively prime integers. Such a multiplication is called *primitive*. To every rational T associated with the class of multiplications will correspond a multiplication whose matrices are $m\boldsymbol{\alpha}$, $m\mathbf{a}$, where m is now an integer, and the index n of T will be $m^{2p}|a|$. For the matrices of all rational transformations there exists an F. S., $\mathbf{a}_1, \ldots, \mathbf{a}_h$, where h is the singularity index of $\boldsymbol{\omega}$. Every $\mathbf{a}$ whose terms are integers is then of the form $\Sigma x_i \mathbf{a}_i$ (x_i integer), Scorza(94), p. 283. All possible indices are $|\Sigma x_i a_i| = f(x_1, \ldots, x_h)$ where f is a homogeneous form of degree $2p$, with integer coefficients, $\geqq 0$ for all x's not all zero,

and in fact >0 when $\boldsymbol{\omega}$ is pure, Scorza(93), p. 6. A more accurate result is given by Scorza(96), p. 198. (See below No. 59). In order that T be a birational transformation it is necessary and sufficient that $|a|=1$.

Additional reference on transformations of V_p when $\boldsymbol{\omega}$ is impure: Cherubino(12.1). For $p=2$, see No. 15.

The preceding discussion can be extended to a rational transformation T of V_p^2 attached to $\boldsymbol{\omega}_2$, into V_p^1 attached to $\boldsymbol{\omega}_1$. In place of (21) we have now a relation $\boldsymbol{\alpha\omega}_1=\boldsymbol{\omega}_2\mathbf{a}$, with $\pm|a|$ as the order of T. Here T is birational if, and only if, $|a|=\pm 1$ that is if $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are equivalent. Assume $|a|\neq 0$. Then $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are isomorphic and to the multiplication $\boldsymbol{\beta\omega}_1=\boldsymbol{\omega}_1\mathbf{b}$ of $\boldsymbol{\omega}_1$ corresponds $(\boldsymbol{\alpha\beta\alpha}^{-1})\boldsymbol{\omega}_2=\boldsymbol{\omega}_2(\mathbf{aba}^{-1})$ of $\boldsymbol{\omega}_2$. Thus the multiplications of $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ correspond to one another but the primitive multiplications, i. e., the fundamental sets, do not in general unless $|a|=\pm 1$, when V_p^1 and V_p^2 are birationally equivalent.

To determine all rational transformations of V_p or all multiplications (21) with the terms of $\mathbf{a}$ integers, we may first obtain a set of h independent multiplications. From this set, an F. S. for the more special type just mentioned is readily derived by well-known arithmetical processes, Scorza(94), p. 282.

§ 6. COMPLEX MULTIPLICATION

44. We have already examined the case $p=1$ in Ch. 16, No. 2, and may refer for more extensive information to Fricke(26), Fueter(29), Weber(117). The case $p=2$ has also been considered at length in § 2 and we have shown there how Humbert in (42) laid down the foundations for many of the later developments. Humbert-Lévy(45); Lévy(57); have developed the properties of a certain geometric configuration attached to an R. M. of genus three in relation to its multiplications. This representation is somewhat related to that used by Scorza(94) or Rosati(82).

Previous to Humbert, only complex multiplications corresponding to *principal* transformations had been considered. For references see Krazer-Wirtinger(48), p. 852; also Krazer(47), p. 220, where these transformations are treated in detail. Here as in No. 16, and with Scorza(94), p. 287, a transformation is *principal* whenever there is a principal $\mathbf{c}$ of $\boldsymbol{\omega}$ such that $\mathbf{aca}'=l\mathbf{c}$. The study of the most general multiplication was initiated by Scorza in (99) and the chief contributions are due to him, to Lefschetz and to Rosati. The latter has always in view his main problem: singular correspondences on a C^p. His work has been considered in various parts of Ch. 16.

45. From now on we associate with the multiplication (21) the uniquely defined rational transformation (22) with constants β all zero,

and apply to one or the other the same symbol T. Let T_1, T_2, be two multiplications with $\boldsymbol{\alpha}_1$, $\mathbf{a}_1$ and $\boldsymbol{\alpha}_2$, $\mathbf{a}_2$ as $\boldsymbol{\alpha}$, $\mathbf{a}$ matrices. Then $m_1\boldsymbol{\alpha}_1+m_2\boldsymbol{\alpha}_2$, $m_1\mathbf{a}_1+m_2\mathbf{a}_2$ (m_1, m_2 rational) belong to a multiplication to be called $m_1T_1+m_2T_2$ and $\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2$, $\mathbf{a}_1\mathbf{a}_2$ to a multiplication called T_1T_2. When Σm_iT_i corresponds to matrices $\boldsymbol{\alpha}$, $\mathbf{a}$, both zero (they always vanish simultaneously) we say that the T's are *dependent* and write $\Sigma m_iT_i=0$. The maximum number of linearly independent T's is the multiplication index h, Nos. 38, 46. Hence the T's are combined like the numbers of a linear associative algebra, in fact a matrix algebra, in h units over the field of rational numbers. These properties are all due to Scorza(94), (96) and he has made a systematic study of the algebra in (96). We return to it in No. 59.

46. We have already considered in No. 38, Scorza's important theorem according to which the numbers of linearly independent rational projectivities and rational reciprocities are both h. It implies that the number of linearly independent multiplications is also h. Scorza's proof actually shows, however, that the reciprocities or the rational matrices of $\boldsymbol{\omega}$ determine the multiplications, and vice versa. It is worth while giving a proof of this important proposition, more in line with the order of ideas of this Report where we have made the matrices themselves the central elements.

Let then $\mathbf{c}$ be a principal matrix of $\boldsymbol{\omega}$ so that $\boldsymbol{\omega}\mathbf{c}\boldsymbol{\omega}'=0$, $|c|\neq 0$. Then from (21) follows $\boldsymbol{\alpha}\boldsymbol{\omega}\mathbf{c}\boldsymbol{\omega}'=\boldsymbol{\omega}\mathbf{a}\mathbf{c}\boldsymbol{\omega}'=0$, therefore $\mathbf{d}=\mathbf{a}\mathbf{c}$ is an ordinary matrix of $\boldsymbol{\omega}$. Conversely with $\boldsymbol{\Omega}=\begin{Vmatrix}\boldsymbol{\omega}\\ \bar{\boldsymbol{\omega}}\end{Vmatrix}$, we have, given $\mathbf{d}$ and since $\mathbf{c}$ is principal,

$$\boldsymbol{\Omega}\mathbf{c}\boldsymbol{\Omega}'=\begin{Vmatrix}0 & \boldsymbol{\gamma}\\ \bar{\boldsymbol{\gamma}} & 0\end{Vmatrix};\qquad \boldsymbol{\Omega}\mathbf{d}\boldsymbol{\Omega}'=\begin{Vmatrix}0 & \boldsymbol{\delta}\\ \bar{\boldsymbol{\delta}} & 0\end{Vmatrix},$$

where $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ are square, of order p. Since $|\gamma|$, $|c|$ are $\neq 0$, $\boldsymbol{\alpha}=\boldsymbol{\delta}\boldsymbol{\gamma}^{-1}$, $\mathbf{a}=\mathbf{d}\mathbf{c}^{-1}$, are uniquely defined and we find

$$\mathbf{A}\boldsymbol{\Omega}=\boldsymbol{\Omega}\mathbf{a};\qquad \mathbf{A}=\begin{Vmatrix}\boldsymbol{\alpha} & 0\\ 0 & \bar{\boldsymbol{\alpha}}\end{Vmatrix},$$

from which (21) and the mutual dependence of $\mathbf{d}$ and $\mathbf{c}$ upon one another follow.

47. The problem that naturally comes next is the classification of R. M.'s of a given genus according to their multiplications. The two chief contributions on that topic are due to Scorza(97), (100), (96) and Lefschetz(53), (51). They are found in the major papers (96) and (51), both published in (1921), the others being merely preliminary short

notes. Scorza's main object is to develop the general theory of the multiplication algebra and particularly the properties of a new invariant, the rank ρ of an R. M. His typical theorems bear on the possible relative values of h, k, ρ, p. Lefschetz on the contrary attacked directly the classification and in particular made a thoroughgoing investigation of two special but very important types of R. M.'s that include all those found in the literature. We shall therefore report on his work and at the same time on related special contributions. In No. 59 we shall then give an account of Scorza's work on the multiplication algebra.

48. Consider first the multiplication T represented by (21). The characteristic equations of $\boldsymbol{\alpha}$ and $\mathbf{a}$ are respectively

$$\phi(x) = |\alpha - xI| = 0; \qquad F(x) = |a - xI| = 0.$$

The roots of ϕ are the *multipliers* of T. When (21) arises from a correspondence on a C^p they are the negatives of the Rosati valences (89), (90). In the notation of No. 46, since $\mathbf{a} = \boldsymbol{\Omega}^{-1}\mathbf{A}\boldsymbol{\Omega}$, necessarily

$$F(x) = |a - xI| = |\alpha - xI| \cdot |\bar{\alpha} - xI| = \phi(x) \cdot \bar{\phi}(x).$$

Therefore if x_1 is a root of ϕ, $\bar{x}_1$ is a root of $\bar{\phi}$ and with the same multiplicity. Hence the real roots of F are all of even multiplicity and its complex roots are associated in pairs of conjugate roots of same multiplicity. This was proved for principal transformations by Frobenius(87), and for the general case by Scorza(94), p. 284. See also Krazer(47), p. 224, Krazer-Wirtinger(48), p. 853.

Incidentally $|a| = |\alpha| \cdot |\bar{\alpha}| \geqq 0$ and can only be zero when one of the multipliers is zero, and then $\boldsymbol{\omega}$ is impure. Conversely when $\boldsymbol{\omega}$ is impure it always possesses a multiplication whose $|a| = 0$ (No. 39).

T is an ordinary multiplication when all the multipliers are equal and then they are rational.

T satisfies the *characteristic* equation $F(T) = 0$, but its *minimum* equation $f(T) = 0$, or equation of least degree q that it satisfies, may be different, although the roots of $F(x) = 0$, $f(x) = 0$ are the same. Here q, called the *degree* of T, is the maximum number of linearly independent powers of T. See in this connection Rosati(89), (90).

49. Let $\boldsymbol{\omega}$ be impure and in the normal form of No. 39. All its **d** matrices (No. 46) are linear combinations of those of degenerate type which involve one or two pure components of the same $\boldsymbol{\Omega}_i$, both of which types are known when the **d** matrices for the pure components are known. Hence the determination of all multiplications of $\boldsymbol{\omega}$ can be reduced to that of all multiplications of the pure components $\boldsymbol{\omega}_i$, and to find all possible multiplications and all R. M.'s that possess complex multiplications we may limit ourselves to pure R. M.'s.

Now for a pure R. M., f is always irreducible and $F \equiv f^r$. Hence when $\boldsymbol{\omega}$ is pure the degree q of f is always a factor of $2p$. It is furthermore a factor of h; and these circumstances narrow the field down a good deal. Observe that F may well have the preceding form for an impure $\boldsymbol{\omega}$. See Lefschetz(51), Part II, Ch. 2, § 1; Scorza(100), (96), Part II, §§ 2, 3, 4.

50. When there is a T of degree $q > 1$ such that $F \equiv f^r$, with the roots of f all simple, $\boldsymbol{\omega}$ can be reduced to a form exhibiting the presence of its multiplication. If the multipliers are $\mu_1, \ldots, \mu_p$, the u's can be so chosen that the equations of the transformation become

$$u'_j = \mu_j u_j \tag{23}$$

and (21) gives then

$$\mu_j \omega_{j\sigma} = \Sigma \omega_{j\nu} a_{\nu\sigma}. \tag{24}$$

Hence if $\omega_{j\sigma}$ is any set of periods so is $\mu_j{}^s \omega_{j\sigma}$. This enables us to reduce $\boldsymbol{\omega}$ to the form

$$\mathbf{H} = ||\xi_{j\sigma} \mu_j{}^s||, \quad (s = 0, 1, \ldots, q-1; \quad \sigma = 1, 2, \ldots, r; \quad j = 1, 2, \ldots, p),$$

which is especially convenient for our purpose because it exhibits plainly the elements independent of T: the terms of the matrix ξ. Given T and its characteristic equation when $r > 1$, ξ is indeed in general variable, an observation first made by Frobenius(87) for the special case of a principal transformation. For $r = 1$, $\mathbf{H}$ was first given by Scorza(94), p. 24, for $r > 1$ by Lefschetz(51), p. 397. Earlier examples for $p = 2$ and cyclic T occur in Bagnera-de Franchis(2).

Among the notable R. M.'s of type $\mathbf{H}$ are those which correspond to principal transformations. They have been repeatedly considered by various authors. See Koenigsberger(45.2); Kronecker(49); Weber(115); Frobenius(87); Scorza(94), p. 287; (96), p. 153; Lefschetz(51), p. 399; Krazer-Wirtinger(48), p. 853. Frobenius showed that the multipliers of such a T have all the same absolute value $\sqrt{\rho}$, where ρ is a rational number. Scorza showed that if $\rho = 1$, T is a cyclic birational transformation, and that conversely when T has that property it is principal with its $\rho = 1$, (94), p. 289. In (96), p. 153, he observes that when $\boldsymbol{\omega}$ is non-singular ($k = 1$) all its transformations are principal.

Rosati showed (89), p. 1007 (see Ch. 16, No. 7) that a symmetric correspondence on a C^p is associated with a T of the R. M. of C^p whose minimum equation has only simple roots all real. When the correspondence is skew symmetric the multipliers are pure imaginary, Rosati(90), p. 14. When the R. M. is pure it falls in the two categories investigated by Lefschetz in (51) and considered below Nos. 55, 56.

51. Lefschetz has extensively investigated, (81), Part II, Ch. 2, the case $r=1$. Then $F\equiv f$, $q=2p$ and the matrix is

$$\mathbf{K}\equiv ||\mu_j{}^s||, \quad (s=0,1,\ldots,2p-1;\quad j=1,2,\ldots,p).$$

The roots of $F=0$ are all distinct and imaginary and the multipliers consist of a set of p of these roots no two of which are conjugate. The properties of $\mathbf{K}$ depend partly upon the equation $F=0$, partly upon the particular choice of roots for multipliers.

Given, a priori, $F(x)=0$ with the same properties as above, select p roots $\mu_1, \mu_2, \ldots, \mu_p$, that do not include a conjugate pair and write down $\mathbf{K}$. In order that it be an R. M. it is necessary and sufficient that the group G of F permute no pair of roots μ_i, μ_j with a conjugate pair. There are $p+t$ pairs that cannot be derived from the pairs of p roots selected by the operations of G. The invariants of $\mathbf{K}$ are $h=2k=2(p+t)$. In order that $\mathbf{K}$ be pure it is then necessary that $t=0$ (No. 40) a condition which is also found to be sufficient. Hence a pure K has the indices $h=2k=2p$.

In G there is a unique substitution (binary) S permuting every root with its conjugate. The necessary and sufficient conditions in order that $\mathbf{K}$ be pure are: (a) S must be permutable with G. (b) It must be possible to derive by means of operations of G every non-conjugate pair of roots from some pair of multipliers. Of these two conditions (b) alone depends upon the choice of multipliers. When (a) is fulfilled every choice of p roots, no two of which are conjugate, gives rise to an R. M. $\mathbf{K}$ (pure or impure).

When $\mathbf{K}$ is pure it is isomorphic to an R. M. of the same type with corresponding

$$F(x)\equiv \prod_{j=1}^{p} (x^2-2\xi_j x+R(\zeta_j)),$$

where R is a polynomial of degree $p-1$ with rational coefficients and the ζ's all real, satisfy an irreducible equation with rational coefficients and are such that $\zeta_j{}^2<R(\zeta_j)$ for every j. The special case where R is a rational number has been considered by Scorza(103).

52. More precise results can be given when G is an abelian group. Then $\mathbf{K}$ is an R. M. provided the ζ's include no pair of conjugate roots. The necessary and sufficient condition in order that the R. M. be pure is that the group G contain no subgroup (other than the identity) leaving the set $\mu_1, \ldots, \mu_p$ invariant. Let there exist such a subgroup and let n be its order. Then $\mathbf{K}$ is isomorphic to an R. M. in the normal form (No. 39) consisting of a single $\mathbf{\Omega}_1$ with n component submatrices each of type $\mathbf{K}$ and genus $\dfrac{p}{n}$, with the associated equation analogous to $F=0$

also abelian. The invariants are $h=2k=2np>2p$, another indication that the matrix is impure.

In order that two different matrices **K** derived from $F(x)=0$ be isomorphic it is necessary and sufficient that G contain some operation (not the identity) permuting their sets of multipliers. The number of essentially distinct R. M.'s belonging to $F(x)=0$ is therefore the same as its number of classes of transitively permutable sets of p roots containing no conjugate pair.

An especially interesting case is that in which T is cyclic and the μ's are roots of unity.

53. In (51), Part II, Ch. 4, Lefschetz investigated the R. M.'s of the curves

$$y^q=(x-a_1)^{a_1}(x-a_2)^{a_2}(x-a_3)^{a_3},$$

where q is an odd prime not a factor of the exponents but a factor of their sum. The genus $p=\frac{1}{2}(q-1)$. The R. M. is pure of type **K** unless the curve is a birational transform of

$$y^q=(x-1)(x-\eta)^{\lambda}(x-\eta^2)^{\lambda^2}$$

with $\dfrac{q-1}{3}=$ an integer; $\eta=e^{\frac{2\pi i}{3}}$; $\lambda^3\equiv 1$, mod q. In this case $h=2k=6p$. See the end of Chapter 15. A number of special cases of matrices **K** occur in the literature. For early references see Krazer-Wirtinger(48), p. 830. To them must be added Scorza(92), Raciti(71), (73). Some special matrices **K** of genus two are also given by Enriques-Severi(22), Bagnera-de Franchis(8), Scorza(94), Part II.

In (96), p. 164, Scorza considers R. M.'s whose rank (see No. 59) $\rho=2p$ is maximum. He shows that when they are pure they are isomorphic to a matrix **K**.

54. In (51), p. 413, (55.1), Lefschetz considers the general properties of the R. M. **H** with irreducible minimum equation $f(x)=0$. The multipliers can be assumed in such an order that those that are equal are all consecutive. Let the root μ_j of $f(x)$ occur r_j times among them and call $\boldsymbol{\xi}_j$ the matrix in r_j rows and r columns made up of the corresponding terms of $\boldsymbol{\xi}$. To a rational matrix **c** of **H** corresponds a square matrix whose terms are polynomials in μ_j, μ_k with rational coefficients, $\boldsymbol{\gamma}(\mu_j, \mu_k)$, such that $\boldsymbol{\xi}_j \cdot \boldsymbol{\gamma}(\mu_j, \mu_k) \cdot \boldsymbol{\xi}'_k=0$. When **c** is alternate

$$\boldsymbol{\gamma}(\mu_j, \mu_k)=-\boldsymbol{\gamma}(\mu_k, \mu_j).$$

In order that **H** be an R. M. it is necessary and sufficient that it possess an alternate **c** such that the hermitian form $\frac{i}{2}\Sigma\gamma_{\rho\sigma}(\mu_j, \bar{\mu}_j)x_\rho\bar{x}_\sigma$ be positive for all sets of x's linear combinations of the rows of ξ_j. If $|\gamma(\mu_j,\mu_k)|=0$ for any one pair of roots, **H** is impure.

When the multiplications of a given R. M. are all permutable, they depend upon the powers of a unique one T, and if q is its degree, $h=q$. The Scorza algebra is then permutable and its elements correspond term for term to the numbers of an algebraic domain of degree q. When the multiplications are not permutable, whatever T, $h>q$. In order that the first case occur for a pure matrix it is necessary that for any T the multipliers be either all real, or else all complex, the group G of $f(x)$ being permutable with the operation S that interchanges conjugate roots.

55. When the multipliers are all real, Lefschetz(51), p. 420, the integers r_j are all equal to $\frac{1}{2}r$. The necessary and sufficient conditions in order that **H** be an R. M. are: There exists a square matrix of order r, $\boldsymbol{\gamma}(\mu_j)$, whose terms are polynomials in μ_j with rational coefficients, such that $\xi_j\cdot\boldsymbol{\gamma}(\mu_j)\cdot\xi'_j=0$ while $i\xi_j\cdot\boldsymbol{\gamma}(\mu_j)\cdot\xi'_j$ is a definite positive hermitian matrix. In short the ξ's behave like R. M.'s whose principal matrices have for terms numbers of the domain $R(\mu_j)$ obtained by adjoining μ_j to the ordinary domain of rationality. The general theorems on R. M.'s can be paraphrased here, giving rise to new theorems obtained by replacing rational matrices by matrices whose terms are in $R(\mu_j)$. In particular there are indices h_1, k_1, analogous to h, k and when $r>2$, $h=qh_1$, $k=qk_1$, while for $r=2$, $h=k=q$. The matrix **H** is pure, if and only if, the ξ's are pure. In order that the multiplication T be of maximum degree it is necessary and sufficient that $h_1=1$ and since $0<k_1\leqq h_1$, $k_1=1$, hence $h=k=q$. Given $f(x)$, **H** depends upon $\frac{1}{4}p(r+2)$ essential parameters.

The case $r=2$ gives rise to a theory analogous to elliptic functions, with modular functions, thetas, and applications to number theory. There are important investigations along that line due to Hecke(30). See also Krazer-Wirtinger(48), p. 857.

56. Lefschetz examines in detail (51), p. 426, the case of a matrix **H** whose f has a group G permutable with the substitution S of No. 54. In this case the integers r_j may all be different. Let r_j, $\bar{r}_j$ correspond to μ_j, $\bar{\mu}_j$. Then for any choice of these integers such that $r_j+\bar{r}_j=r$ (for every j) there exists a corresponding R. M. and it depends upon $\Sigma r_j\bar{r}_j$ continuous parameters, the sum in this expression including only one term for each conjugate pair of roots. Lefschetz gives explicitly the values of h, k. When G is abelian and $r>2$, $h=2k=q$ and all multiplications of **H** depend upon the powers of T. One type discussed, for which $r_j=r'_j=\frac{1}{2}r$, and in general $h=2q$, $k=\frac{1}{2}q$, includes a noteworthy class discovered by Scorza(97), (96), p. 179. It is characterized by $q=2$, $k=1$,

$h=4$. Humbert(42), had shown that for $p=2$, complex multiplication could only occur when the R. M. is singular, i. e., when $k>1$. Scorza showed by his example that this is true only for $p=2$, the special R. M.'s that he discovered being, however, the only exception.

57. The R. M.'s of the curves

$$y^q=\Pi(x-a_i)^{a_i}, \qquad q \text{ an odd prime}$$

fall under the type **H**. Their properties are investigated at length by Lefschetz(51), Part II, Ch. 4, under the assumption that the a's are arbitrary numbers. Let $r+2$ be the number of branch points of $y(x)$. The curve may be birationally transformed into one of the same type with $r+1$ factors at the right and neither the a's nor their sum divisible by q. The Jacobi variety V_p possesses a cyclic birational transformation T:

$$u'_j=\epsilon^{n_j}u_j\,; \qquad \epsilon=e^{\frac{2\pi}{q}},$$

the exponents n_j being readily determined when the a's are known. The degree of T is $q-1$ and the R. M. is of type **H** with

$$\mu_j=\epsilon^{n_j}, \qquad f(x)\equiv 1+x+\ldots+x^{q-1},$$

and the present r playing the same part as before. We have already considered $r=1$ in No. 53. For $r>1$ the R. M. is pure with $h=2k=q-1$, unless the curve is birationally transformable into

$$y^q=\frac{x-a_1}{x-a_2}\left(\frac{x-a_3}{x-a_4}\right)^a,$$

when it is impure, with $h=2k=6p$. Lefschetz determines the complete birational group of the curve and of V_p. With the one exception just mentioned, the multiplications all depend upon the powers of T, just as for the general R. M. of the class. This is a remarkable fact in view of the restrictions here present. For we may always take $a_1=0$, $a_2=1$, hence there are only $r-1$ essential parameters which is always less than $\Sigma r_j\bar{r}_j$, the number of parameters of the general matrix of the class.

58. In (51), p. 432, Lefschetz applies his general results to the classification of R. M.'s with $p=2$, 3 into classes of non-isomorphic matrices. They all fall under the types **H**, **K**, just described, or else have no complex multiplications, when $h=k=1$. These general R. M.'s may also be thought of as of type **H** with $q=1$. The matrices are described in each case, the number and range of the essential parameters given and the values of h, k determined. The one R. M. of genus two of type **K** (51), pp. 405 and 432, may be reduced to a simpler form. For if $\phi(x)$ is any polynomial with rational coefficients there is a T whose multipliers are

$\phi(\mu_j)$, with a corresponding new isomorphic form for the matrix. By choosing ϕ properly, the multipliers can be taken as

$$\mu_1 = i\sqrt{n+m\sqrt{d}}, \qquad \mu_2 = i\sqrt{n-m\sqrt{d}},$$

where m, n, d are positive integers such that m, n are prime to one another, d has no square factor, and n^2-dm^2 is not a perfect square. See also for $p=3$, Scorza(98), (96), p. 167.

59. In (100), (96), Scorza introduces various linear algebras, related to the multiplications of an R. M. $\boldsymbol{\omega}$, and investigates their chief properties. These algebras are isomorphically invariant. The most important one is the algebra over the rational domain whose elements are the multiplications T. Scorza defines the *rank* ρ of $\boldsymbol{\omega}$ (not to be confused with the rank of an attached V_p). ρ is the maximum of the degree of T, or degree of the generic T, and is an isomorphic invariant integer. In any case $\rho \leqq 2p$. Scorza examines the nature of the algebra in relation to the normal form of No. 39, and finds that its properties are known whenever those of the individual components $\boldsymbol{\Omega}_i$, and ultimately those of the pure components $\boldsymbol{\omega}_i$, are known. He gives a number of theorems on ρ. For a pure matrix it is a factor of $2p$ and also of h. Furthermore $\frac{h}{\rho}$ is a factor of ρ, hence $h \leqq \rho^2$. He also proves in this connection that when $\boldsymbol{\omega}$ is pure h is a factor of $4p^2$. If ρ is prime $h=\rho$ or ρ^2. Scorza also examines the special cases where ρ is maximum $(\rho=2p)$, $\rho=2$, or ρ is prime, also the case where p is prime and gives the corresponding values of h, k. He investigates at length (96), p. 179, the non-singular matrices mentioned in No. 56, proves their existence for every $p>2$, shows that their $\rho=2$, while their invariants h, k can assume the sets of values $k=1, 2$; $h=2, 4$ in all four possible combinations, with $k=1$ possible only when $p \neq 2$. In the same paper, p. 198, Scorza also proves this theorem: If $\boldsymbol{\omega}$ is pure, its V_p can only possess rational $(1, n)$ transformations, whose n can be represented by integer values of the variables s, in the form $f(x_1, \ldots, x_h)^{\frac{2p}{\rho}}$, where f is a certain homogeneous polynomial of degree ρ. The form f itself can represent either 1 or -1 and when $\frac{2p}{\rho}$ is odd and $\boldsymbol{\omega}$ pure the form is >0 when the variables are not all zero.

As closely related to the properties of ρ see the theorems on the degree q of T given by Lefschetz(51), p. 395.

§ 7. ADDITIONAL TOPICS

60. *Jacobi Varieties.*—We have already observed (No. 3) that they are very special abelian varieties, owing to the fact that they depend upon $3p-3$ moduli only instead of $\frac{p(p+1)}{2}$ like the general abelian V_p. The actual effect of this reduction in the number of moduli is as yet wholly obscure and there is no inkling of anything like a theory of the Jacobi V_p, but only scattered results. They bear mostly upon the interrelations of the properties of the period matrix and the singular correspondences on the related C^p, and in this connection have been treated in Chapter 16. In regard to the connections between the hermitian birational transformations of V_p and those of C^p see Comessatti(18.1); Rosenblatt(91); also Ch. 16, No. 15.

61. *Varieties of Rank* >1.—Here also we have scattered results rather than a general theory. This is partly due to our ignorance of the properties of algebraic varieties of more than two dimensions. However, there would appear to be no difficulty in extending a good part of the results due to Enriques-Severi, and Bagnera-deFranchis (§2 of this chapter). For example we believe that by rather simple considerations of Analysis Situs it is possible to prove the following extension of the important theorem of Enriques-Severi(22), p. 331, this chapter No. 19: An I_n on a V_p of rank one whose coincidence locus has at most $p-2$ dimensions is generated by a finite birational group of V_p. However, although there is no complete theory for the varieties whose rank >1, there are a number of results which we shall now describe.

62. The generalized Kummer variety W_p, image of the I_2 generated on V_p of rank one by the binary transformation $u'_j=-u_j$, has been investigated by Wirtinger(121), (122). He found that it has most properties of the Kummer surface. See also Coble(14.1), especially for the case $p=3$, in connection with his investigations on modular functions. In (19) Wirtinger, by proving the functional independence of the thetas of order two, has indirectly shown that the general Kummer variety can be parametrically represented by making the homogeneous coordinates proportional to the elements of a maximal linearly independent set of even θ's of order two. The representation is (1, 1) without exception. The carrying space of W_p is on an S_n $n=2^p-1$ and its order is $2^p \cdot p!$. For further references see Krazer(47), p. 368; Krazer-Wirtinger(48), pp. 669, 753; Raciti(72).

63. In (51), Part II, Ch. 3, Lefschetz examines cyclic birational transformations of a V_p of rank one, and related varieties of higher rank. If

the multiplications of V_p all depend upon the powers of a single one, the discovery of the birational transformations is reducible to that of the units of a certain Dedekind order (ring) of a definite algebraic domain. This is applied to the case $p=2$. See in this connection Spampinato(109.1), (109.2) and for a related discussion for correspondences on curves, Rosati(84), p. 37.

Any cyclic transformation T of V_p has parametric equations

$$u'_j=\epsilon^{n_j}u_j+\eta_j, \qquad \epsilon=e^{\frac{2\pi i}{n}},$$

where n is the order of T. The resulting variety W_p of rank n, image of the I_n whose groups are made up of any point of V_p and its transforms, has $F(1)$ singular points, where $F(T)=0$ is the characteristic equation of T. Lefschetz also determines the connectivity and torsion indices of W_p. When there are no isolated singular points some of the latter are >1 and the surfaces which they carry yield simple examples of surfaces whose $\sigma>1$. He also examines the number of linearly independent integrals of the first kind of various multiplicities and proves this theorem: When $\boldsymbol{\omega}$ is pure and the characteristic equation reducible W_p cannot be a locus of linear spaces. This he also extends to the case where the characteristic equation is irreducible and the order n is a prime number >7.

64. *Real Abelian Varieties.*—In (56), Lefschetz examines their properties having especially in view the number and properties of the real folds. By considering the effect upon the linear cycles of the (1, 1) transformation S of V_p which permutes its pairs of conjugate points he finds that the R. M. $\boldsymbol{\omega}$ is isomorphic to the form

$$\mathbf{L}\equiv||\Omega_{j1}, \ldots, \Omega_{jp}; i\Omega_{j,\,p+1}, \ldots, i\Omega_{j,\,2p}||$$

where all quantities $\Omega_{j\mu}$ are real. The double of every period system of $\boldsymbol{\omega}$ is expressible as a sum of multiples of those just written, so that $2\boldsymbol{\omega}$ is equivalent to $\mathbf{L}$. If there are $p-s$ distinct periods whose imaginary part is $\frac{i}{2}\Sigma r_\mu\Omega_{h,\,p+\mu}$ the variety possesses 2^s real folds. An example is given to show that there are actual varieties for every s from 0 to p, extremes included. The folds are all continuously and birationally transformable into one another on V_p and constitute therefore homologous p-cycles. Topologically they are p dimensional rings. Lefschetz applies this to real Jacobi varieties and obtains a new proof of Harnack's theorem according to which a real curve of genus p can have at most $p+1$ real branches. He also determines the number ρ' of algebraically distinct real hypersurfaces of V_p. It may be considered as the analogue of the Picard number ρ when only real curve systems are considered. These systems are defined by the condition that the transformation S defined above, leaves them invariant. The value of ρ' is equal to the number of linearly independent rational matrices attached to $\mathbf{L}$ that are of the form $\left\|\begin{matrix} 0 & \mathbf{m} \\ -\mathbf{m}' & 0 \end{matrix}\right\|$. When $\mathbf{L}$ is pure $\rho' \leqq p$. This is to be compared with the limit resulting for ρ itself from the fact that $\rho = k \leqq 2p$. Lefschetz examines also the possible real types of hyperelliptic surfaces and the corresponding values of h, k, ρ'.

In (12) Cherubino shows that in order that there exist a real V_p attached to $\boldsymbol{\omega}$ it is necessary and sufficient that $\boldsymbol{\omega}$ and $\bar{\boldsymbol{\omega}}$ be equivalent. They satisfy then a relation $\boldsymbol{\alpha}\boldsymbol{\omega} = \bar{\boldsymbol{\omega}}\mathbf{a}$, $|a| = (-1)^p$. He investigates the isomorphisms between $\boldsymbol{\omega}$ and $\bar{\boldsymbol{\omega}}$ (whether there exists a real V_p or not) and develops for them a theory quite similar to Scorza's for ordinary multiplication. He also obtains the number of distinct classes of real varieties belonging to a non-singular $\boldsymbol{\omega}$ $(k=1)$, distinct in the sense that they are not reducible to one another by real birational transformations.

Comessatti has considered the same questions, (15), (15.1), (18), and showed that the number in question is 2^{s+1}, where s is the same integer as introduced previously by Lefschetz(56). Among the classes in question only two have real points. In (15.1) Comessatti also considers various types of hyperelliptic surfaces and generalized Kummer varieties, particularly from the point of view of their distribution into distinct real classes in the above sense. See also Cherubino(12.2).

65. *Additional References.*—A number of questions more or less closely related to those treated in this chapter are taken up in Krazer-Wirtinger(48), pp. 843, 853. There are also a number of papers due to Rosati not described in this chapter, but listed under his name in the bibliography at the end, likewise for some of the papers, mostly short notes, listed under Scorza. The former author's work is directly or indirectly applicable to our topic and much of it has been described in Chapter 16. The material in Scorza's short notes is always contained in his major papers to which we have largely confined the quotations of this chapter.

Humbert(40); Rémy(74), (77), (78); Godeaux(29.1), (29.2), (29.3) have investigated the surface F obtained as follows: to every point pair of a curve C^3 corresponds a point of F, to every point of F two point pairs constituting a canonical group of C^3. F is on a generalized Kummer V_3 whose other surfaces are studied by Rémy.

Krazer has given a very complete treatment of ordinary transformations(47), Ch. 5, also Krazer-Wirtinger(48), with very complete references. To the references given by Krazer-Wirtinger(48), p. 648, on half-integer characteristics should be added Coble(14), (14.2), where there is found a very comprehensive representation by certain points on a finite projective space.

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